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Some New Bounds for α-Adjacency Energy of Graphs

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Abstract: Let G be a graph with the adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of G. Nikiforov first defined the matrix $A_\alpha(G)$ as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, $0 \le \alpha \le 1$, which shed new light on $A(G)$ and $Q(G) = D(G) + A(G)$, and yielded some surprises. The $\alpha-$adjacency energy $E^{A_\alpha}(G)$ of G is a new invariant that is calculated from the eigenvalues of $A_\alpha(G)$. In this work, by combining matrix theory and the graph structure properties, we provide some upper and lower bounds for $E^{A_\alpha}(G)$ in terms of graph parameters (the order n, the edge size m, etc.) and characterize the corresponding extremal graphs. In addition, we obtain some relations between $E^{A_\alpha}(G)$ and other energies such as the energy $E(G)$. Some results can be applied to appropriately estimate the α-adjacency energy using some given graph parameters rather than by performing some tedious calculations.

Keywords: adjacency matrix; energy; α-adjacency matrix; α-adjacency energy

MSC: 05C50; 05C12; 15A18

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1. Introduction

All graphs in this paper are simple, finite, and undirected. Let G be a graph with a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Denote by $n = n(G) = |V(G)|$ the order of G and $m = m(G) = |E(G)|$ the number of edges of G. Let d_i be the i-th largest degree of the vertex of G, $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix, and $A(G)$ the adjacency matrix of G. The matrix $A_\alpha(G)$ of G is defined in [1] as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), 0 \le \alpha \le 1. \quad (1)$$

Clearly,

$$A(G) = A_0(G), D(G) = A_1(G), Q(G) = 2A_{1/2}(G),$$

where $Q(G)$ is the signless Laplacian matrix of G. In this way, $A(G)$, $Q(G)$, and $D(G)$ were viewed from a new perspective, which resulted in many interesting problems (see [1] for more details).

The energy $E(G)$ of a graph G was introduced by Gutman [2], i.e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|,$$

where $\lambda_1(G) \ge \lambda_2(G) \ge \cdots \ge \lambda_n(G)$ are the eigenvalues of $A(G)$, which are called the adjacency eigenvalues of G. This quantity has a long-known chemical application (see the surveys in [3–5] for details).

Note that $A_\alpha(G)$ is a real and symmetric matrix and all its eigenvalues are real and denoted by $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$. They are also called the α-adjacency eigenvalues of G. The α-adjacency energy $E^{A_\alpha}(G)$ of a graph G is defined in [6] as

$$E^{A_\alpha}(G) = \sum_{i=1}^{n} \left|\rho_i(G) - \frac{2\alpha m}{n}\right|.$$

The adjacency and α-adjacency eigenvalues of G obey the following relations:

$$\sum_{i=1}^{n} \lambda_i(G) = 0; \sum_{i=1}^{n} \lambda_i^2(G) = 2m, \tag{2}$$

$$\sum_{i=1}^{n} \rho_i(G) = 2\alpha m; \sum_{i=1}^{n} \rho_i^2(G) = 2(1-\alpha)^2 m + \alpha^2 \sum_{i=1}^{n} d_i^2. \tag{3}$$

Clearly,

$$E^{A_0}(G) = E(G), 2E^{A_{1/2}}(G) = QE(G),$$

where $QE(G)$ is the signless Laplacian energy of G. So, it is of great interest to study the $\alpha-$adjacency energy.

In this paper, we provide some new upper and lower bounds for $E^{A_\alpha}(G)$ and characterize the extremal graphs that attain these bounds. We also consider the relations between the $\alpha-$adjacency energy and the other energies of a graph.

2. Upper Bounds for α-Adjacency Energy of Graphs

For any matrix A, A^* is the conjugate transpose of A. The singular values of a matrix A are defined as the square roots of the eigenvalues of A^*A and the energy of A is the sum of its singular values and is denoted by $E(A)$.

Lemma 1 ([7])**.** *Let $A, B \in \mathbb{R}^{n\times n}$ and let $C = A + B$. Then,*

$$E(C) \leq E(A) + E(B).$$

Moreover, the equality holds if and only if there exists an orthogonal matrix P such that PA and PB are both positive semidefinite matrices.

The following lemmas provide some basic properties of the positive semidefinite matrices.

Lemma 2 ([8])**.** *If $A \in \mathbb{R}^{n\times n}$ and there exist positive semidefinite matrices $X, Y \in \mathbb{R}^{n\times n}$ and orthogonal matrices $P, Q \in \mathbb{R}^{n\times n}$, such that $A = PX = YQ$. Moreover, $X = |A|$, $Y = (AA^T)^{\frac{1}{2}}$ are unique matrices that satisfy these equalities. In addition, the matrices P and Q are uniquely determined if and only if A is nonsingular.*

Lemma 3 ([9])**.** *If $A = (a_{ij})_{n\times n}$ is a positive semidefinite matrix and $a_{ii} = 0$ for some i, $a_{ij} = 0 = a_{ji}$, $j = 1, 2, \ldots, n$.*

In 2006, Gutman and Zhou [10] studied the Laplacian energy of graph G,

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i(G) - \frac{2m}{n}\right|,$$

where $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ are Laplacian eigenvalues of G. Furthermore, they first introduced the auxiliary "eigenvalues". Similarly, let γ_i, $i = 1, 2, \ldots, n$, be defined via

$$\gamma_i = \rho_i(G) - \frac{2\alpha m}{n}. \tag{4}$$

Then, in analogy with (2) and combined with (3), we have

$$\sum_{i=1}^{n} \gamma_i = 0; \sum_{i=1}^{n} \gamma_i^2 = 2M, \tag{5}$$

where

$$M = (1-\alpha)^2 m + \frac{1}{2}\alpha^2 \sum_{i=1}^{n} \Big(d_i - \frac{2m}{n} \Big)^2.$$

The following upper bound was proven as Theorem 2.6 in [11]. Next, we give the extremal graph as a complement to Theorem 2.6.

Theorem 1. *Let G be a graph with n vertices and m edges, $\alpha \in [0, 1)$. Then,*

$$E^{A_\alpha}(G) \leq \sqrt{2Mn}. \tag{6}$$

The equality holds if and only if either $G \cong nK_1$ or $G \cong mK_2$.

Proof. We apply similar proof as that shown in Theorem 2 [10]. Consider the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} \Big(|\gamma_i| - |\gamma_j| \Big)^2, \tag{7}$$

and by calculation,

$$S = 2n \sum_{i=1}^{n} {\gamma_i}^2 - 2\Big(\sum_{i=1}^{n} |\gamma_i| \Big) \Big(\sum_{j=1}^{n} |\gamma_j| \Big) = 4nM - 2\Big(E^{A_\alpha}(G) \Big)^2.$$

Since $S \geq 0$, thus $4nM - 2(E^{A_\alpha}(G))^2 \geq 0$ and (6) holds.

Note that the equality in (6) is obtained if and only if $S = 0$ in (7), meaning that all $|\gamma_i|-$ values are all equal. Therefore, we conclude that G has, at most, two distinct $\alpha-$adjacency eigenvalues.

From Nikiforov's results in [1], a connected graph G has the same $\alpha-$adjacency eigenvalue if and only if G is a null graph, i.e., $G \cong nK_1$.

In addition, a connected graph G has only two distinct $\alpha-$adjacency eigenvalues if and only if G is a complete graph, with $\alpha \neq 1$ [1], i.e., $G \cong tK_k$, $tk = n$.

Nikiforov in [1] gave the $\alpha-$adjacency eigenvalues of K_k, i.e., $\rho_1(K_k) = k - 1$ and $\rho_i(K_k) = \alpha k - 1$ for any $2 \leq i \leq k$. Since all $|\gamma_i|-$values are the same, then

$$k - 1 - \alpha(k-1) = \alpha(k-1) - \alpha k + 1,$$

i.e., $k = 2$, with $\alpha \neq 1$.

We complete the proof. $\square$

Theorem 2. *Let G be a connected graph with n vertices and m edges, $\alpha \in [0, 1)$. Then,*

$$E^{A_\alpha}(G) \leq (1-\alpha)E(G) + \alpha \sum_{i=1}^{n} \Big| d_i - \frac{2m}{n} \Big|. \tag{8}$$

The equality holds if and only if G is regular.

Proof. Since $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, then we obtain

$$A_\alpha(G) - \frac{2\alpha m}{n} I_n = \alpha\Big(D(G) - \frac{2m}{n} I_n\Big) + (1-\alpha)A(G).$$

Using Lemma 1, we have

$$\begin{aligned}&E\Big(A_\alpha(G) - \frac{2\alpha m}{n} I_n\Big)\\ \leq&E\Big((1-\alpha)A(G)\Big) + E\Big(\alpha\Big(D(G) - \frac{2m}{n} I_n\Big)\Big)\\ =&(1-\alpha)E(G) + \alpha\sum_{i=1}^{n}\Big|d_i - \frac{2m}{n}\Big|.\end{aligned}$$

Thus, (8) follows from $E^{A_\alpha}(G) = E\Big(A_\alpha(G) - \frac{2\alpha m}{n} I_n\Big)$.

Suppose that the equality holds in (8). Then, using Lemma 1, there must exist an orthogonal matrix P such that

$$X = P\Big(\alpha\Big(D(G) - \frac{2m}{n} I_n\Big)\Big),\ Y = P\Big((1-\alpha)A(G)\Big)$$

are both positive and semidefinite. Hence,

$$P^T X = \alpha\Big(D(G) - \frac{2m}{n} I_n\Big),\ P^T Y = (1-\alpha)A(G),$$

and using Lemma 2, we obtain

$$X = \Big|\alpha\Big(D(G) - \frac{2m}{n} I_n\Big)\Big|,\ Y = \Big|(1-\alpha)A(G)\Big|.$$

So, $X = \mathrm{diag}\Big(|\alpha a_1|, |\alpha a_2|, \ldots, |\alpha a_n|\Big)$, where $a_i = d_i - \frac{2m}{n}$, $i = 1, 2, \ldots, n$, and $a_1 = d_1 - \frac{2m}{n} \geq 0$.

Suppose that G is not regular, then, $a_1 > 0$. Let

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{pmatrix}.$$

Since $P^T X = \alpha\Big(D(G) - \frac{2m}{n} I_n\Big)$, then, we have

$$\begin{pmatrix} \alpha a_1 & & & \\ & \alpha a_2 & & \\ & & \ddots & \\ & & & \alpha a_n \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix}\begin{pmatrix} \alpha|a_1| & & & \\ & \alpha|a_2| & & \\ & & \ddots & \\ & & & \alpha|a_n| \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} \alpha a_1 & & & \\ & \alpha a_2 & & \\ & & \ddots & \\ & & & \alpha a_n \end{pmatrix} = \begin{pmatrix} \alpha|a_1|p_{11} & \alpha|a_2|p_{21} & \cdots & \alpha|a_n|p_{n1} \\ \alpha|a_1|p_{12} & \alpha|a_2|p_{22} & \cdots & \alpha|a_n|p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha|a_1|p_{1n} & \alpha|a_2|p_{2n} & \cdots & \alpha|a_n|p_{nn} \end{pmatrix},$$

which implies that $p_{11} = 1$ and $p_{1i} = 0$, $i = 2, \dots, n$. So,

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

and

$$\begin{aligned} Y &= P\Big((1-\alpha)A(G)\Big) \\ &= (1-\alpha)\begin{pmatrix} 1 & 0 & \cdots & 0 \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{pmatrix} \\ &= (1-\alpha)\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ * & \cdots & & * \\ \vdots & & \ddots & \vdots \\ * & \cdots & & * \end{pmatrix}. \end{aligned}$$

Since $Y = P\Big((1-\alpha)A(G)\Big)$ is positive and semidefinite, using Lemma 3, we obtain $a_{1j} = 0$, $j = 2, \dots, n$, which is a contradiction with the connection of G. Thus, the result holds. $\square$

As a special case $\alpha = 1/2$, the following corollary can be obtained easily from Theorem 2.

Corollary 1. *Let G be a connected graph with n vertices and m edges. Then,*

$$QE(G) \le E(G) + \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|.$$

The equality occurs if and only if G is regular.

The Zagreb index $Zg(G)$ of a graph G is given in [11] as $Zg(G) = \sum_{i=1}^{n} d_i^2$.

Theorem 3. *Let G be a graph with n vertices and m edges, $\alpha \in [0,1)$. Then,*

$$E^{A_\alpha}(G) \le (1-\alpha)E(G) + \alpha\sqrt{nZg(G) - 4m^2}. \tag{9}$$

For G being connected, the equality holds if and only if G is regular.

Proof. By applying the Cauchy–Schwarz inequality, we have

$$\sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right| \le \sqrt{n\sum_{i=1}^{n}\Big(d_i - \frac{2m}{n}\Big)^2} = \sqrt{nZg(G) - 4m^2}. \tag{10}$$

Combining (8) with (10), we have

$$E^{A_\alpha}(G) \leq (1-\alpha)E(G) + \alpha \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right| \tag{11}$$

$$\leq (1-\alpha)E(G) + \alpha \sqrt{nZg(G) - 4m^2}. \tag{12}$$

If G is connected, for the equality, it implies that both equalities hold in (11) and (12), i.e., G is regular according to Theorem 2 and $|d_i - \frac{2m}{n}|$ is mutually equal for all $i = 1, 2, \ldots, n$. So, the result holds. □

3. Lower Bounds for α-Adjacency Energy of Graphs

Let $\Phi(G;x) = \det|xI - A_\alpha(G)| = \Pi_{i=1}^{n}(x - \rho_i(G))$ be the characteristic polynomial of $A_\alpha(G)$.

Theorem 4. *Let G be a graph with n vertices and m edges, $0 \leq \alpha < 1$, then,*

$$E^{A_\alpha}(G) \geq n \left| \Phi\left(G; \frac{2\alpha m}{n}\right) \right|^{\frac{1}{n}} \tag{13}$$

with the equality if and only if either $G \cong nK_1$ or $G \cong mK_2$.

Proof. From the geometric-arithmetic mean inequality, we obtain

$$\frac{E^{A_\alpha}(G)}{n} = \frac{1}{n} \sum_{i=1}^{n} \left| \rho_i - \frac{2\alpha m}{n} \right|$$

$$\geq \left(\prod_{i=1}^{n} \left| \rho_i - \frac{2\alpha m}{n} \right| \right)^{\frac{1}{n}} = \left| \Phi(G; \frac{2\alpha m}{n}) \right|^{\frac{1}{n}},$$

which directly yields (13).

The equality occurs if and only if for any $i, j, 1 \leq i, j \leq n$, the equality $|\rho_i - \frac{2\alpha m}{n}| = |\rho_j - \frac{2\alpha m}{n}|$ is satisfied. Therefore, we conclude that G has, at most, two distinct $\alpha-$adjacency eigenvalues. Using the similar proof of Theorem 1, the result follows. □

Theorem 5. *Let G be a graph with n vertices and m edges, then,*

$$E^{A_\alpha}(G) \geq \sqrt{2M + n(n-1) \left| \Phi\left(G; \frac{2\alpha m}{n}\right) \right|^{\frac{2}{n}}},$$

where $M = (1-\alpha)^2 m + \frac{1}{2}\alpha^2 Zg(G) - \frac{2\alpha^2 m^2}{n}$. If $G \cong nK_1$ or $G \cong mK_2$, the equality holds.

Proof. Consider

$$\left(E^{A_\alpha}(G)\right)^2 = \left(\sum_{i=1}^{n} |\gamma_i|\right)^2 = 2M + \sum_{i \neq j} |\gamma_i||\gamma_j|.$$

By applying the geometric-arithmetic mean inequality, we obtain

$$\sum_{i \neq j} |\gamma_i||\gamma_j| \geq n(n-1) \left[\prod_{i \neq j} |\gamma_i||\gamma_j| \right]^{\frac{1}{n(n-1)}}$$

$$= n(n-1) \left[\prod_{i=1}^{n} |\gamma_i|^{2(n-1)} \right]^{\frac{1}{n(n-1)}}$$

$$= n(n-1) \left| \Phi\left(G; \frac{2\alpha m}{n}\right) \right|^{\frac{2}{n}}.$$

By combining the above results, we obtain

$$\left(E^{A_\alpha}(G)\right)^2 \geq 2M + n(n-1)\left|\Phi\left(G; \frac{2\alpha m}{n}\right)\right|^{\frac{2}{n}},$$

i.e.,

$$E^{A_\alpha}(G) \geq \sqrt{2M + n(n-1)\left|\Phi\left(G; \frac{2\alpha m}{n}\right)\right|^{\frac{2}{n}}}.$$

For $G \cong nK_1$ or $G \cong mK_2$, it can be easily verified that the equality holds. □

Lemma 4 ([12]). *Let $B = (b_{ij})_{n\times n}$ be a non-negative matrix, $n \geq 2$, the largest eigenvalue of B be $\rho(B)$, and suppose $a = \min_i b_{ii}$, then,*

$$\rho(B) \geq \max_i\left\{\frac{b_{ii}+a}{2} + \left(\frac{(b_{ii}-a)^2}{4} + \sum_{i\neq j} b_{ij}b_{ji}\right)^{\frac{1}{2}}\right\}.$$

Moreover, if B is irreducible, $n \geq 3$, and there exists more than one non-zero off-diagonal entry in at least two rows (two columns) of B , and the inequality strictly holds.

Let Δ and δ be the maximum degree and the minimum degree of G, respectively.

Theorem 6. *Let G be a connected graph of order $n \geq 3$, $0 \leq \alpha \leq 1$, and $\rho_1(G)$ the largest eigenvalue of $A_\alpha(G)$, then,*

$$\rho_1(G) \geq \frac{\alpha(\Delta+\delta) + \sqrt{\alpha^2(\Delta-\delta)^2 + 4(1-\alpha)^2\Delta}}{2} \tag{14}$$

with the equality if and only if $G \cong K_{1,n-1}$.

Proof. Using Lemma 4, we obtain

$$\rho_1(G) \geq \frac{\alpha\Delta + \alpha\delta}{2} + \left(\frac{(\alpha\Delta - \alpha\delta)^2}{4} + (1-\alpha)^2\Delta\right)^{\frac{1}{2}}.$$

Since G is connected, the matrix $A_\alpha(G)$ is irreducible. Suppose that G has at least two vertices with degrees larger than 1, then (14) strictly holds according to Lemma 4. Note that $\rho_1(K_{1,n-1}) = \frac{\alpha n + \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}}{2}$ [1], so the lower bound attains if and only if $G \cong K_{1,n-1}$. □

Theorem 7. *Let G be a connected graph with n vertices and m edges, $n \geq 3$, $0 < \alpha \leq 1$. Then,*

$$E^{A_\alpha}(G) \geq \alpha(\Delta+\delta) + \sqrt{\alpha^2(\Delta-\delta)^2 + 4(1-\alpha)^2\Delta} - \frac{4\alpha m}{n}. \tag{15}$$

The equality holds if and only if $G \cong K_{1,n-1}$.

Proof. Let $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$ be the eigenvalues of $A_\alpha(G)$ and σ the positive integer such that $\rho_\sigma(G) \geq \frac{2\alpha m}{n}$ and $\rho_{\sigma+1}(G) < \frac{2\alpha m}{n}$, then,

$$\begin{aligned}
E^{A_\alpha}(G) =& 2\left(\sum_{i=1}^{\sigma} \rho_i(G) - \sigma \cdot \frac{2\alpha m}{n}\right) \\
\geq& 2\left(\rho_1(G) - \frac{2\alpha m}{n}\right) \\
\geq& \alpha(\Delta+\delta) + \sqrt{\alpha^2(\Delta-\delta)^2 + 4(1-\alpha)^2\Delta} - \frac{4\alpha m}{n}. \text{ (From (14))}
\end{aligned}$$

All the inequalities occurring above become equalities in the cases of $G \cong K_{1,n-1}$ (according to Theorem 6) and $\sigma = 1$. For the graph $K_{1,n-1}$, $\rho_2(G) = \alpha$ [1] satisfies $\alpha < \frac{2\alpha m}{n}$ for $\alpha \neq 0$. Thus, the equality is obtained in (15) if and only if $G \cong K_{1,n-1}$. This proof is completed. □

As an application of Theorem 7, we provide the graph $K_{1,5} + e$, which is obtained from $K_{1,5}$ by adding an edge between two pendent vertices of $K_{1,5}$. Through simple computation, we know that $E^{A_\alpha}(K_{1,5} + e) \geq 2\alpha + 2\sqrt{9\alpha^2 - 10\alpha + 25}$, which provides an estimation for $E^{A_\alpha}(K_{1,5} + e)$.

Let $f(\delta) = \alpha(\Delta + \delta) + \sqrt{\alpha^2(\Delta - \delta)^2 + 4(1-\alpha)^2\Delta} - \frac{4\alpha m}{n}$, and by comparison, we find that $f(\delta) \geq f(1)$. For a connected graph, we have $\delta \geq 1$. Then, the following corollary can be obtained using Theorem 7 and the results are the same as Theorem 3.4 [11].

Corollary 2 ([11])**.** *Let G be a connected graph with n vertices and m edges, $n \geq 3$, $0 < \alpha \leq 1$. Then,*

$$E^{A_\alpha}(G) \geq \alpha(\Delta + 1) + \sqrt{\alpha^2(\Delta + 1)^2 + 4\Delta(1 - 2\alpha)} - \frac{4\alpha m}{n}. \tag{16}$$

The equality holds if and only if $G \cong K_{1,n-1}$.

4. The Relation between α-Adjacency Energy and Other Energies

As we know, the energy $E(G) \geq 0$ with the equality holds if and only if $m = 0$, which is a direct analog for the $\alpha-$adjacency energy $E^{A_\alpha}(G)$. Indeed, it is evident from (1) that $E^{A_\alpha}(G) \geq 0$ and it is obvious (from the proof of Theorem 1) that $E^{A_\alpha}(G) = 0$ if and only if $m = 0$.

Theorem 8 ([11])**.** *If G is a regular graph with n vertices , $0 \leq \alpha < 1$, then,*

$$E^{A_\alpha}(G) = (1 - \alpha)E(G).$$

Theorem 9. *If the disconnected graph G has two components G_1 and G_2, and the average vertex degree of G_1 is the same as G_2, then,*

$$E^{A_\alpha}(G) = E^{A_\alpha}(G_1) + E^{A_\alpha}(G_2).$$

Proof. Let G_1, G_2 be graphs on n_i vertices and m_i edges for $i = 1, 2$, then, $n = |V(G)| = n_1 + n_2$ and $m = |E(G)| = m_1 + m_2$. Since $\frac{2m_1}{n_1} = \frac{2m_2}{n_2} = \frac{2m}{n}$, then,

$$\begin{aligned}
E^{A_\alpha}(G) &= \sum_{i=1}^{n_1+n_2} \left|\rho_i(G) - \frac{2\alpha m}{n}\right| \\
&= \sum_{i=1}^{n_1} \left|\rho_i(G_1) - \frac{2\alpha m_1}{n_1}\right| + \sum_{i=1}^{n_2} \left|\rho_i(G_2) - \frac{2\alpha m_2}{n_2}\right| \\
&= E^{A_\alpha}(G_1) + E^{A_\alpha}(G_2).
\end{aligned}$$

We complete the proof. □

If the condition $\frac{2m_1}{n_1} = \frac{2m_2}{n_2}$ is not satisfied, it may due to one of the following three cases: $E^{A_\alpha}(G) > E^{A_\alpha}(G_1) + E^{A_\alpha}(G_2)$, $E^{A_\alpha}(G) < E^{A_\alpha}(G_1) + E^{A_\alpha}(G_2)$, or $E^{A_\alpha}(G) = E^{A_\alpha}(G_1) + E^{A_\alpha}(G_2)$. This requires further study.

In particular, if G_2 consists of n_2 isolated vertices, then,

$$E^{A_\alpha}(G) = \sum_{i=1}^{n_1} \left|\rho_i(G_1) - \frac{2\alpha m}{n_1 + n_2}\right| + n_2 \cdot (\frac{2\alpha m}{n_1 + n_2}).$$

Theorem 10. *Let G be a graph with n vertices. Then,*

$$|E^{A_\alpha}(G) - \alpha LE(G)| \leq E(G). \tag{17}$$

If $G \cong nK_1$, the equality holds.

Proof. Note that

$$\begin{aligned} A_\alpha(G) - \frac{2\alpha m}{n} I_n &= \alpha\Big(D(G) - A(G) - \frac{2m}{n} I_n\Big) + A(G) \\ &= \alpha\Big(L(G) - \frac{2m}{n} I_n\Big) + A(G), \end{aligned}$$

using Lemma 1, we obtain

$$\begin{aligned} E^{A_\alpha}(G) &= E\Big(A_\alpha(G) - \frac{2\alpha m}{n} I_n\Big) = E\Big(\alpha\Big(L(G) - \frac{2m}{n} I_n\Big) + A(G)\Big) \\ &\leq E\Big(\alpha\Big(L(G) - \frac{2m}{n} I_n\Big)\Big) + E(A(G)) = \alpha LE(G) + E(G) \end{aligned}$$

and

$$\begin{aligned} \alpha LE(G) &= E\Big(\alpha\Big(L(G) - \frac{2m}{n} I_n\Big)\Big) = E\Big(\Big(A_\alpha(G) - \frac{2\alpha m}{n} I_n\Big) - A(G)\Big) \\ &\leq E\Big(A_\alpha(G) - \frac{2\alpha m}{n} I_n\Big) + E(-A(G)) = E^{A_\alpha}(G) + E(G). \end{aligned}$$

Thus, the result follows. Finally, if $G \cong nK_1$, $E^{A_\alpha}(G) = LE(G) = E(G) = 0$, then the equality holds. □

The following theorems give two Nordhaus–Gaddum-type bounds in terms of the order n.

Theorem 11. *Let G be a graph with n vertices, $n \geq 2$, $\overline{G}$ the complement graph of G, then,*

$$E^{A_\alpha}(G) + E^{A_\alpha}(\overline{G}) < \sqrt{n^2(n-1)[2(1-\alpha)^2 + \alpha^2(n-1)]}.$$

Proof. Let

$$M = M(G) = (1-\alpha)^2 m(G) + \frac{1}{2}\alpha^2 \sum_{i=1}^{n}\Big(d_i - \frac{2m(G)}{n}\Big)^2$$

and

$$M(\overline{G}) = (1-\alpha)^2 m(\overline{G}) + \frac{1}{2}\alpha^2 \sum_{i=1}^{n}\Big(d_i - \frac{2m(G)}{n}\Big)^2.$$

In fact,

$$\sum_{i=1}^{n} d_i^2 \leq (n-1)(\sum_{i=1}^{n} d_i) = 2(n-1)m(G)$$

with the equality if and only if either $G \cong nK_1$ or $G \cong K_n$, then, we have

$$\begin{aligned}
&M(G) + M(\overline{G}) \\
=&(1-\alpha)^2(m(G)+m(\overline{G})) + \alpha^2 \sum_{i=1}^{n}\Big(d_i - \frac{2m(G)}{n}\Big)^2 \\
=&(1-\alpha)^2 \cdot \frac{n(n-1)}{2} + \alpha^2\Big(\sum_{i=1}^{n} d_i^2 - \frac{4m^2(G)}{n}\Big) \\
\leq&(1-\alpha)^2 \cdot \frac{n(n-1)}{2} + \alpha^2[2(n-1)m(G) - \frac{4m^2(G)}{n}] \\
\leq&(1-\alpha)^2 \cdot \frac{n(n-1)}{2} + \alpha^2 \cdot \frac{n(n-1)^2}{4} \\
=&\frac{n(n-1)}{4}[2(1-\alpha)^2 + \alpha^2(n-1)].
\end{aligned}$$

For $n \geq 2$, since the size of the edges of K_n and $\overline{K_n}$ is different from $\frac{n(n-1)}{4}$, we have

$$M(G) + M(\overline{G}) < \frac{n(n-1)}{4}[2(1-\alpha)^2 + \alpha^2(n-1)]. \tag{18}$$

In combination with $E^{A_\alpha}(G) \leq \sqrt{2Mn}$ of Theorem 1, it is easy to see that the inequality

$$\begin{aligned}
E^{A_\alpha}(G) + E^{A_\alpha}(\overline{G}) &\leq \sqrt{4n[M(G)+M(\overline{G})]} \\
&< \sqrt{n^2(n-1)[2(1-\alpha)^2 + \alpha^2(n-1)]}
\end{aligned}$$

holds. □

Lemma 5 ([11])**.** *Let G be a connected graph with n vertices and m edges and $\rho_1(G)$ the largest eigenvalues of $A_\alpha(G)$, then,*

$$\rho_1(G) \geq \frac{2m}{n}.$$

The equality holds if and only if G is regular.

Theorem 12. *Let G be a graph with n vertices and $\overline{G}$ be the complement graph of G, then,*

$$E^{A_\alpha}(G) + E^{A_\alpha}(\overline{G}) \geq 2(1-\alpha)(n-1). \tag{19}$$

The equality occurs if and only if G and $\overline{G}$ are both regular with only one positive adjacency eigenvalue, respectively.

Proof. Note that

$$\begin{aligned}
&E^{A_\alpha}(G) + E^{A_\alpha}(\overline{G}) \\
=&\sum_{i=1}^{n}\Big|\rho_i(G) - \frac{2\alpha m(G)}{n}\Big| + \sum_{i=1}^{n}\Big|\rho_i(\overline{G}) - \frac{2\alpha m(\overline{G})}{n}\Big| \\
\geq&2\Big(\rho_1(G) - \frac{2\alpha m(G)}{n}\Big) + 2\Big(\rho_1(\overline{G}) - \frac{2\alpha m(\overline{G})}{n}\Big) \\
\geq&2\Big(\frac{2m(G)}{n} - \frac{2\alpha m(G)}{n}\Big) + 2\Big(\frac{2m(\overline{G})}{n} - \frac{2\alpha m(\overline{G})}{n}\Big) \text{(by Lemma 5)} \\
=&2(1-\alpha)(n-1).
\end{aligned}$$

All equalities occur if and only if G is regular (according to Lemma 5). In addition,

$$E^{A_\alpha}(G) = 2\Big(\rho_1(G) - \frac{2\alpha m(G)}{n}\Big), \rho_2\Big(A_\alpha(G) - \frac{2\alpha m(G)}{n} I_n\Big) < 0,$$

and

$$E^{A_\alpha}(\overline{G}) = 2\Big(\rho_1(\overline{G}) - \frac{2\alpha m(\overline{G})}{n}\Big), \rho_2\Big(A_\alpha(\overline{G}) - \frac{2\alpha m(\overline{G})}{n} I_n\Big) < 0.$$

Thus,

$$A_\alpha(G) - \frac{2\alpha m(G)}{n} I_n = (1-\alpha)A(G)$$

and

$$A_\alpha(\overline{G}) - \frac{2\alpha m(\overline{G})}{n} I_n = (1-\alpha)A(\overline{G}).$$

Both G and $\overline{G}$ have only one positive eigenvalue. This proof is completed. □

5. Conclusions

In this paper, we considered some properties of α-adjacency energy. In particular, we obtained some new upper bounds in terms of the graph parameters associated with the structure of the graph (Theorems 2 and 9) and some new lower bounds (Theorems 4 and 7). Moreover, extremal graphs have been provided within these theorems. It will be interesting to explore more properties of this spectral graph invariant in the future. There are also other theorems such as Theorem 10, where graphs satisfying the equality cannot be found yet so further research is needed.

The α-adjacency energy merges the energy (the case where $\alpha = 0$) and the signless Laplacian energy (the case where $\alpha = 1/2$). Therefore, if we choose an appropriate value of α in some upper or lower bounds, the results will be clear and intuitive and, at the same time, can enrich spectral graph theory.

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