mathematics

Article

A Class of Efficient Sixth-Order Iterative Methods for Solving the Nonlinear Shear Model of a Reinforced Concrete Beam

José J. Padilla [1], Francisco I. Chicharro [2], Alicia Cordero [2], Alejandro M. Hernández-Díaz [3] and Juan R. Torregrosa [2,*]

[1] Departamento de Ingeniería Civil, UCAM Universidad Católica de Murcia, 30107 Guadalupe, Spain; jjpadilla@ucam.edu
[2] Instituto Universitario de Matemàtica Multidisciplinar, Universitat Politècnica de València, 46022 València, Spain; frachilo@mat.upv.es (F.I.C.); acordero@mat.upv.es (A.C.)
[3] Área de Mecánica de Medios Continuos y Teoría de Estructuras, Universidad de La Laguna, 38200 La Laguna, Spain; ahernadi@ull.edu.es
* Correspondence: jrtorre@mat.upv.es

Abstract: In this paper, we present a three-step sixth-order class of iterative schemes to estimate the solutions of a nonlinear system of equations. This procedure is designed by means of a weight function technique. We apply this procedure for predicting the shear strength of a reinforced concrete beam. The values for the parameters of the nonlinear system describing this problem were randomly selected inside the prescribed ranges by technical standards for structural concrete. Moreover, some of these parameters were fixed taking into consideration the solvability region of the adopted steel constitutive model. The effectiveness of the new class is also compared with other current schemes in terms of the computational efficiency and numerical performance, with very good results. The advantages of this new class come from the low computational cost, due to the existence of an only inverse operator.

Keywords: nonlinear systems; iterative methods; reinforced concrete; shear behaviour; convergence order; efficiency

MSC: 65F10

Citation: Padilla, J.J.; Chicharro, F.I.; Cordero, A.; Hernández-Díaz, A.M.; Torregrosa, J.R. A Class of Efficient Sixth-Order Iterative Methods for Solving the Nonlinear Shear Model of a Reinforced Concrete Beam. *Mathematics* **2024**, *12*, 499. https://doi.org/10.3390/math12030499

Academic Editor: Patricia J. Y. Wong

Received: 29 December 2023
Revised: 26 January 2024
Accepted: 1 February 2024
Published: 5 February 2024

1. Introduction

Reinforced and prestressed concrete beams represent a structural type that resists internal stresses in a relatively complex manner due to their constitutive nature. Prior to the cracking of the concrete, the shear loads are carried by a set of diagonal compressive stresses complemented by another set of diagonal tensile stresses acting perpendicular to the first ones. Once the concrete tensile strength is reached, cracks form in the direction normal to the diagonal tensile stresses while pre-existing cracks spread and change inclination. Then, the ability of concrete to transmit diagonal tensile stresses is significantly reduced and the appropriate reinforcement is necessary to create a new system of internal stresses that carry the shear acting on the beam after cracking.

Between 1899 and 1902, Ritter [1] and Mörsch [2] proposed a truss model for explaining the field of forces in a cracked reinforced concrete beam, with the principal compressive stresses acting as diagonal members at 45º and the stirrups acting as vertical tension members. This model neglected the tensile stresses in the cracked concrete. In 1910, the first ACI Code modified Mörsch's 45º truss model through the addition of concrete in order to compensate for the conservatism of the model and to account for the fact that the crack angle is usually less than 45º [3]. Between 1904 and 1922, Talbot and Withey demonstrated that the stirrup stresses were lower than those predicted by the 45º truss model [4].

Thus, before using the equilibrium equations, the inclination of the diagonal compressive struts should be known. In 1929, Wagner [5] treated a similar problem by studying the post-buckling shear response of thin metal girders, and he assumed that the angle of inclination of the diagonal tensile stresses coincided with the corresponding value of the diagonal tensile strains. Between 1974 and 1978, Collins and Mitchell developed a shear design model for reinforced concrete that, based on Wagner's assumption, predicted the inclination of the compressive struts by considering the strains in the transverse and longitudinal reinforcement, and in the diagonally stressed concrete [6,7]; this last approach became known as the Compression Field Theory (CFT).

In 1982, Vecchio and Collins found that the principal compressive stress in the concrete is a function not only of the principal compressive strain, but also of the coexisting principal tensile strain [8]. In 1986, they published, as a further development of the CFT, the so-called Modified Compression Field Theory (MCFT) [9], which accounts for the influence of the tensile stresses in the cracked concrete. This theory assumes a bilinear constitutive model for the steel and requires the checking of the local bar stresses at the cracks, ensuring thus that the smeared steel stresses between adjacent cracks are lower than the yield value.

Since the MCFT, other alternative approaches have been also developed due to the consideration given to other strength mechanisms, such as the dowel action of the reinforcement intersecting the cracks or the friction between the crack faces [10]. Likewise, different procedures have been proposed to treat the shear response of a reinforced concrete member in a continuum mechanics context (i.e., to account for tensile stresses in the diagonally cracked concrete), such as the developments of Hsu and his co-workers at the University of Houston [11,12], in the framework of the so-called Rotating-Angle Softened Truss Model (RA-STM). The RA-STM proposes a constitutive relationship for the reinforcement as stiffened by the concrete (i.e., the embedded model bar); due to this alternative concept, the steel stress does not exceed the yielding point and the local checking of cracks is no longer necessary. The last contribution in this line is the so-called Refined Compression Field Theory (RCFT) [13,14], where the embedded bar stress–strain relationship is obtained from the concrete tension stiffening model considered in the MCFT, so the crack check is avoided and a new formulation with respect to the traditional MCFT one is no longer needed. Moreover, the numerical results obtained from the RCFT lead to a better fitting of the experimental results, particularly in the region near the peak point in the shear–strain response, where MCFT significantly deviates from the experimental data. This last approach to the constitutive modeling of the steel reinforcement is the one considered in this work.

As it is justified in Section 2, CFT mechanical models involve several types of nonlinearities, among other reasons, due to the constitutive relationships of the reinforced concrete. These models usually require in their implementation the application of iterative fixed-point methods to solve the nonlinear systems that appear. In fact, the most efficient way to solve a nonlinear problem is usually to choose between accuracy and computational cost [15]. Moreover, in this work, the previous determination of a solvability region using algebraic procedures is also necessary in order to improve the efficiency of the numerical solver, as indicated in Section 2.

Solving systems of nonlinear equations is an important problem in science and engineering, as has been previously described. The objective is to find the roots of the nonlinear system $F(x) = 0$, F being a multidimensional function, $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, on D convex set, of size $n \times n$, $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T$ being f_i, $i = 1, 2, \ldots, n$, the functional coordinates of F.

One of the most commonly used methods is the classical Newton's method, which has a quadratic order of convergence and iterative expression:

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0, 1, 2, \ldots, \tag{1}$$

where $F'(x^{(k)})$ is the Jacobian matrix of F at k-th iteration.

Several Newton-type procedures, by using different techniques, have been published in the last few years, some introductory texts to this area are [16–20]. Their main aim is accelerating their convergence or increasing their efficiency with differently designed techniques used by numerous authors in the literature (see, for example, [21–26]). In what follows, we are going to recall some of them for comparison purposes.

All the schemes we are going to mention use, in their iterative expression, the Jacobian matrix of function F and have, under the usual conditions, a convergence order 6. We will compare these methods, from the point of view of the results of the convergence order and computational efficiency, with the methods proposed in this paper that also have an order of 6 and only use $[F'(x)]^{-1}$ in their expressions.

In [27], by using the weight function procedure, the authors designed a Jarratt-type method for solving nonlinear systems, denoted by $M2_6$, the iterative expression of which is

$$\left\{\begin{array}{lcl} y^{(k)} & = & x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}),\ k = 0, 1, \ldots, \\ z^{(k)} & = & x^{(k)} - \left(\frac{5}{8}I + \frac{3}{8}([F'(y^{(k)})]^{-1}F'(x^{(k)}))^2\right)[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} & = & z^{(k)} - \left(\frac{-9}{4}I - \frac{15}{8}[F'(x^{(k)})]^{-1}F'(y^{(k)}) + \frac{11}{8}[F'(y^{(k)})]^{-1}F'(x^{(k)})\right)[F'(y^{(k)})]^{-1}F(z^{(k)}), \end{array}\right. \tag{2}$$

where I denotes the identity matrix of size $n \times n$. This method aims to evaluate the Jacobian matrix in two points and uses two inverse operators. These elements increase the number of operations per iteration.

In order to reduce the number of inverse operators, Narang et al. in [28], from a Chebyshev–Halley-type family, constructed a class of iterative schemes of the sixth order. One of its members, denoted by $M_{6,2}(1/2, 0)$, has the following iterative expression:

$$\left\{\begin{array}{lcl} y^{(k)} & = & x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} & = & x^{(k)} - \left(\frac{1}{2}G(x^{(k)})\right)H(G(x^{(k)}))[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} & = & z^{(k)} - \left(I + \frac{3}{2}G(x^{(k)})\right)[F'(x^{(k)})]^{-1}F(z^{(k)}),\ k = 0, 1, \ldots, \end{array}\right. \tag{3}$$

where $G(x^{(k)}) = I - [F'(x^{(k)})]^{-1}F'(y^{(k)})$ and $H(G(x^{(k)})) = I - \frac{1}{4}G(x^{(k)}) + \frac{11}{8}(G(x^{(k)}))^2$.

Behl et al. in [29], using the indeterminate parameter procedure, designed a family of iterative sixth-order methods for solving systems of nonlinear equations. One of its members, denoted by PM1, has the following iterative expression:

$$\left\{\begin{array}{lcl} y^{(k)} & = & x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} & = & y^{(k)} - \left(4I - 3[F'(x^{(k)})]^{-1}F'(y^{(k)}) + \frac{9}{8}([F'(x^{(k)})]^{-1}F'(y^{(k)}))^{-2}\right)[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} & = & z^{(k)} - \left(\frac{5}{2}I - \frac{3}{2}[F'(x^{(k)})]^{-1}F'(y^{(k)})\right)[F'(x^{(k)})]^{-1}F(z^{(k)}),\ k = 0, 1, \ldots \end{array}\right. \tag{4}$$

Finally, Yaseen and Zafar presented in [30] a Jarratt-type scheme of three steps for solving nonlinear systems, denoted by $FS6$, with sixth-order convergence and iterative expression:

$$\left\{\begin{array}{lcl} y^{(k)} & = & x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} & = & x^{(k)} - \left(\frac{5}{8}[U_k]^{-1} + \frac{3}{8}U_k\right)[F'(y^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} & = & z^{(k)} - \left(\frac{-13}{2}I + \frac{9}{2}[V_k]^{-1} + 3V_k\right)[F'(x^{(k)})]^{-1}F(z^{(k)}),\ k = 0, 1, \ldots, \end{array}\right. \tag{5}$$

where $U_k = [F'(y^{(k)})]^{-1}F'(x^{(k)})$ and $V_k = [F'(x^{(k)})]^{-1}F'(y^{(k)})$.

The rest of the paper is organized as follows. In Section 2, we describe the nonlinear system obtained for predicting the shear strength of a reinforced concrete beam. The efficient method for estimating its solution is presented in Section 3, as well as its convergence order. Section 4 is devoted to the efficiency analysis. The numerical performance of our proposed methods are studied on academical problems and on the nonlinear shear model described in Section 2. Finally, some conclusions are exposed.

2. Problem Statement

In [3], the authors proposed this stress–strain relationship for concrete cracked in tension:

$$\sigma_1 = \begin{cases} E_c \varepsilon_1, & \varepsilon_1 \le \varepsilon_{ct}, \\ \dfrac{\alpha f_{ct}}{1+\sqrt{500\varepsilon_1}}, & \varepsilon_1 > \varepsilon_{ct}, \end{cases} \tag{6}$$

where σ_1 represents the contribution of tensile stresses in the concrete between the cracks or tension stiffening effect, ε_1 is the principal tensile strain, E_c being the modulus of elasticity of the concrete, ε_{ct} the strain related to the strength of the tensile, f_{ct}. Coefficient α is equal to 1.0 in cases of fast and non-cyclic loads and for deformed bars.

Regarding the concrete behaviour in compression, Vecchio and Collins formulated in [9], using the Modified Compression Field Theory (MCFT), the following relationship between diagonal compressive strain, ε_2, and the diagonal (or principal) compressive stress, σ_2:

$$\begin{aligned} &\sigma_2 = f_{2max}\left[2\left(\tfrac{\varepsilon_2}{\varepsilon_c}\right) - \left(\tfrac{\varepsilon_2}{\varepsilon_c}\right)^2\right], \\ &\text{with } f_{2max} = \tfrac{f_c}{0.8+170\varepsilon_1} \le f_c, \end{aligned} \tag{7}$$

where ε_c is the compressive stress related to the compressive strength of concrete in a cylindrical test f_c, f_{2max} is the maximum compressive stress in a diagonally cracked web and ε_1 is the coexisting principal tensile stress.

In CFT procedures, a perfect bond between concrete and steel is assumed; in consequence, any deformation developed by the reinforcement is identical to the one experienced by the surrounding concrete in the same direction; thus, a single average strain tensor of the composite material is adopted. The following relationship is considered regarding the compatibility of the strains in the reinforcement and the diagonally stressed concrete:

$$\tan^2\theta = \frac{\varepsilon_x - \varepsilon_2}{\varepsilon_t - \varepsilon_2} = \frac{\varepsilon_1 - \varepsilon_t}{\varepsilon_1 - \varepsilon_x}, \tag{8}$$

where ε_x is the mean longitudinal strain and ε_t is the mean transversal strain on the web of a beam oriented according to the orthogonal $x - t$ direction (see Figure 1). The strain ε_2 is aligned in the direction of the compressive struts, at angle θ to the longitudinal axis (x) of the beam. Moreover, due to strain tensor, the main tensile strain is

$$\varepsilon_1 = \varepsilon_x + \varepsilon_t + \varepsilon_2. \tag{9}$$

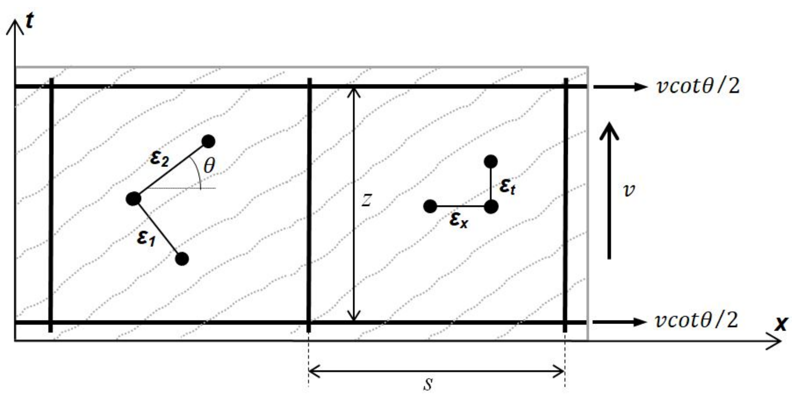


Figure 1. Strain compatibility between diagonally stressed concrete and the reinforcement in the cracked web of a reinforced concrete beam.

On the other hand, in CFT models, the equilibrium between the external loads and the internal forces is governed by the following equations:

$$\sigma_2 = \frac{\nu}{b_w z}(\tan\theta + \cot\theta) - \sigma_1, \tag{10}$$

$$2A_{st}\sigma_{st} = (\sigma_2 \sin^2\theta - \sigma_1 \cos^2\theta) b_w s, \tag{11}$$

$$4A_{sx}\sigma_{sx} + A_p\sigma_p = (\sigma_2 \cos^2\theta - \sigma_1 \sin^2\theta) b_w z = \frac{\nu}{\tan\theta} - \sigma_1 b_w z, \tag{12}$$

where θ is the angle of the main tensile stress, z is the flexural lever arm, s is the stirrup spacing, ν is the internal shear force, and b_w is the web width; A_{sx}, A_{st} and A_p are the cross-section surfaces for the longitudinal bars, the stirrup legs, and the prestressed reinforcement, respectively, and σ_{sx}, σ_{st} and σ_p are the related mean tensile stresses. The angles of the inclination of the principal strains coincide with the angles of the inclination of the principal stresses; this is known as EPA assumption or as Wagner's hypothesis [31].

Regarding the stress–strain relationship of the steel reinforcement, beyond the type of steel to consider (such as, for example, mild steel or stainless steel), CFT methods mainly differ in terms of the treatment of the steel behavior [9,11,13]. In this work, one of the most recent approaches to steel behaviour is adopted: the RCFT, previously introduced in Section 1, which is based on the concept of an embedded bar model that takes into account the concrete tension stiffening effect between cracks. The latter theory allows us to apply, in the most general case, the following mean stress–strain model for each type of steel reinforcement of the beam (i.e., longitudinal reinforcement and transverse stirrups):

$$\begin{aligned}
&\sigma_{s,i} = \begin{cases} f_{y,i} - \frac{\lambda_i A_{c,i}}{A_{s,i}} \frac{f_{ct}}{1+\sqrt{3.6 M_i \varepsilon_{s,i}}} & \text{if} \quad \varepsilon_{s,i} \geq \varepsilon_{max,i} \\ E_s \varepsilon_{s,i} & \text{if} \quad \varepsilon_{s,i} < \varepsilon_{max,i} \end{cases} \\
&i = \{x, t\} \\
&\text{in which} \\
&\varepsilon_{max,i} = \frac{f_{y,i}}{E_s} - \frac{\frac{\lambda_i A_{c,i} f_{ct}}{1+\sqrt{3.6 M_i \varepsilon_{max,i}}}}{E_s A_{s,i}} \\
&M_i = \frac{\lambda_i A_{c,i}}{\sum \pi \phi_i},
\end{aligned} \tag{13}$$

where the subscripts x and t refer to the longitudinal and the transverse reinforcement, respectively (then, (13) actually involves two equations); f_y is the steel yield stress, E_s is the elastic modulus of the steel, $\sigma_{s,av}$ is the average tensile stress in the steel, $\varepsilon_{s,av}$ is the average strain in the reinforcing bar, $\varepsilon_{max,i}$ is the apparent yield strain (cf. [13]), M is the joint parameter, A_s is the cross-section of the steel bars (longitudinal or transverse), A_c is the area of concrete attached to the bar that participates in the tensile stiffening effect; this is usually considered equal to the rectangular area surrounding the bar of diameter ϕ and over a distance no greater than 7.5ϕ from the center of the bar, and finally, λ_i is the coefficient for fixing the numerical solvability of the steel constitutive model.

In the case of prestressed concrete members, the following two additional equations are required:

$$\varepsilon_p = \varepsilon_x + \Delta\varepsilon_p, \tag{14}$$

$$\sigma_p = \begin{cases} E_p \varepsilon_p & , \quad \varepsilon_p \leq \frac{f_{py}}{E_p}, \\ f_{py} & , \quad \varepsilon_p > \frac{f_{py}}{E_p}, \end{cases} \tag{15}$$

where (14) represents the strain compatibility, $\Delta\varepsilon_p$ and ε_p being the strain imposed by the prestressing system and the strain of the prestressing strand, respectively, and Equation (15) represents the stress–strain relationship for the prestressing steel, f_{py} and E_p being its yield stress and elastic modulus, respectively.

Equation (13) is based on the concept of force equilibrium between a general section (or non-cracked section, where both the steel and the surrounding concrete contribute) and a cracked section (where only the reinforcement resists the internal forces; please see Figure 2). The greatest value of the area A_c in order to preserve the solvability of the embedded steel constitutive model proposed by the RCFT (i.e., in order to preserve the internal equilibrium of forces, in such a way that as the concrete participation increases, the steel stress diminishes) is obtained by the application of the following coefficient [32]:

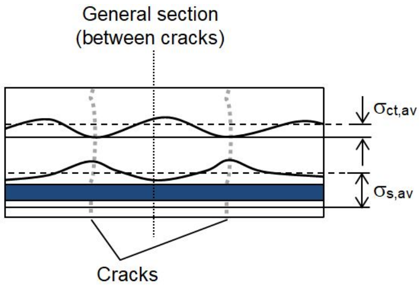


Figure 2. Average stress profiles ($\sigma_{ct,av}$ and $\sigma_{st,av}$) for an embedded reinforcement constitutive model including several cracks.

$$\lambda_{max,i} = \frac{A_s \cdot f_y}{A_c \cdot f_{ct}} \cdot \left(\frac{2}{3} + \frac{\sqrt{(1 + 10.8 \cdot M \cdot \epsilon_y)^3}}{48.6 \cdot M \cdot \epsilon_y} \right), \tag{16}$$

where the coefficient $\lambda_{max,i}$ represents the boundary of the solvability region for the embedded steel constitutive model in the i-direction (i.e., the maximum value of the coefficient λ in the i-direction in order to preserve the solvability), and ϵ_y is the strain corresponding to the steel yield stress (i.e., $\epsilon_y = f_y / E_s$). For certain design cases, the previous boundary may lay within the design range prescribed by technical codes for the tension stiffening area, A_c.

In summary, for a given value of tensile principal strain in concrete, ε_1, where such strain works as an input parameter, the shear model to predict the load–deformation response of a prestressed concrete beam is derived from the nonlinear system defined by (7)–(15), containing up to 10 equations (notice that (13) is actually two equations in turn) in the 10 unknowns (θ, ε_x, ε_t, ν, ε_2, σ_2, $\sigma_{s,x}$, $\sigma_{s,t}$, ε_p, and σ_p).

Two thousand solutions obtained from solving the nonlinear system of Equations (7)–(15) has been obtained from a set of input vectors uniformly generated. The range of the input parameters considered to this aim are presented in Table 1. These solutions were obtained using Newton's method and considering the same initial approximation of all the cases.

Table 1. Ranges for input parameters of the nonlinear system of Equations (7)–(15), with E_p = 190,000 MPa and $f_{py} = 1674$ MPa.

Input	Range
E_s(MPa)	[195,000, 205,000]
f_y(MPa)	$[350, 500]$
ϕ_x(mm)	$[6, 40]$
ϕ_t(mm)	$[6, 40]$
$\Delta\varepsilon_p(-)$	$[0.10 f_{py}/E_p, 0.90 f_{py}/E_p]$
$f_c(\text{mm}^2)$	$[25, 50]$
$b_w(\text{mm})$	$[100, 1000]$
$s(\text{mm})$	$[15\phi_t, 600]$
$A_p(\text{mm}^2)$	$[300, 1200]$
$\lambda_x(\text{mm})$	$[0.1\lambda_{max,x}, 0.9\lambda_{max,x}]$
$\lambda_t(\text{mm})$	$[0.1\lambda_{max,t}, 0.9\lambda_{max,t}]$
$\varepsilon_1(-)$	$[0.0001, 0.01]$

3. Development and Convergence of the Method

By using the weight matrix function procedure, we present a class of three-step iterative methods with the following iterative expression:

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - G(\mu^{(k)})b[F'(x^{(k)})]^{-1}F(y^{(k)})), \\ x^{(k+1)} &= z^{(k)} - G(\mu^{(k)})[F'(x^{(k)})]^{-1}\big(iF(z^{(k)}) + hF(y^{(k)})\big),\ k = 0,1,2,\dots, \end{aligned} \tag{17}$$

where $\mu = [F'(x)]^{-1}F(y)$ is the variable of the weight function G, and b, i, and h are free parameters.

On the other hand, with F being a sufficiently differentiable Fréchet function, we can regard $\xi + m \in \mathbb{R}^n$ as being in the neighbourhood of the zero of F, ξ. Using Taylor developments and $F'(\xi)$ being nonsingular,

$$F(\xi + m) = F'(\xi)\left[h + \sum_{q=2}^{p-1} C_q m^q\right] + O(m^p), \tag{18}$$

where $C_q = \dfrac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$ for $q \geq 2$. Also, $C_q h^q \in \mathbb{R}^n$, as $F^{(q)}(\xi) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\xi)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$. Therefore,

$$F(\xi + m) = F'(\xi)\left[I + \sum_{q=2}^{p-1} qC_q m^{q-1}\right] + O(m^{p-1}), \tag{19}$$

being $qC_q m^{q-1} \in \mathcal{L}(\mathbb{R}^n)$. For more details of this notation, see [33].

Indeed, following the notation introduced by Artidiello et al. in [34], the matrix function $G : X \to X$ can be defined in such a way that its Fréchet derivatives holds

(a) $G'(u)(v) = G_1uv$, being $G' : X \to \mathcal{L}(X)$, $G_1 \in \mathbb{R}$
(b) $G''(u,v)(w) = G_2uvw$, being $G_2 : X \times X \to \mathcal{L}(X)$, $G_2 \in \mathbb{R}$

when $X = \mathbb{R}^{n\times n}$ is the Banach space of real $n \times n$ matrices, and $\mathcal{L}(X)$ is the set of linear operators defined in X.

In the next result, we present the convergence of the family (17).

Theorem 1. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differential Fréchet function defined on a convex neighborhood D of $\xi \in \mathbb{R}^n$, and a zero of F. Also, let $G : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be a sufficiently differentiable matrix function. Suppose that $F'(\xi)$ is nonsingular, and that $x^{(0)}$ is a seed sufficiently close to ξ. Therefore, the sequence $\{x^{(k)}\}_{k\geq 0}$ from (17) converges to ξ with an order of convergence six if $b = \dfrac{1}{G_0}$, $h = 0$, $i = \dfrac{1}{G_0}$, $G_1 = -G_0$, and $|G_2| < \infty$, $G_0 = G(I)$ and I being the identity matrix of size $n \times n$. In this case, the error equation is*

$$\begin{aligned} e^{(k+1)} = \Bigg(& 24C_2^5 - 4G_2(37C_2^5 - 6C_3 - 6C_2C_3C_2^2C_3C_2 + 3C_3C_2C_3)\frac{1}{G_0} \\ & + 4G_2(32C_2^5G_0 + C_2^5G_2 - 6G_0C_3C_2^3 - 6G_0C_2C_3C_2^2 - 6G_0C_2^2C_3C_2 + 3G_0C_3C_2C_3)\left(\frac{1}{G_0}\right)^2 \\ & + 2C_2^5G_0^2(3G_0 - G_2)\left(\frac{1}{G_0}\right)^3\Bigg) {e^{(k)}}^6 + O({e^{(k)}}^7), \end{aligned}$$

where $e^{(k)} = x^{(k)} - \xi$ and $C_q = \dfrac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q = 2,3,\dots$

Proof. By means of the Taylor expansion of $F(x^{(k)})$ and $F'(x^{(k)})$ about ξ, we obtain

$$F(x^{(k)}) = F'(\xi)\left[e^{(k)} + \sum_{i=2}^{6} C_i {e^{(k)}}^i\right] + \mathcal{O}({e^{(k)}}^7),$$

and

$$F'(x^{(k)}) = F'(\xi)\left[I + + \sum_{i=2}^{5} iC_i {e^{(k)}}^{i-1}\right] + \mathcal{O}({e^{(k)}}^6).$$

We can deduce that

$$[F'(x^{(k)})]^{-1} = \left[I + + \sum_{i=2}^{5} X_i {e^{(k)}}^{i-1}\right][F'(\xi)]^{-1} + \mathcal{O}({e^{(k)}}^6),$$

where $X_2 = -2C_2$, $X_3 = -3C_3 + 4C_2^2$, $X_4 = -4C_4 + 6C_2C_3 + 6C_3C_2 - 8C_2^3$ and

$$\begin{aligned} X_5 &= -5C_5 + 8C_2C_4 - 12C_2^2C_3 + 9C_3^2 + 8C_4C_2 - 12C_2C_3C_2 + 16C_2^4 - 12C_3C_2^2, \\ X_6 &= -6C_6 + 10C_2C_5 + 12C_4C_3 - 18C_2C_3^2 - 18C_3C_2C_3 + 24C_2^3C_3 + 12C_3C_4 \\ &\quad - 16C_2^2C_4 + 10C_5C_2 - 16C_2C_4C_2 - 18C_3^2C_2 + 24C_2^2C_3C_2 - 16C_4C_2^2 \\ &\quad + 24C_2C_3C_2^2 + 24C_3C_2^3 - 32C_2^5. \end{aligned}$$

Then,

$$\begin{aligned} y^{(k)} - \xi &= C_2{e^{(k)}}^2 - 2(C_2^2 - C_3){e^{(k)}}^3 \\ &\quad - (4C_2C_3 + 3C_3C_2 - 4C_2^3 - 3C_4){e^{(k)}}^4 \\ &\quad - (-4C_5 + 6C_2C_4 - 8C_2^2C_3 + 6C_3^2 + 4C_4C_2 - 6C_2C_3C_2 + 8C_2^4 - 6C_3C_2^2){e^{(k)}}^5 \\ &\quad + \mathcal{O}({e^{(k)}}^6), \end{aligned}$$

and

$$(y^{(k)} - \xi)^2 = C_2^2{e^{(k)}}^4 + (2C_2C_3 + 2C_3C_2 - 4C_2^3){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6).$$

Moreover,

$$\begin{aligned} F(y^{(k)}) = F'(\xi)\Big[&C_2{e^{(k)}}^2 + 2(C_3 - C_2^2){e^{(k)}}^3 + (3C_4 + 5C_2^3 - 3C_3C_2 - 4C_2C_3){e^{(k)}}^4 + \\ &(-12C_2^4 - 6C_3^2 + 4C_5 - 6C_2C_4 + 10C_2^2C_3 + 6C_3C_2^2 - 4C_4C_2 + 8C_2C_3C_2){e^{(k)}}^5\Big] \\ &+ \mathcal{O}({e^{(k)}}^6). \end{aligned}$$

So,

$$\begin{aligned} F'(y^{(k)}) = F'(\xi)\Big[&I + 2C_2^2{e^{(k)}}^2 + 4(C_2C_3 - 4C_2^3){e^{(k)}}^3 + (6C_2C_4 + 8C_2^2C_3 + 3C_3C_2^2){e^{(k)}}^4\Big] \\ &+ \mathcal{O}({e^{(k)}}^5), \end{aligned}$$

and the expansion of the variable $\mu^{(k)}$ is

$$\begin{aligned}\mu^{(k)} =& -2C_2e^{(k)} + (6C_2^2 - 3C_3){e^{(k)}}^2 + (-16C_2^3 - 4C_4 + 10C_2C_3 + 6C_3C_2){e^{(k)}}^3 \\ &+ (40C_2^4 + 9C_3^2 - 5C_5 + 14C_2C_4 - 28C_2^2C_3 - 15C_3C_2^2 + 8C_4C_2 - 18C_2C_3C_2){e^{(k)}}^4 \\ &+ (-96C_2^5 - 6C_6 - 30C_2C_3^2 + 18C_2C_5 - 40C_2^2C_4 + 72C_2^3C_3 + 36C_3C_2^3 + 12C_3C_4 \\ &- 12C_3^2C_2 - 24C_4C_2^2 + 12C_4C_3 + 10C_5C_2 + 42C_2C_3C_2^2 \\ &- 24C_2C_4C_2 + 48C_2^2C_3C_2 - 24C_3C_2C_3){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6).\end{aligned}$$

For the weight function G, we set

$$G(\mu^{(k)}) = G(I) + G_1(\mu^{(k)} - I) + \frac{1}{2}G_2(\mu^{(k)} - I)^2 + \mathcal{O}(\mu^{(k)} - I)^3,$$

that is,

$$\begin{aligned}G(\mu^{(k)}) =& G(I) - 2C_2G_1e^{(k)} + (6C_2^2G_1 - 3C_3G_1 + 2C_2^2G_2){e^{(k)}}^2 \\ &+ (-16C_2^3G_1 - 4C_4G_1 - 12C_2^3G_2 + 10G_1C_2C_3 + 3G_2C_2C_3 + 6G_1C_3C_2 \\ &+ 3G_2C_3C_2){e^{(k)}}^3 + \mathcal{O}({e^{(k)}}^4).\end{aligned}$$

We denote that $S = [F'(x^{(k)})]^{-1}F(y^{(k)})$. So, its Taylor development can be expressed as

$$\begin{aligned}S =& C_2{e^{(k)}}^2 + (-4C_2^2 + 2C_3){e^{(k)}}^3 + (13C_2^3 + 3C_4 - 8C_2C_3 - 6C_3C_2){e^{(k)}}^4 \\ &+ (-38C_2^4 - 12C_3^2 + 4C_5 - 12C_2C_4 + 26C_2^2C_3 + 18C_3C_2^2 - 8C_4C_2 + 20C_2C_3C_2){e^{(k)}}^5 \\ &+ \mathcal{O}({e^{(k)}}^6).\end{aligned}$$

So,

$$\begin{aligned}z^{(k)} =& (C_2 - bC_2G_0{e^{(k)}}^2 + (-2C_2^2 + 2C_3 + 4bC_3G_0){e^{(k)}}^3 \\ &+ (4C_2^3 + 3G_0 - 13bC_2^3G_0 - 3bC_4G_0 - 14bC_2^3G_2 - 2bC_2^3G_2 - 4C_2C_3 \\ &+ 8bGC_2C_3 + 4bG_1C_2C_3 - 3C_3C_2 + 6bG_0C_2 + 3bG_1C_3C_2){e^{(k)}}^4 \\ &+ (-8C_2^4 - 6C_3^2 + 38bC_2^4G_0 + 12bC_3^2G_0 + 66bC_2^4G_1 + 6bC_3^2G_1 + 20bC_2^4G_2 \\ &+ 4C_5 - 4bG_0C_5 - 6C_2C_4 + 12bG_0C_2C_4 + 6bG_1C_2C_4 + 8C_2^2C_3 - 26bG_0C_2^2C_3 \\ &- 28bG_1C_2^2C_3 - 4bG_2C_2^2C_3 + 6C_3C_2^2 - 18bG_0C_3C_2^2 - 18bG_1C_3C_2^2 - 3bG_2C_3C_2^2 \\ &- 4C_4C_2 + 8bG_0C_4C_2 + 4bG_1C_4C_2 + 6C_2C_3C_2 - 20bG_0C_2C_3C_2 - 22bG_1C_2C_3C_2 \\ &- 3bG_2C_2C_3C_2){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6),\end{aligned}$$

and

$$\begin{aligned}F(z^{(k)}) =& (C_2 - bC_2G_0){e^{(k)}}^2 + (-2C_2^2 + 2C_3 + 4bC_2G_0 - 2bC_3G_0 - 2bC_2^2G_1){e^{(k)}}^3 \\ &+ (5C_2^3 + 3C_4 - 15bC_2^3G_0 - 3bC_4G_0 + b^2c_2^3G_0^2 - 14bC_2^3G_1 - 2bC_2^3G_2 - 4C_2C_3 \\ &+ 8bG_0C_2C_3 + 4bG_1C_2C_3 - 3C_3C_2 + 6bG_0C_3C_2 + 3bG_1C_3C_2){e^{(k)}}^4 \\ &+ (-8C_2^4 - 6C_3^2 + 38bC_2^4G_0 + 12bC_3^2G_0 + 66bC_2^4G_1 + 6bC_3^2G_1 + 20bC_2^4G_2 + 4C_5 \\ &- 4bG_0C_5 - 6C_2C_4 + 12bG_0C_2C_4 + 6bG_1C_2C_4 + 8C_2^2C_3 - 26bG_0C_2^2C_3 \\ &- 28bG_1C_2^2C_3 - 4bG_2C_2^2C_3 + 6C_3C_2^2 - 18bG_0C_3C_2^2 - 18bG_1C_3C_2^2 - 3bG_2C_3C_2^2 \\ &- 4C_4C_2 + 8bG_0C_4C_2 + 4bG_1C_4C_2 + 6C_2C_3C_2 - 20bG_0C_3C_2 - 22bG_1C_2C_3C_2 \\ &- 3bG_2C_2C_3C_2){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6).\end{aligned}$$

Now, we denote $Sc = [F'(x^{(k)})]^{-1}F(z^{(k)})$. Therefore, its Taylor expansion is

$$\begin{aligned}
Sc &= (C_2 - bC_2G_0){e^{(k)}}^2 + (-4C_2^2 + 2C_3 + 6bC_2^2G_0 - 2bC_3G_0 + 2bC_2^2G_1){e^{(k)}}^3 \\
&\quad + (13C_2^3 + 3C_4 - 27bC_2^3G_0 - 3bC_4G_0 + b^2C_2^3G_0^2 - 18bC_2^3G_1 - 2bC_2^3G_2 - 8C_2C_3 \\
&\quad + 12bG_0C_2C_3 + 4bG_1C_2C_3 - 6C_3C_2 + 9bGC_3C_2 + 9bG_0C_3C_2 + 3bG_1C_3C_2){e^{(k)}}^4 \\
&\quad + (-34C_2^4 - 12C_3^2 + 92bC_2^4G + 18bC_3^2G_0 - 2b^2C_2^4G_0^2 + 102bC_2^4G_1 + 6bC_3^2G_1 \\
&\quad + 24bC_2^4G_2 + 4C_5 - 4bG_0C_5 - 12C_2C_4 + 18bG_0C_2C_4 + 6bG_1C_2C_4 + 24C_2^2C_3 \\
&\quad - 50bG_0C_2^2C_3 - 36bG_1C_2^2C_3 - 4bG_2C_2^2C_3 + 18C_3C_2^2 - 36bG_0C_3C_2^2 - 24bG_1C_3C_2^2 \\
&\quad - 3bG_2C_3C_2^2 - 8C_4C_2 + 12bG_0C_4C_2 + 4bG_1C_4C_2 + 18C_2C_3C_2 - 38bG_0C_2C_3 \\
&\quad - 28bG_1C_2C_3C_2 - 3bG_2C_2C_3C_2){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6).
\end{aligned}$$

If $Ss = i \cdot Sc + h \cdot S$, then it is expanded as

$$\begin{aligned}
Ss &= (C_2h + C_2i - bC_2Gi){e^{(k)}}^2 \\
&\quad + (-4C_2^2h + 2C_3h - 4C_2^2i + 2C_3i + 6bC_2^2G_0i - 2bC_3G_0i + 2bC_2^2G_1i){e^{(k)}}^3 \\
&\quad + (13C_2^3h + 3C_4h + 13C_2^3i + 3C_4i - 27bC_2^3G_0i - 3bC_4G_0i + b^2C_2^3G_0^2i - 18bC_2^3G_0i \\
&\quad - 2bC_2^3G_2i - 8hC_2C_3 - 8iC_2C_3 + 12bG_0iC_2C_3 + 4bG_1iC_2C_3 - 6hC_3C_2 - 6iC_3C_2 \\
&\quad + 9bG_0iC_3 + 3bG_1iC_3C_2){e^{(k)}}^4 \\
&\quad + (-38C_2^4h - 12C_3h - 34C_2^4i - 12C_3^2i + 92bC_2^4G_0i + 18bC_3^2G_0i - 2bC_2^4G_0^2i + 102bC_2^4G_0i \\
&\quad + 6bC_3^2G_0i + 24bC_2^4G_2i + 4hC_5 + 4iC_5 - 4bGiiC_5 - 12hC_2C_4 - 12iC_2C_4 + 18bGiC_2C_4 \\
&\quad + 6bG_1iC_2C_4 + 26hC_2C_3 + 24iC_2^2C_3 - 50bG_0iC_2^2C_3 - 36bG_1iC_2^2C_3 - 4bG_2iC_2^2C_3 \\
&\quad + 18G_0C_3C_2^2 + 18iC_3C_2^2 - 36bGiC_3C_2^2 - 24bG_1iC_3C_2^2 - 3bG_2iC_3C_2^2 - 8hC_4C_2 - 8iC_4C_2 \\
&\quad + 12bG_0iC_4C_2 + 4bG_1iC_4C_2 + 20hC_2C_3C_2 + 18iC_2C_3C_2 - 38bGiC_2C_3C_2 - 28bG_0iC_2C_3C_2 \\
&\quad - 3bG_2iC_2C_3C_2){e^{(k)}}^5 + \mathcal{O}({e^{(k)}}^6).
\end{aligned}$$

Then, the error equation is

$$\begin{aligned}
e^{(k+1)} &= (C_2 - bC_2G_0 - C_2G_0h - C_2G_0i + bC_2G_0^2i){e^{(k)}}^2 \\
&\quad + (-2C_2^2 + 2d + 4bC_2^2G_0 - 2bdG_0 + 2bC_2^2G_0 + 4C_2^2G_0h - 2dG_0h \\
&\quad + 2C_2^2G_0h + 4C_2^2G_0i - 2dG_0i - 6bC_2^2G_0^2i + 2bdG_0^2i + 2C_2^2G_1i - 4bC_2^2G_0G_1i){e^{(k)}}^3 \\
&\quad + (4C_2^3 + 3C_4 - 13bC_2^3G_0 - 3bC_4G_0 - 14bC_2^3G_1 - 2bC_2^3G_2 - 13C_2^3G_0h - 3C_4G_0h \\
&\quad - 14C_2^3G_1h - 2C_2^3G_2h - 13C_2^3Gi - 3C_4G_0i + 27bC_2^3G_0^2i + 3bC_4G_0^2i - b^2C_2^3G_0^3i \\
&\quad - 14C_2^3G_1i + 36bC_2^3G_0G_1i + 4bC_2^3G_1^2i - 2C_2^3G_2ii + 4bC_2^3G_0G_2ii - 4C_2C_3 + 8bG_0C_2C_3 \\
&\quad + 4bG_1C_2C_3 + 8G_0hC_2C_3 + 4G_1hC_2C_3 + 8G_0iC_2C_3 - 12bG_0^2iC_2C_3 + 4G_1iC_2C_3 \\
&\quad - 8bG_0G_1iC_2C_3 - 3C_3C_2 + 6bG_0C_3C_2 + 3bG_1C_3 + 6G_0hC_3C_2 + 3G_1hC_3C_2 \\
&\quad + 6G_0iC_3C_2 - 9bG_0^2iC_3C_2 + G_1iC_3C_2 - 6bG_0G_1iC_3C_2){e^{(k)}}^4 + M5{e^{(k)}}^5 \\
&\quad + M6{e^{(k)}}^6 + \mathcal{O}({e^{(k)}}^7).
\end{aligned}$$

By fixing $b = \dfrac{1}{G_0}$, $h = 0$, $i = \dfrac{1}{G_0}$, and $G_1 = -G_0$, the error equation becomes

$$\begin{aligned} e^{(k+1)} = \Bigg(& 24C_2^5 - 4G_2(37C_2^5 - 6C_3 - 6C_2C_3C_2^2C_3C_2 + 3C_3C_2C_3)\frac{1}{G_0} \\ & + 4G_2(32C_2^5G_0 + C_2^5G_2 - 6G_0C_3C_2^3 - 6G_0C_2C_3C_2^2 - 6G_0C_2^2C_3C_2 \\ & + 3G_0C_3C_2C_3)\left(\frac{1}{G_0}\right)^2 + 2C_2^5G_0^2(3G_0 - G_2)\left(\frac{1}{G_0}\right)^3\Bigg)e^{(k)^6} + \mathcal{O}(e^{(k)^7}). \end{aligned}$$

With this, the proof is finished. □

Let us notice that the order of convergence of this class of iterative methods can be increased up to 7 for specific values of G_2, depending on G_0 also being free. However, in order to reduce the computational cost, we set $G_2 = 0$, $G_0 = I$, and therefore the matrix weight function to be used in the iterative expression is

$$G(\mu^{(k)}) = 2I - \mu^{(k)} = 2I - [F'(x^{(k)})]^{-1}F'(y^{(k)}).$$

Therefore, the family is reduced to an iterative method of only order 6, denoted by O6, the iterative expression of which is

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}),\ k = 0,1,2,\ldots, \\ z^{(k)} &= y^{(k)} - \left[2I - [F'(x^{(k)})]^{-1}F'(y^{(k)})\right][F'(x^{(k)})]^{-1}F(y^{(k)}), \\ x^{(k+1)} &= z^{(k)} - \left[2I - [F'(x^{(k)})]^{-1}F'(y^{(k)})\right][F'(x^{(k)})]^{-1}F(z^{(k)}). \end{aligned}$$

Let us remark that this scheme has especially good properties, due to the existence of only one inverse operator. This yields that all the linear systems to be solved per iteration have the same coefficient matrix and therefore, the computational cost can be reduced by its LU factorization and the solution of several triangular linear systems. This is discussed in depth in the next section, in comparison with the introduced known procedures.

4. Efficiency Indices

To compare the iterative methods used, we use the computational efficiency index, CI, defined as [35]

$$CI = \rho^{\frac{1}{d + op}},$$

where d is the number of functional evaluations and op is the number of products/quotients per iteration.

In each iteration, five linear systems are solved with the same coefficient matrix, there are two matrix–vector products and, with respect to functional evaluations, we have two evaluations of Jacobian matrices and three of functions. The computational cost of method O6 is

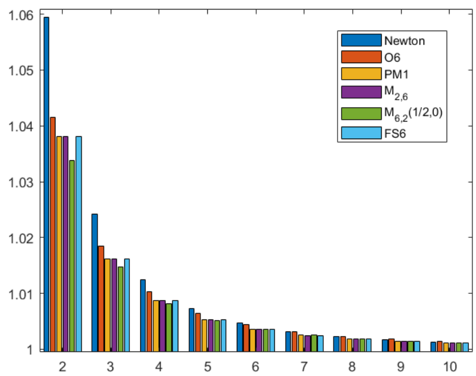
$$\frac{1}{3}n^3 + 9n^2 + \frac{8}{9}n, \tag{20}$$

In Table 2, the rest of the CI corresponding to the comparison methods are exposed. The way in which they have been calculated is similar to that of the O6 method.

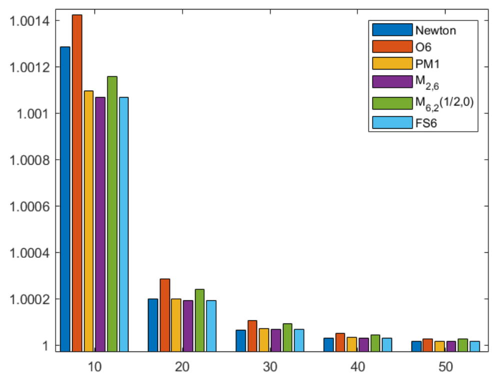
Table 2. Comparisons of CI.

Method	CI
Newton	$2^{\frac{1}{\frac{1}{3}n^3+2n^2+\frac{2}{3}n}}$
O6	$6^{\frac{1}{\frac{1}{3}n^3+9n^2+\frac{8}{9}n}}$
PM1	$6^{\frac{1}{\frac{2}{3}n^3+11n^2+\frac{4}{3}n}}$
$M_{2,6}$	$6^{\frac{1}{\frac{2}{3}n^3+10n^2+\frac{4}{3}n}}$
$M_{6,2}(1/2,0)$	$6^{\frac{1}{\frac{1}{3}n^3+12n^2+\frac{5}{3}n}}$
FS6	$6^{\frac{1}{\frac{2}{3}n^3+10n^2+\frac{4}{3}n}}$

The results are represented in the semi-logarithmic scale; see Figure 3 for a better visualization of the differences between the indices (CI) for the methods used and several sizes (n) of the systems.



(**a**) Sizes from 2 to 10



(**b**) Sizes from 10 to 50 with a step of 10

Figure 3. Computational efficiency indices.

In Figure 3a, we can observe that, for $2 \leq n \leq 7$, the best CI index corresponds to the Newton method, O6 being the best for $n \geq 8$. In Figure 3b, we can check that for bigger systems, $n \geq 10$; the best CI remains as O6.

5. Numerical Performance

We analyze the performance of the methods described above to check their efficiency and compare it with other known methods. The results from Tables 3–6 correspond to the calculations made with Matlab R2022b, by using variable precision arithmetics with 1200 digits of mantissa, on a PC equipped with an *Intel* Core™i5-5200U CPU 2.20GHz. In all the tables, we show the residual errors $\|x^{(k+1)} - x^{(k)}\|$ and $\|F(x^{(k+1)})\|$ of the last iteration satisfying the stopping criterium $\|x^{(k+1)} - x^{(k)}\| < 10^{-300}$ or $\|F(x^{(k+1)})\| < 10^{-300}$, and the CPU time obtained as the mean of 20 executions (e-time). Moreover, a computational estimation of the order of convergence is obtained by the means of ACOC, introduced as

$$\rho \approx ACOC = \frac{\ln \frac{\|x^{(k+1)}-x^{(k)}\|}{\|x^{(k)}-x^{(k-1)}\|}}{\ln \frac{\|x^{(k)}-x^{(k-1)}\|}{\|x^{(k-1)}-x^{(k-2)}\|}}. \tag{21}$$

5.1. Example

We consider the nonlinear system, $F_1(x) = (f_1(x), f_2(x), \dots, f_n(x))^T = 0$, such that

$$f_i(x) = x_i - \cos\left(2x_i - \sum_{j=1}^{4} x_j - x_i\right), i = 1, 2, 3, 4 \dots, 20, \tag{22}$$

with seed $x^{(0)} = (0.75, 0.75, \dots, 0.75)^T$, and in this case, $\alpha \approx (0.519, 0.519, \dots, 0.519)^T$.

Table 3. Numerical results for Example 5.1.

Method	**Iteration**	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	ρ	**e-Time**
Newton	8	3.1586×10^{-160}	2.297×10^{-320}	2.0	0.98
O6	4	3.4133×10^{-217}	0.0	6.0	0.99
FS6	4	1.682×10^{-201}	1.614×10^{-1207}	6.0	1.02
PM1	4	6.0584×10^{-186}	4.036×10^{-1115}	6.0	0.99
$M2_6$	4	4.7636×10^{-127}	1.891×10^{-635}	5.0	1.00
$M_{6,2}(1/2, 0)$	4	1.4631×10^{-189}	4.603×10^{-1137}	6.0	1.02

In Table 3, it can be observed that the number of iterations of all the sixth-order schemes are equal and the time is very similar in all of the methods; however, the best residual is obtained by the proposed scheme, O6. The ACOC estimates the theoretical order of the convergence accurately in all the cases.

5.2. Example

The second example is given by $F_2(x) = (g_1(x), g_2(x), \dots, g_n(x))^T = 0$, such that

$$g_i(x) = x_i - 2\ln\left(1 + \sum_{j=1}^{n} x_j - x_i\right),\ i = 1, 2, \dots, 20, \tag{23}$$

with seed $x^{(0)} = (1, 1, \dots, 1)^T$ and $\alpha \approx (9.376, 9.376, \dots, 9.376)^T$.

Table 4. Numerical results for Example 5.2.

Method	**Iterations**	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	ρ	**e-Time (Sec)**
Newton	11	1.1642×10^{-199}	3.409×10^{-401}	2.0	9.99
O6	5	3.3111×10^{-100}	1.032×10^{-608}	6.0	10.24
FS6	5	5.1171×10^{-73}	1.310×10^{-445}	6.0	11.00
PM1	6	2.169×10^{-291}	3.749×10^{-1755}	6.0	10.58
$M2_6$	6	5.3026×10^{-198}	4.327×10^{-996}	6.0	11.07
$M_{6,2}(1/2, 0)$	6	6.2633×10^{-289}	1.325×10^{-1206}	6.0	10.79

5.3. Example

Let us define now the nonlinear system $F_3(x) = (h_1(x), h_2(x), \dots, h_n(x))^T = 0$, such that

$$h_i(x) = arctan(x_i) + 1 - 2\left(\sum_{j=1}^{n} x_j^2 - x_i^2\right),\ i = 1, 2, \dots, n, \tag{24}$$

with seed $x^{(0)} = (0.5, 0.5, \dots, 0.5)^T$, $n = 20$, and $\alpha \approx (0.1758, 0.1758, \dots, 0.1758)^T$.

Of note, in Table 5, O6 and FS6 provide a solution satisfying the stopping criterium in a lower or equal number of iterations than the rest of the schemes. Indeed, the value of the residual errors in O6 and FS6 highly improve that of Newton's. This is the reason why their residuals are not as close to zero as those of the other schemes.

Table 5. Numerical results for Example 5.3.

Method	Iterations	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	ρ	e-Time (Sec)
Newton	10	1.2449×10^{-154}	1.322×10^{-307}	2.0	1.22
O6	5	1.3563×10^{-218}	2.414×10^{-1207}	6.0	1.24
FS6	4	4.4455×10^{-58}	4.283×10^{-344}	6.0	1.20
PM1	5	1.2252×10^{-218}	8.687×10^{-1208}	6.0	1.27
$M2_6$	5	6.2256×10^{-173}	5.016×10^{-861}	5.0	1.39
$M_{6,2}(1/2, 0)$	5	2.5983×10^{-252}	3.704×10^{-1208}	6.0	1.39

Regarding the applied problem described in Section 2, the underlying data of the nonlinear shear model of a reinforced concrete beam are provided by random values with few digits inside the prescribed ranges by technical standards for structural concrete; moreover, some of these parameters were fixed taking into consideration the solvability region of the adopted steel constitutive model. The stopping criterium is $\|x^{(k+1)} - x^{(k)}\| < 10^{-6}$ or $\|F(x^{(k+1)})\| < 10^{-6}$. The initial estimation used is $\theta = 34$, $\varepsilon_x = 0.0001$, $\varepsilon_t = 200,000$, $\nu = 0.0001$, $\varepsilon_2 = 0.0001$, $\sigma_2 = 0.0001$, $\sigma_{s,x} = 200$, $\sigma_{s,t} = 7$, $\varepsilon_p = 200$ and $\sigma_p = 100$. The results provided by the new and existing schemes appear in Table 6. The ACOC does not appear in this table, as it yields to unstable data in all cases.

Table 6. Problem statement Section 2.

Method	Iterations	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	e-Time (Sec)
Newton	5	0.0342	2.448×10^{-10}	18.3242
O6	3	9.0862	2.253×10^{-29}	21.6703
FS6	3	218.18	4.071×10^{-16}	22.1258
PM1	4	3.8554	3.138×10^{-28}	21.9594
$M2_6$	4	7.3926	9.621×10^{-21}	28.0727
$M_{6,2}(1/2, 0)$	3	0.0423	3.710×10^{-31}	22.4797

However, the best methods in terms of the number of iteration are O6, FS6, and $M_{6,2}(1/2, 0)$, all with three iterations. Among the sixth-order methods, the lowest e-time corresponds to our proposed scheme, O6. Although with this initial estimation, the e-time of Newton's method is the best, small changes in some of the coordinates of the seed yields to better results of O6 than Newton's scheme. This good performance allows us to assure the reliance and robustness of our proposed procedure.

6. Conclusions

In this article, we have developed a vectorial parametric family of numerical methods of the sixth order to solve nonlinear systems. In particular, it is applied on a constitutive equation of reinforced concrete (6). The order of the convergence of the new class (O6) is proven, and a particular member of the family is selected with better computational properties, as only one inverse operator is needed. Its efficiency is compared to other existing methods with the same order of convergence, and also with Newton's scheme, in terms of the computational efficiency index. For the size of the system $n \geq 8$, the proposed method, O6, gives the best results. In the numerical tests, all the comparison procedures need the same or more iterations and achieve lower precision results in the same or shorter execution time to achieve the required tolerance. This confirms the accuracy, robustness, and applicability of the proposed scheme.

Author Contributions: Conceptualization, J.J.P.; methodology, F.I.C.; software, A.C. and F.I.C.; formal analysis, J.R.T.; investigation, A.C.; writing—original draft preparation, J.J.P. and A.M.H.-D.; writing—review and editing, A.C. and J.R.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors would like to thank the anonymous reviewers for their useful comments and suggestions that have improved the final version of the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Ritter, W. *Die Bauweise Hennebique*; Schweiserische Bauzeitung: Zürich, Switzerland 1899; Volume 33, pp. 41–43.
2. Mörsch, E. *Concrete-Steel Construction*; English Translation by EP Goodrich; Engineering News Publishing Company: New York, NY, USA, 1909.
3. Collins, M.P.; Mitchell, D. *Prestressed Concrete Structures*; Prentice Hall: Englewood Cliffs, NJ, USA, 1991; Volume 9.
4. Hognestad, E. *What Do We Know About Diagonal Tension and Web Reinforcement in Concrete? A Historical Study*; Engineering Experiment Station; Circular Series No. 64; University of Illinois Urbana: Champaign, IL, USA, 1952.
5. Wagner, H. Ebene Blechwandtrager mit sehr dunnem Stegblech (Metal beams with very thin webs). *Z. Flugtech. Mot.* **1929**, *20*.
6. Mitchell, D.; Collins, M.P. Diagonal compression field theory-a rational model for structural concrete in pure torsion. *J. Proc.* **1974**, *71*, 396–408.
7. Collins, M.P. Towards a rational theory for RC members in shear. *J. Struct. Div.* **1978**, *104*, 649–666. [CrossRef]
8. Vecchio, F.; Collins, M. *The Response of Reinforced Concrete to In-Plane Shear and Normal Stresses*; University of Toronto: Toronto, Canada, 1982.
9. Vecchio, F.J.; Collins, M.P. The modified compression-field theory for reinforced concrete elements subjected to shear. *ACI J.* **1986**, *83*, 219–231.
10. Martín-Pérez, B.; Pantazopoulou, S.J. Effect of bond, aggregate interlock and dowel action on the shear strength degradation of reinforced concrete. *Eng. Struct.* **2001**, *23*, 214–227. [CrossRef]
11. Belarbi, A.; Hsu, T. Constitutive laws of concrete in tension and reinforcing bars stiffened by concrete. *Struct. J.* **1994**, *91*, 465–474.
12. Pang, X.; Hsu, T. Behavior of reinforced concrete membrane elements in shear. *Struct. J.* **1995**, *92*, 665–679.
13. Gil-Martín, L.; María, L.; Hernández-Montes, E.; Aschheim, M.; Pantazopoulou, S. Refinements to compression field theory, with application to wall-type structures. *Am. Concr. Inst. Spec. Publ.* **2009**, *265*, 123–142.
14. Palermo, M.; Gil-Martín, L.; Hernandez-Montes, E.; Aschheim, M. Refined compression field theory for plastered straw bale walls. *Constr. Build. Mater.* **2014**, *58*, 101–110. [CrossRef]
15. Pérez-Aracil, J.; Camacho-Gómez, C.; Hernández-Díaz, A.; Pereira, E.; Camacho, D.; Salcedo-Sanz, S. Memetic coral reefs optimization algorithms for optimal geometrical design of submerged arches. *Swarm Evol. Comput.* **2021**, *67*, 100958. [CrossRef]
16. Amat, S.; Busquier, S. *Advances in Iterative Methods for Nonlinear Equations*; SEMA SIMAI Springer Series; Springer: Cham, Switzerland, 2016.
17. Ortega, J.; Rheinboldt, W. *Iterative Solution of Nonlinear Equations in Several Variables*; Academic Press: New York, NY, USA, 1970.
18. Traub, J. *Iterative Methods for the Solution of Equation*; Chelsea Publishing Company: New York, NY, USA, 1982.
19. Potra, F.; Pták, V. *Nondiscrete Induction and Iterarive Processes*; Pitman Publishing: Boston, MA, USA, 1984.
20. Petković, M.S.; Neta, B.; Petković, L.D.; Džunić, J. *Multipoint Methods for Solving Nonlinear Equations*; Elsevier: Amsterdam, The Netherlands, 2013.
21. Junjua, M.; Akram, S.; Yasmin, N.; Zafar, F. A new Jarratt-type fourth-order method for solving system of nonlinear equations and applications. *J. Appl. Math.* **2015**, *2015*, 805278. [CrossRef]
22. Khirallah, M.Q.; Hafiz, M.A. Solving system of non-linear equations using family of Jarratt methods. *Int. J. Differ. Equ. Appl.* **2013**, *12*, 69–83.
23. Singh, H.; Sharma, J.; Kumar, S. A simple yet efficient two-step fifth-order weighted-Newton method for nonlinear models. *Numer. Algorithms* **2022**, *93*, 203–225. [CrossRef]
24. Xiao, X.; Yin, H. Increasing the order of convergence for iterative methods to solve nonlinear systems. *Calcolo* **2016**, *53*, 285–300. [CrossRef]
25. Singh, H.; Sharma, J. Simple and efficient fifth order solvers for systems of nonlinear problems. *Math. Model. Anal.* **2023**, *28*, 1–22. [CrossRef]
26. Alzahrani, A.; Behl, R.; Alshomrani, A. Some higher-order iteration functions for solving nonlinear models. *Appl. Math. Comput.* **2018**, *334*, 80–93. [CrossRef]
27. Hueso, J.L.; Martínez, E.; Teruel, C. Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems. *J. Comput. Appl. Math.* **2015**, *275*, 412–420. [CrossRef]
28. Narang, M.; Bhatia, S.; Kanwar, V. New two-parameter Chebyshev–Halley-like family of fourth and sixth-order methods for systems of nonlinear equations. *Appl. Math. Comput.* **2016**, *275*, 394–403. [CrossRef]
29. Behl, R.; Maroju, P.; Motsa, S. Efficient family of sixth-order methods for nonlinear models with its dynamics. *Int. J. Comput. Methods* **2019**, *16*, 1840008. [CrossRef]
30. Yaseen, S.; Zafar, F. A new sixth-order Jarratt-type iterative method for systems of nonlinear equations. *Arab. J. Math.* **2022**, *11*, 585–599. [CrossRef]

31. Hernández-Díaz, A.; Gil-Martín, L. Analysis of the equal principal angles assumption in the shear design of reinforced concrete members. *Eng. Struct.* **2012**, *42*, 95–105. [CrossRef]
32. Hernández-Díaz, A.; García-Román, M. Computing the refined compression field theory. *Int. J. Concr. Struct. Mater.* **2016**, *10*, 143–147. [CrossRef]
33. Cordero, A.; Hueso, J.; Martínez, E.; Torregrosa, J. A modified Newton-Jarratt's composition. *Numer. Algorithms* **2010**, *55*, 87–99. [CrossRef]
34. Artidiello, S.; Cordero, A.; Torregrosa, J.R.; Vassileva, M.P. Multidimensional generalization of iterative methods for solving nonlinear problems by means of weight-function procedure. *Appl. Math. Comput.* **2015**, *268*, 1064–1071. [CrossRef]
35. Cordero, A.; Torregrosa, J.R. On interpolation variants of Newton's method for functions of several variables. *J. Comput. Appl. Math.* **2010**, *234*, 34–43. [CrossRef]