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Article

Chern-Simons Path Integrals in $S^2 \times S^1$

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Abstract: Using torus gauge fixing, Hahn in 2008 wrote down an expression for a Chern-Simons path integral to compute the Wilson Loop observable, using the Chern-Simons action S_{CS}^{κ}, κ is some parameter. Instead of making sense of the path integral over the space of $\mathfrak{g}$-valued smooth 1-forms on $S^2 \times S^1$, we use the Segal Bargmann transform to define the path integral over B_i, the space of $\mathfrak{g}$-valued holomorphic functions over $\mathbb{C}^2 \times \mathbb{C}^{i-1}$. This approach was first used by us in 2011. The main tool used is Abstract Wiener measure and applying analytic continuation to the Wiener integral. Using the above approach, we will show that the Chern-Simons path integral can be written as a linear functional defined on $C(B_1^{\times^4} \times B_2^{\times^2}, \mathbb{C})$ and this linear functional is similar to the Chern-Simons linear functional defined by us in 2011, for the Chern-Simons path integral in the case of $\mathbb{R}^3$. We will define the Wilson Loop observable using this linear functional and explicitly compute it, and the expression is dependent on the parameter κ. The second half of the article concentrates on taking κ goes to infinity for the Wilson Loop observable, to obtain link invariants. As an application, we will compute the Wilson Loop observable in the case of $SU(N)$ and $SO(N)$. In these cases, the Wilson Loop observable reduces to a state model. We will show that the state models satisfy a Jones type skein relation in the case of $SU(N)$ and a Conway type skein relation in the case of $SO(N)$. By imposing quantization condition on the charge of the link L, we will show that the state models are invariant under the Reidemeister Moves and hence the Wilson Loop observables indeed define a framed link invariant. This approach follows that used in an article written by us in 2012, for the case of $\mathbb{R}^3$.

Keywords: Chern-Simons; path integral; non-abelian gauge; framed link invariants; Jones polynomial; state models

MSC classifications: 2010: 81T08, 81T13, 60H99

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1. Introduction

This is an unplanned sequel to [1,2]. Let M be a 3-manifold and G be a compact connected semisimple Lie group. Without loss of generality we will assume that G is a Lie subgroup of $U(\bar{N})$, $\bar{N} \in \mathbb{N}$. We will identify the Lie algebra $\mathfrak{g}$ of G with a Lie subalgebra of the Lie algebra $\mathfrak{u}(\bar{N})$ of $U(\bar{N})$ throughout this article. Suppose we write $\mathrm{Tr} \equiv \mathrm{Tr}_{\mathrm{Mat}(\bar{N},\mathbb{C})}$. Then we can define a positive, non-degenerate bilinear form by $\langle A, B\rangle = -\mathrm{Tr}_{\mathrm{Mat}(\bar{N},\mathbb{C})}[AB]$ for $A, B \in \mathfrak{g}$.

Let $\mathfrak{v} \subseteq \mathfrak{g}$ be a subspace. The vector space of all smooth $\mathfrak{v}$-valued 1-forms on a manifold Σ (need not be a 3-manifold) will be denoted by $\mathcal{A}_{\Sigma,\mathfrak{v}}$. We will identify the space of connection 1-forms on the trivial principal fiber bundle $P(M, G)$ with group G and base manifold M with $\mathcal{A}_{M,\mathfrak{g}} \equiv \mathcal{A}$.

Denote the group of all smooth G-valued mappings on M by $\mathcal{G}$, called the gauge group. The gauge group induces a gauge transformation on $\mathcal{A}$, $\mathcal{A} \times \mathcal{G} \to \mathcal{A}$ given by

$$A \cdot \Omega := A^{\Omega} = \Omega^{-1} d\Omega + \Omega^{-1} A\Omega$$

for $A \in \mathcal{A}$, $\Omega \in \mathcal{G}$. The orbit of an element $A \in \mathcal{A}$ under this operation will be denoted by $[A]$ and the set of all orbits by $\mathcal{A}/\mathcal{G}$.

For $A \in \mathcal{A}$, the Chern-Simons action is given by

$$S^{\kappa}_{CS}(A) = \frac{\kappa}{4\pi}\int_M \mathrm{Tr}_{\mathrm{Mat}(\bar{N},\mathbb{C})}\left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right],\ \kappa \neq 0. \tag{1}$$

Note that $\kappa \in \mathbb{Z}$ so that $\exp(iS^{\kappa}_{CS}([A]))$ is invariant under gauge transformation even though $S^{\kappa}_{CS}([A])$ is not.

The interest in Chern-Simons path integrals is the evaluation of Wilson Loop observables, that is we want to compute

$$Z(M, \kappa, q; l^i, \rho_i) := \frac{1}{Z_{\mathrm{CS}}}\int_{[A]\in\mathcal{A}/\mathcal{G}} \prod_{k=1}^{n} W(l^k; q)([A]) e^{iS^{\kappa}_{CS}([A])} D[A], \tag{2}$$

where

$$Z_{CS} = \int_{[A]\in\mathcal{A}/\mathcal{G}} e^{iS^{\kappa}_{CS}([A])} D[A],$$

is a normalising constant.

Here, $L = \{l^k\}_{k=1}^n$ is a link in M with non-intersecting (closed) curves l^k and

$$W(l^k; q)(A) := \mathrm{Tr}_{\rho_k}\mathcal{T}\exp\left[q\int_{l^k} A\right] \tag{3}$$

is the Wilson loop associated to l^k. And, $D[A]$ is some heuristic Lebesgue measure on $\mathcal{A}/\mathcal{G}$, Tr_{ρ_k} is the matrix trace for some representation $\rho_k : \mathfrak{g} \to \mathfrak{u}(N_k)$, $N_k \in \mathbb{N}$, and $\mathcal{T}$ is the time ordering operator.

Note that $W(l^k; q)(A)$ is the holonomy operator of A, computed along the loop l^k. The integral in Equation (2) will be known as the Wilson Loop observable associated to the link L and q will be called the charge of the link. When L consists of only one curve, the link is termed as a knot.

It was argued in [3] that if one can make sense of the RHS of Equation (2), then one can define a suitable generalization of the Jones polynomial of the link L in M. The objective of this article is to compute the right hand side of Equation (2) for the case of $M = S^2 \times S^1$ in the non-abelian case. The case when the manifold is $S^2 \times S^1$ is also singled out in [3] and is the next simple case to consider after $M = \mathbb{R}^3$.

The main purpose of this article is to define a Chern-Simons path integral in $S^2 \times S^1$ using torus gauge fixing and non-abelian gauge group. We will further show how link invariants appear from these path integrals in the second half of this article.

The case $M = \mathbb{R}^3$ was worked out by [1,2] for the abelian and non-abelian gauge group G respectively. Using axial gauge fixing, it suffices to only consider connections which are zero in the z-direction.

Unfortunately, in the case of $S^2 \times S^1$, it is not possible to make the connection disappear in the S^1 direction. On our compact Lie group, fix a maximal torus T and let $\mathfrak{t}$ be the Lie algebra of T. Under torus gauge fixing, we can choose the connection such that it takes values in $\mathfrak{t}$ in the S^1 direction. This was accomplished by Hahn in [4] and he wrote down an expression for the Chern-Simons path integral in Expression 6. We will try to make sense of this expression instead.

Using local coordinates, we will work on $\mathbb{R}^2 \times [0, 1)$, which we will call it the classical space. The link L is mapped inside $\mathbb{R}^2 \times [0, 1)$, called a truncated link. Now, consider $\mathbb{C}^3$, whereby $\mathbb{C}^3$ is a complexification of $\mathbb{R}^3$. We will refer $\mathbb{C}^3$ as a quantum space. After 'scaling' the truncated link and embed it inside $\mathbb{C}^3$, the Wilson Loop observable will then be defined on this quantum space. Details to be given later.

Over this quantum space, we will explain how to construct two Wiener spaces. The first Wiener space will be the space of analytic 2-tuple $\mathfrak{g}$-valued functions over the quantum space. The second Wiener space will be space of analytic 4-tuple $\mathfrak{g}$-valued functions over the quantum space. The Chern-Simons path integral is defined as an integral over the product space of these 2 Wiener spaces. For the Wilson Loop observable, we will explicitly work out this integral for the truncated link embedded inside the quantum space.

The link invariants that we are interested in will only appear when we take the limit of the Wilson Loop observable as κ goes to infinity. This limit can be computed easily from a truncated link diagram, by projecting L on $\mathbb{R}^2$. By assigning ± 1 to crossings on this link diagram, we can write down a formula for the Wilson Loop observable directly from this link diagram. Furthermore, we will show that the Wilson Loop observable is equal to a state model for links when the representation is the same for all curves in L.

Two diagrams represent the same link up to ambient isotopy if the 2 diagrams can be obtained from each other by applying Reidemeister moves. It is not true that the state model defines a link invariant. The state model for links has to satisfy certain algebraic equations to be a link invariant, including the Yang Baxter Equation (34). This will impose quantization conditions on the charge q of the link.

As an application, we will work out explicitly for the gauge groups $SU(N)$ and $SO(N)$. We will show that using gauge group $SU(N)$, the Wilson Loop observable will satisfy a Homfly skein relation Equation (38), with $l = e^{-\pi i q^2/N}$ and $m = 2i \sin(\pi q^2)$. For gauge group $SO(N)$, the Wilson Loop observable will satisfy a Conway-type skein relation, with $z = 2i \sin(\pi q^2/2)$. For both cases, q^2 is quantized to take only a discrete number of values.

This article is organized as follows. In Section 2, we will explain Hahn's heuristic expression for the Chern-Simons path integral using torus gauge fixing. This section will contain mainly definitions. In Section 3, we will give a heuristic but equivalent definition, whereby the path integral will be defined on. In Section 4, we will compute some simple functional integrals, which motivates the definition of the Chern-Simons path integral. This is an extension to the path integral considered in [1]. In Section 5, we need to introduce some important linear operators which are necessary in defining the Chern-Simons path integral. In Section 6, we will give our definition of the Chern-Simons path integral. As an application, we will define the Wilson Loop observable given in Equation (2) and compute it.

The second half of this article concentrates on taking the limit as κ goes to infinity of the Wilson Loop observable. In Section 7, we will define a link diagram for a framed link L. In Section 8, we will compute the limit of the Wilson Loop observable. In Section 9, we will obtain framed link invariants in the case of gauge group $SU(N)$ and $SO(N)$. We will make some ending remarks in Section 10.

We end this section by stating some notations which will be assumed throughout this article.

Notation 1. *Suppose we have two Hilbert spaces, H_1 and H_2. We consider the tensor product $H_1 \otimes H_2$. The inner product on the tensor product $H_1 \otimes H_2$ is given by*

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle_{H_1 \otimes H_2} = \langle u_1, v_1 \rangle_{H_1} \langle u_2, v_2 \rangle_{H_2}.$$

This definition of the inner product on the tensor product of Hilbert spaces will be assumed throughout this article.

Now consider the direct product $H_1 \times H_2$. The inner product on $H_1 \times H_2$ is defined by

$$\langle (u^1, u^2), (v^1, v^2) \rangle_{H_1 \times H_2} := \sum_{i=1}^{2} \langle u^i, v^i \rangle_{H_i}.$$

This definition of the inner product on the direct product of Hilbert spaces will also be assumed throughout this article.

If $H_1 = H_2 = H$, we abbreviate by writing $H \times H \equiv H^{\times^2}$.

Finally, we always use $\langle \cdot, \cdot \rangle$ to denote an inner product.

2. Some Definitions and Notations

From this point onwards, we only consider the 3-manifold $S^2 \times S^1$. On S^2, fix a north pole $\mathbf{n}$ and let the south pole $\mathbf{s}$ sit on the origin of $\mathbb{R}^2$. We use the stereographic projection $X : S^2 \rightarrow \mathbb{R}^2$ as local coordinates. Let $x = (x_+, x_-)$ be local coordinates on $\mathbb{R}^2$.

On S^1, let i_{S^1} denote the mapping $u \in [0,1] \mapsto \exp(2\pi i u) \in \{z \in \mathbb{C} | \ |z| = 1\} \cong S^1$ and we set $t_0 := i_{S^1}(0) \in S^1$. The restriction of i_{S^1} onto $[0,1)$, which is a bijective mapping $[0,1) \rightarrow S^1$, will also be denoted by i_{S^1} and its inverse will be denoted by $i_{S^1}^{-1}$. The tangent vector of S^1 at the point $i_{S^1}(u)$, induced by the curve i_{S^1}, will be denoted by $i'_{S^1}(u)$, for $u \in [0,1]$. Finally, $\dfrac{\partial}{\partial t}$ will denote the vector field on S^1 given by $\dfrac{\partial}{\partial t}(i_{S^1}(t)) = i'_{S^1}(t)$ for $t \in [0,1]$ and dt, the real-valued 1-form on S^1 is dual to $\dfrac{\partial}{\partial t}$.

For the rest of this article, instead of working in $S^2 \times S^1$, we work in local coordinates $(X, i_{S^1}^{-1})$. All the formulas in the sequel will be written using these local coordinates.

2.1. Quasi-Axial and Torus Gauge Fixing

Let $\mathcal{A}$ be the vector space of (smooth) $\mathfrak{g}$-valued 1-forms on $\mathbb{R}^2 \times [0,1)$. We further impose the condition that it vanishes at infinity. Now, we write $\mathcal{A} = \mathcal{A}^{\perp} \oplus \mathcal{A}^{||}$, where

$$\mathcal{A}^{\perp} := \left\{ A \in \mathcal{A} \,\middle|\, A\left(\frac{\partial}{\partial t}\right) = 0 \right\}, \ \mathcal{A}^{||} := \{ B \otimes dt \mid B \in C^{\infty}(\mathbb{R}^2 \times [0,1), \mathfrak{g}) \}.$$

For every $A \in \mathcal{A}$, $A^{\perp}$ and $A^{||}$ will denote the unique elements of $\mathcal{A}^{\perp}$, respectively $\mathcal{A}^{||}$ such that $A = A^{\perp} + A^{||}$ holds. For a given $A \in \mathcal{A}$, we set $A_0 := A\left(\frac{\partial}{\partial t}\right) \in C^{\infty}(\mathbb{R}^2 \times [0,1), \mathfrak{g})$, *i.e.*, A_0 is the element of $C^{\infty}(\mathbb{R}^2 \times [0,1), \mathfrak{g})$ given by $A^{||} = A_0 \otimes dt$.

Let T be a maximal torus of G and denote the Lie algebra of T by $\mathfrak{t}$. An element $A \in \mathcal{A}$ will be called "quasi-axial" (respectively "in the T-torus gauge") if the functions $A_0((\sigma, \cdot))$, $\sigma \in \mathbb{R}^2$ are constant (respectively constant and $\mathfrak{t}$-valued). We will denote the set of all quasi-axial elements (respectively all elements in the T-torus gauge) of $\mathcal{A}$ by $\mathcal{A}^{\mathrm{qax}}$ (respectively $\mathcal{A}^{\mathrm{qax}}(T)$). Thus, we have

$$\mathcal{A}^{\mathrm{qax}} = \mathcal{A}^{\perp} \oplus \{ B \otimes dt \mid B \in C^{\infty}(\mathbb{R}^2, \mathfrak{g}) \}, \ \mathcal{A}^{\mathrm{qax}}(T) = \mathcal{A}^{\perp} \oplus \{ B \otimes dt \mid B \in C^{\infty}(\mathbb{R}^2, \mathfrak{t}) \}.$$

The following proposition is Proposition 5.2 taken from [4], the proof is omitted. We present the proposition using local coordinates X.

Proposition 1. *Let $A \in \mathcal{A}^{\mathrm{qax}}$ and let $A^{\perp} \in \mathcal{A}^{\perp}$ and $B \in C^{\infty}(\mathbb{R}^2, \mathfrak{g})$ be given by $A = A^{\perp} + B \otimes dt$. Then we have*

$$\begin{aligned} S_{CS}^{\kappa}(A) =& S_{CS}^{\kappa}(A^{\perp} + B \otimes dt) \\ =& -\frac{\kappa}{4\pi} \int_0^1 dt \left[\int_{\mathbb{R}^2} \mathrm{Tr}\left[A^{\perp}(t) \wedge \left(\frac{\partial}{\partial t} + \mathrm{ad}(B) \right) \cdot A^{\perp}(t) \right] - 2 \int_{\mathbb{R}^2} \mathrm{Tr}\left[A^{\perp}(t) \wedge dB \right] \right]. \end{aligned}$$

Definition 1. *(Regular elements) Let G_{reg} denote the set of regular elements of G, i.e., the set of all $g \in G$ which are contained in a unique maximal torus of G. Similarly, let $\mathfrak{g}_{\mathrm{reg}}$ denote the set of regular elements of $\mathfrak{g}$,* i.e., *the set of all $B \in \mathfrak{g}$ which are contained in a unique maximal Abelian Lie subalgebra of $\mathfrak{g}$. We set $\mathfrak{g}'_{\mathrm{reg}} := \exp^{-1}(G_{\mathrm{reg}})$.*

It is not difficult to see that $g \in G_{\mathrm{reg}}$ (resp. $B \in \mathfrak{g}_{\mathrm{reg}}$) if and only if the set of fixed points of $\mathrm{Ad}(g)$ (resp. the kernel of $\mathrm{ad}(B)$) is a maximal Abelian Lie subalgebra of $\mathfrak{g}$. Thus, $\mathfrak{g}'_{\mathrm{reg}} \subset \mathfrak{g}_{\mathrm{reg}}$.

Hahn in [4] was able to write 2 expressions for Expression 2 on the subspace $\mathcal{A}^{\mathrm{qax}}$ and $\mathcal{A}^{\mathrm{qax}}(T)$. Let $L = \{l^k\}_{k=1}^n$ be a link. Using quasi-axial gauge fixing, we have the following expression taken from Equation (6.3) in [4], ($\sim$ means up to a constant.)

$$\begin{aligned} Z(M, \kappa, q; l^i, \rho_i) =& \int \prod_{k=1}^n W(l^k; q)(A) \frac{1}{Z} \exp(i S_{CS}^{\kappa}(A)) DA \\ \sim& \int_{C^{\infty}(\mathbb{R}^2, \mathfrak{g}'_{reg})} \int_{\mathcal{A}^{\perp}} \prod_{k=1}^n W(l^k; q)(A^{\perp} + B \otimes dt) \exp(i S_{CS}^{\kappa}(A^{\perp} + B \otimes dt)) DA^{\perp} \tilde{\Delta}[B] \tilde{D}B \\ =& \int_{C^{\infty}(\mathbb{R}^2, \mathfrak{g}'_{reg})} \left[\int_{\mathcal{A}^{\perp}} \prod_{k=1}^n W(l^k; q)(A^{\perp} + B \otimes dt) d\mu_B^{\perp}(A^{\perp}) \right] \tilde{\Delta}[B] \tilde{D}B, \end{aligned} \tag{4}$$

where $DA^{\perp}$ is the informal "Lebesgue measure" on $\mathcal{A}^{\perp}$ and

$$\tilde{D}B = \det\left(\sum_{n=0}^{\infty}\frac{(\mathrm{ad}(B))^n}{(n+1)!}\right)DB,$$

DB is the informal Lebesgue measure on $C^{\infty}(\mathbb{R}^2,\mathfrak{g})$. For $B \in C^{\infty}(\mathbb{R}^2,\mathfrak{g}'_{reg})$,

$$d\mu_B^{\perp}(A^{\perp}) := \exp\left(iS_{CS}^{\kappa}(A^{\perp}+B\otimes dt)\right)DA^{\perp},$$

and from Proposition 1, we will write

$$\begin{aligned} S_{CS}^{\kappa}(A) &= S_{CS}^{\kappa}(A^{\perp}+B\otimes dt),\ A\in\mathcal{A}^{\mathrm{qax}} \\ &= -\int_0^1 dt\,\left[\frac{1}{2}\left\langle A^{\perp}(t),\left(\frac{\partial}{\partial t}+\mathrm{ad}(B)\right)\cdot A^{\perp}(t)\right\rangle_{\mathbb{R}^2,\mathfrak{g}} - \left\langle A^{\perp}(t), dB\right\rangle_{\mathbb{R}^2,\mathfrak{g}}\right], \end{aligned} \tag{5}$$

with $\langle\cdot,\cdot\rangle_{\mathbb{R}^2,\mathfrak{g}}$ denotes the bilinear form on the vector space of smooth $\mathfrak{g}$-valued 1-forms on $\mathbb{R}^2$, $\mathcal{A}_{\mathbb{R}^2,\mathfrak{g}}$, given by

$$\langle A, A'\rangle_{\mathbb{R}^2,\mathfrak{g}} := \frac{\kappa}{2\pi}\int_{\mathbb{R}^2}\mathrm{Tr}(A\wedge A')$$

for $A,\ A'\in\mathcal{A}_{\mathbb{R}^2,\mathfrak{g}}\subset\mathcal{A}_{\mathbb{R}^2,\mathrm{Mat}(\bar{N},\mathbb{C})}$. Similar definition for $\langle\cdot,\cdot\rangle_{\mathbb{R}^2,\mathfrak{t}}$, with $A,\ A'\in\mathcal{A}_{\mathbb{R}^2,\mathfrak{t}}$. Here, $\mathcal{A}_{\mathbb{R}^2,\mathrm{Mat}(\bar{N},\mathbb{C})}$ is the vector space of $\mathrm{Mat}(\bar{N},\mathbb{C})$-valued 1-forms on $\mathbb{R}^2$. Finally, for $A = A^{\perp}+B\otimes dt\in\mathcal{A}^{\mathrm{qax}}$, we have

$$\left|\det\left(\frac{\partial}{\partial t}+\mathrm{ad}(B)\right)\right| := \tilde{\Delta}[B].$$

Here, $\dfrac{\partial}{\partial t}+\mathrm{ad}(B)$ is viewed as an operator on $C^{\infty}(\mathbb{R}^2\times S^1,\mathfrak{g})$.

Definition 2. *(Maximal Torus)*

1. *Let T be a fixed maximal torus of G. The Lie algebra of T will be denoted by $\mathfrak{t}$. Moreover, we set $T_{\mathrm{reg}} := T\cap G_{\mathrm{reg}}$ and $t'_{\mathrm{reg}} := \mathfrak{t}\cap\mathfrak{g}'_{\mathrm{reg}}$. Note that $\exp^{-1}(T_{reg})\subset\mathfrak{t}$.*
2. *Let $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ denote the scalar product $(A,B)\in\mathfrak{g}\times\mathfrak{g}\mapsto -\mathrm{Tr}(AB)\in\mathbb{R}$ on $\mathfrak{g}$ and let $\mathfrak{t}$ be the Lie algebra of T. Let $\mathfrak{g}_0$ be the $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$.*

Suppose we write $\mathcal{A}^{\perp} = \hat{\mathcal{A}}^{\perp}\oplus\mathcal{A}_c^{\perp}$, whereby

$$\begin{aligned} \hat{\mathcal{A}}^{\perp} :=& \{A^{\perp}\in\mathcal{A}^{\perp}\mid \pi_{\mathcal{A}_{\mathbb{R}^2,\mathfrak{t}}}(A^{\perp})(0)=0\}, \\ \mathcal{A}_c^{\perp} :=& \{A^{\perp}\in\mathcal{A}^{\perp}\mid A^{\perp}(t)=A^{\perp}(0)\in\mathcal{A}_{\mathbb{R}^2,\mathfrak{t}},\ \forall t\in[0,1)\}\cong\mathcal{A}_{\mathbb{R}^2,\mathfrak{t}}. \end{aligned}$$

Here, $\pi_{\mathcal{A}_{\mathbb{R}^2,\mathfrak{t}}}$ is the projection operator onto the second term in the direct sum $\mathcal{A}_{\mathbb{R}^2,\mathfrak{g}}\cong\mathcal{A}_{\mathbb{R}^2,\mathfrak{g}_0}\oplus\mathcal{A}_{\mathbb{R}^2,\mathfrak{t}}$. And, $\mathcal{A}_{\mathbb{R}^2,\mathfrak{t}}$ (respectively $\mathcal{A}_{\mathbb{R}^2,\mathfrak{g}_0}$)denotes the vector space of $\mathfrak{t}$-valued ($\mathfrak{g}_0$-valued) smooth 1-forms on $\mathbb{R}^2$.

Let $\hat{A}^{\perp}\in\hat{\mathcal{A}}^{\perp}$, $A_c^{\perp}\in\mathcal{A}_c^{\perp}$. Note that $\mathrm{ad}(B)\cdot A_c^{\perp}=0$. For $\hat{A}^{\perp}+A_c^{\perp}+B\otimes dt\in\mathcal{A}^{\mathrm{qax}}(T)$, we have the following torus gauge analogue of Equation (4), taken from Equation (6.6) in [4],

$$\begin{aligned} Z(M,\kappa,q;l^i,\rho_i) = \frac{1}{Z}\int_{C^{\infty}(\mathbb{R}^2,\mathfrak{t}'_{reg})}&\left[\int_{\mathcal{A}_c^{\perp}}\left[\int_{\hat{\mathcal{A}}^{\perp}}\prod_{k=1}^{n}W(l^k;q)(\hat{A}^{\perp}+A_c^{\perp}+B\otimes dt)d\mu_B^{\perp}(\hat{A}^{\perp})\right]\right. \\ &\left.\times\exp\left(i\langle A_c^{\perp},dB\rangle_{\mathbb{R}^2,\mathfrak{g}}\right)DA_c^{\perp}\right]\tilde{\Delta}[B]\hat{D}B, \end{aligned} \tag{6}$$

where
$$Z = \int_{C^\infty(\mathbb{R}^2,\mathfrak{t}'_{reg})} \int_{\mathcal{A}_c^\perp} \left[\int_{\hat{\mathcal{A}}^\perp} d\mu_B^\perp(\hat{A}^\perp)\right] \exp\left(i\langle A_c^\perp, dB\rangle_{\mathbb{R}^2,\mathfrak{t}}\right) \tilde{\Delta}[B] DA_c^\perp \hat{D}B \tag{7}$$
and $L = \{l^k\}_{k=1}^n$ is a link. In this case, do note that
$$\begin{aligned}\hat{D}B &= \det\left(\sum_{n=0}^\infty \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right) \det\left(-\mathrm{ad}(B)|_{\mathfrak{g}_0}\right) DB \\ &= \det\left(\mathrm{id}_{\mathfrak{g}_0} - \exp\left(\mathrm{ad}(B)|_{\mathfrak{g}_0}\right)\right) DB := Y(B)DB,\end{aligned}$$
with DB denoting the "Lebesgue measure" on $C^\infty(\mathbb{R}^2, \mathfrak{t})$. Now, T_{reg} is dense in T and since $\exp : \mathfrak{t} \rightarrow T$ is a local homeomorphism, we can conclude immediately that $t'_{reg} = \exp^{-1}(T_{reg})$ is dense in the vector space, $\mathfrak{t}$. Thus, we will in the rest of the article, replace $C^\infty(\mathbb{R}^2, \mathfrak{t}'_{reg})$ with $C^\infty(\mathbb{R}^2, \mathfrak{t})$ in Equations (6) and (7).

And with $\sim$ denoting equality up to a multiplicative constant independent of B,
$$\tilde{\Delta}[B] = \left|\det\left(\frac{\partial}{\partial t} + \mathrm{ad}(B)\right)\right|,$$
where the operator $\dfrac{\partial}{\partial t} + \mathrm{ad}(B)$ in the numerator is defined on $C^\infty(\mathbb{R}^2 \times [0,1), \mathfrak{g})$. For $B \in C^\infty(\mathbb{R}^2, \mathfrak{t})$,
$$d\mu_B^\perp(\hat{A}^\perp) := \exp\left(iS_{CS}^\kappa(\hat{A}^\perp + B \otimes dt)\right) D\hat{A}^\perp, \tag{8}$$
whereby a direct calculation using Equation (5) gives
$$\begin{aligned}S_{CS}^\kappa(A) &= S_{CS}^\kappa(\hat{A}^\perp + B\otimes dt) \\ &= -\frac{1}{2}\int_0^1 dt \left\langle \hat{A}^\perp(t), \left(\frac{\partial}{\partial t} + \mathrm{ad}(B)\right) \cdot \hat{A}^\perp(t)\right\rangle_{\mathbb{R}^2,\mathfrak{g}}.\end{aligned}$$

We refer the reader to [4] for the derivation of these expressions as our main focus in this article is to make sense of Expression 6. For ease of notations, we omit κ on the RHS of Expression Equation (6), but the reader should note its dependence on κ.

2.2. Infinite Dimensional Determinant

Let us first digress a little and discuss the function Y, which is defined as
$$\begin{aligned}Y(B) &= \det\left(\sum_{n=0}^\infty \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right) \det\left(-\mathrm{ad}(B)|_{\mathfrak{g}_0}\right) DB \\ &= \det\left(\mathrm{id}_{\mathfrak{g}_0} - \exp\left(\mathrm{ad}(B)|_{\mathfrak{g}_0}\right)\right).\end{aligned}$$

Note that $\mathrm{ad}(B)$ is skew symmetric and thus the operator $\exp[\mathrm{ad}(B)]$ is unitary on $C^\infty(\mathbb{R}^2) \otimes \mathfrak{g}_0$, thus it is not a compact operator and hence $\exp[\mathrm{ad}(B)]$ is not trace class. Therefore we cannot define $\det[I - \exp[\mathrm{ad}(B)]]$ as a Fredholm determinant.

Alternatively, we can interpret $\hat{D}B$ as a product form, *i.e.*,
$$\hat{D}B = \bigotimes^{C^\infty(\mathbb{R}^2,\mathbb{R})} \mu_{\mathfrak{t}},$$

where $\mu_{\mathfrak{t}}$ is a suitable measure on $\mathfrak{t}$. More precisely, we should have $\mu_{\mathfrak{t}}(v) = \det[I|_{\mathfrak{g}_0} - \exp[\mathrm{ad}(v)]|_{\mathfrak{g}_0}]$, where $v \in \mathfrak{t}$. This suggests the following heuristic formula

$$\hat{D}B = \prod_{k=1}^{\infty} \det\left[I|_{\mathfrak{g}_0} - \exp[\mathrm{ad}(b_k)]|_{\mathfrak{g}_0}\right] DB_k,$$

whereby $\{b_k\}_{k=1}^{\infty}$ is some orthonormal basis in $L^2(\mathbb{R}^2) \otimes \mathfrak{t}$ and DB_k is Lebesgue measure on the subspace spanned by b_k. However, the term $\det\left[I|_{\mathfrak{g}_0} - \exp[\mathrm{ad}(b_k)]|_{\mathfrak{g}_0}\right]$ is still ill-defined and we need to resolve this.

Note that $\mathrm{ad}(B)$ is a skew symmetric operator, *i.e.*, $\langle \mathrm{ad}(B)X, Y\rangle = -\langle X, \mathrm{ad}(B)Y\rangle$. Let N be the dimension of $\mathfrak{g}$ and $\{E_i\}_{i=1}^{R}$ be an orthonormal basis in $\mathfrak{t}$, and $\mathrm{ad}(E_i) : \mathfrak{g}_0 \mapsto \mathfrak{g}_0$ is simultaneously diagonalizable. Suppose that $\lambda_1^i, \ldots \lambda_{N-R}^i$ are the complex eigenvalues of $\mathrm{ad}(E_i)|_{\mathfrak{g}_0}$ and let $\{b_k\}_{k=1}^{\infty}$ be an orthonormal basis in $L^2(\mathbb{R}^2)$. Then we write for $B \in L^2(\mathbb{R}^2) \otimes \mathfrak{t}$,

$$\begin{aligned} &\det[I|_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B))|_{\mathfrak{g}_0}] \\ &= \prod_{k=1}^{\infty} \prod_{l=1}^{N-R} \left[1 - \exp\left[\sum_{j=1}^{R} \lambda_l^j \langle B, b_k \otimes E_j\rangle \otimes E_j\right]\Bigg|_{\mathfrak{g}_0}\right], \end{aligned} \tag{9}$$

where $\langle B, b_k \otimes E_j\rangle = -\frac{\kappa}{2\pi}\mathrm{Tr}\int_{\mathbb{R}^2} b_k(x)[B(x)E_j]dx$, dx is Lebesgue measure. That is, we interpret the determinant as an infinite product.

Unfortunately, the infinite product given in Equation (9) converges to 0. Furthermore, if we use Definition 8, we observe that the normalizing constant in Equation (6) can be shown to be 0. See Remark 4.

As such, we will drop the term $Y(B)$ in future for reasons cited above. Another reason for dropping this term is that we really do not need this term to define the link invariants in the second half of this article.

3. Heuristic Argument

Notation 2. *Throughout the rest of this article, we adopt the following notation. For $x \in \mathbb{R}^2$, we let $\phi_\kappa(x) = \kappa^2 e^{-\kappa^2|x|^2/2}/2\pi$, which is a Gaussian measure with variance $1/\kappa^2$. And let $\varsigma = (\kappa/2\pi)^{-1/2}$.*

We let p_r denote the 2-tuple (m_1, m_2), $m_1, m_2 \geq 0$ are integers with $\sum_{j=1}^{2} m_j = r$. And we write $p_r! := m_1!m_2!$. For $z = (z_1, z_2) \in \mathbb{C}^2$, $z^{p_r} := z_1^{m_1} z_2^{m_2}$. Let $\mathcal{P}_r$ denote the set of all such 2-tuples, i.e.,

$$\mathcal{P}_r = \left\{(m_1, m_2)\Big| \sum_{j=1}^{2} m_j = r\right\}.$$

Let $\mathcal{P} = \bigcup_{r=0}^{\infty} \mathcal{P}_r$.

Consider the Schwartz space $\mathcal{S}_\kappa(\mathbb{R}^2)$, with the Gaussian function ϕ_κ, $\sqrt{\phi_\kappa}(x) = \kappa e^{-\kappa^2|x|^2/4}/(2\pi)^{1/2}$. The inner product $\langle \cdot, \cdot \rangle$ is given by $\langle f, g\rangle = \kappa \int_{\mathbb{R}^2} f \cdot g \, d\lambda/2\pi$, λ is Lebesgue measure on $\mathbb{R}^2$. Let $\overline{\mathcal{S}}_\kappa(\mathbb{R}^2)$ be the smallest Hilbert space containing $\mathcal{S}_\kappa(\mathbb{R}^2)$, using this inner product.

The Hermite polynomials $\{h_i\}_{i\geq 0}$ form an orthogonal set on $L^2(\mathbb{R}, d\mu)$ with the Gaussian measure $d\mu(x_0) \equiv e^{-x_0^2/2}dx_0/\sqrt{2\pi}$. Let $H_{p_r}(x) := h_i(x_+)h_j(x_-)$, $p_r = (i, j)$ with $i + j = r$, be a product of Hermite polynomials and $H_{p_r}^{\kappa} = H_{p_r}(\kappa \cdot)$.

We have the normalized Hermite polynomials $H_{p_r}/\sqrt{p_r!}$ with respect to the Gaussian measure $e^{-(|x_+|^2+|x_-|^2)/2}d\lambda/(2\pi)$. Then

$$\bigcup_{r=0}^{\infty}\left\{\varsigma H_{p_r}(\kappa x_+,\kappa x_-)\sqrt{\phi_\kappa}/\sqrt{p_r!} : p_r \in \mathcal{P}_r\right\}$$

is an orthonormal basis for $\overline{\mathbb{S}}_\kappa(\mathbb{R}^2)$.

Definition 3. *Define a transformation* $\eta(t) = \frac{1}{8}\left(\frac{1}{1-t} - \frac{1}{t}\right)$, $t \in (0,1)$ *and* $\eta_\kappa := \kappa\eta$. *Thus* $\eta'(t) = \frac{1}{8}\left(\frac{1}{(1-t)^2} + \frac{1}{t^2}\right)$ *and* $\eta'_\kappa = \kappa\eta'$. *Observe that* $\eta_\kappa(1/2) = 0$ *for any* κ *and that* $\eta'(1/2) = 1$.

For each Hermite polynomial h_n, we will define a function $\tilde{h}_{n,\kappa}$ on $[0,1)$. Define

$$\tilde{h}_{n,\kappa}(t) = h_n\left(\kappa\eta_\kappa(t)\right)\sqrt{\frac{\kappa}{\sqrt{2\pi}}e^{-\kappa^2\eta_\kappa(t)^2/4}\sqrt{\eta'_\kappa(t)}},\ \tilde{h}_{n,\kappa}(0) := 0.$$

Note that $\tilde{h}_{n,\kappa}$ approaches 0 as $t \to 0^+$ or $t \to 1^-$.

Now define a real subspace $V_\kappa \subset L^2([0,1))$, spanned by $\{\tilde{h}_{i,\kappa}\}_{i\geq 0}$. We make V_κ into an inner product space by defining an inner product,

$$\begin{aligned}&\left\langle f\left(\kappa\eta_\kappa\right)\sqrt{\frac{\kappa}{\sqrt{2\pi}}e^{-\kappa^2\eta_\kappa^2/4}\sqrt{\eta'_\kappa}}, g\left(\kappa\eta_\kappa\right)\sqrt{\frac{\kappa}{\sqrt{2\pi}}e^{-\kappa^2\eta_\kappa^2/4}\sqrt{\eta'_\kappa}}\right\rangle\\ &:= \int_0^1 f\left(\kappa\eta_\kappa(t)\right)g\left(\kappa\eta_\kappa(t)\right)\frac{\kappa}{\sqrt{2\pi}}e^{-\kappa^2\eta_\kappa(t)^2/2}\eta'_\kappa(t)\,dt = \int_{-\infty}^{\infty} f(\kappa t)g(\kappa t)\frac{\kappa}{\sqrt{2\pi}}e^{-\kappa^2t^2/2}dt,\end{aligned}$$

whereby f and g are polynomials. Complete the inner product space V_κ into a Hilbert space, denoted by $\overline{\mathbb{S}}_\kappa([0,1))$. Clearly, $\{\tilde{h}_{n,\kappa}/\sqrt{n!}\}_{n\geq 0}$ is an orthonormal basis.

Remark 1. *We remark that the constant function 1 is not inside* $\overline{\mathbb{S}}_\kappa([0,1))$. *Furthermore, it is not necessary to consider all the* L^2 *functions on* $[0,1)$. *To obtain the link invariants later,* $\overline{\mathbb{S}}_\kappa([0,1))$ *is good enough for our consideration.*

Definition 4. *Let* $\overline{\mathbb{S}}_\kappa(\mathbb{R}^2\times[0,1))$ *be the smallest Hilbert space containing* $\overline{\mathbb{S}}_\kappa(\mathbb{R}^2)\otimes\overline{\mathbb{S}}_\kappa([0,1))$. *We define*

$$\begin{aligned}\hat{\mathscr{A}}^{\perp} &= \left\{\hat{A}^{\perp} = \alpha_+^{\perp}\otimes dx_+ + \alpha_-^{\perp}\otimes dx_- :\ \alpha_\pm^{\perp} \in \overline{\mathbb{S}}_\kappa(\mathbb{R}^2\times[0,1))\otimes\mathfrak{g}\right\},\\ \mathscr{A}_c^{\perp} &= \left\{A_c^{\perp} = b_+\otimes dx_+ + b_-\otimes dx_- :\ b_\pm \in \overline{\mathbb{S}}_\kappa(\mathbb{R}^2)\otimes\mathfrak{g}\right\},\end{aligned}$$

and

$$\mathscr{A}^{||} := \left\{B\otimes dt :\ b \in \overline{\mathbb{S}}_\kappa(\mathbb{R}^2)\otimes\mathfrak{g}\right\}.$$

We will write $\mathscr{A}^{\perp} = \hat{\mathscr{A}}^{\perp}\oplus\mathscr{A}_c^{\perp}$ *and* $\mathscr{A} = \mathscr{A}^{\perp}\oplus\mathscr{A}^{||}$. *Observe that* $\mathscr{A}_c^{\perp} \cong \mathscr{A}^{||}\times\mathscr{A}^{||}$.

With this definitions, we are going replace Equation (6) with

$$\begin{aligned}Z(M,\kappa,q;l^i,\rho_i) = \frac{1}{Z}\int_{\mathscr{A}^{||}}\Bigg[\int_{\mathscr{A}_c^{\perp}}\Bigg[\int_{\hat{\mathscr{A}}^{\perp}}\prod_{k=1}^{n}W(l^k;q)(\hat{A}^{\perp}+A_c^{\perp}+Bdt)d\mu_B^{\perp}(\hat{A}^{\perp})\Bigg]\\ \times\exp\left(i\langle A_c^{\perp}, dB\rangle_{\mathbb{R}^2,\mathfrak{g}}\right)DA_c^{\perp}\Bigg]\tilde{\Delta}[B]Y(B)DB.\end{aligned}\tag{10}$$

Henceforth, we will try to make sense of the RHS of Equation (10).

Remark 2. *Note that we replace $\mathscr{A}_c^{\perp}$ and $\mathscr{A}^{||}$ to be $\mathfrak{g}$-valued forms instead of $\mathfrak{k}$-valued forms.*

Let $\Lambda^q(T^*\mathbb{R}^2)$ be the q exterior power of the cotangent bundle over $\mathbb{R}^2$. Let $\Gamma^q(\mathbb{R}^2)$ denote the space of C^∞ sections in $\Lambda^q(T^*\mathbb{R}^2)$. We use local coordinates $X = (x_+, x_-)$. To define an L^2 space on the space of q-forms, we have to introduce a metric g on $T\mathbb{R}^2$. We pick the standard metric $ds^2 = dx_+^2 + dx_-^2$. This metric defines an inner product on $\Lambda^q(T^*\mathbb{R}^2)$ which we denote by $\langle\cdot,\cdot\rangle_q$ and we can define a volume form $\omega = dx_+ \wedge dx_-$. (See [5] for details.) Therefore, we can define a Hodge star operator $*$ acting on k-forms, $* : \Lambda^k(T^*\mathbb{R}^2) \to \Lambda^{2-k}(T^*\mathbb{R}^2)$, such that for $u, v \in \Lambda^k(T^*\mathbb{R}^2)$,

$$u \wedge *v = \langle u, v\rangle_q \omega.$$

Note that because $\dim \mathbb{R}^2 = 2$, we have that $**v = -v$ if $v \in \Lambda^1(T^*\mathbb{R}^2)$; $**v = v$ if $v \in \Lambda^0(T^*\mathbb{R}^2)$. We define an L^2 inner product on sections of real-valued q-forms, $\Gamma^q(\mathbb{R}^2)$ by

$$\langle u, v\rangle_g := \frac{\kappa}{2\pi}\int_{\mathbb{R}^2} u \wedge *v,\ u, v \in \Gamma^q(\mathbb{R}^2).$$

By the choice of the metric, note that $\overline{\mathbb{S}}_\kappa(\mathbb{R}^2)$ is a sub Hilbert space inside $\Gamma^0(\mathbb{R}^2)$. Let

$$\Omega^1(\mathbb{R}^2) := \{u_+ \otimes dx_+ + u_- \otimes dx_-\ :\ u_\pm \in \overline{\mathbb{S}}_\kappa(\mathbb{R}^2)\}.$$

Then, $\Omega^1(\mathbb{R}^2) \cong \overline{\mathbb{S}}_\kappa(\mathbb{R}^2)^{\times^2}$ is a Hilbert space.

We will also write

$$\langle u, v\rangle_{g,\mathfrak{g}} := -\frac{\kappa}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}[u \wedge *v],\ u, v \in \overline{\mathbb{S}}_\kappa(\mathbb{R}^2) \otimes \mathfrak{g} \oplus \Omega^1(\mathbb{R}^2) \otimes \mathfrak{g}. \tag{11}$$

Now, there are 2 Hilbert spaces H_1 and H_2 that we need to consider for the Chern-Simons integral, which we will each make H_i into a direct product $H_i^{\times^2}$, for $i = 1, 2$.

The first Hilbert space H_1 is $\overline{\mathbb{S}}_\kappa(\mathbb{R}^2) \otimes \overline{\mathbb{S}}_\kappa([0,1)) \otimes \mathfrak{g}$. Take the direct product $H_1^{\times^2} \cong \hat{\mathscr{A}}^{\perp}$. This is similar to the construction used in [1].

The second Hilbert space H_2 that we need to consider is $H_2 = \mathscr{A}_c^{\perp} = \Omega^1(\mathbb{R}^2) \otimes \mathfrak{g} \cong (\overline{\mathbb{S}}_\kappa(\mathbb{R}^2) \otimes \mathfrak{g})^{\times^2}$. Now we need to take the direct product $H_2^{\times^2}$, which is isomorphic to the direct product of 4 copies of $\overline{\mathbb{S}}_\kappa(\mathbb{R}^2) \otimes \mathfrak{g}$.

3.1. Heuristic Argument

Lemma 1. *Now write $\hat{A}^{\perp} = \hat{A}_+^{\perp} \otimes dx_+ + \hat{A}_-^{\perp} \otimes dx_- \in \hat{\mathcal{A}}^{\perp}$. Then,*

$$\int_0^1 \left\langle \hat{A}^{\perp}(t), \left(\frac{\partial}{\partial t} + \mathrm{ad}(B)\right) \cdot \hat{A}^{\perp}(t) \right\rangle_{\mathbb{R}^2,\mathfrak{g}} dt = \frac{\kappa}{\pi}\int_0^1 \int_{\mathbb{R}^2} \mathrm{Tr}\left[\hat{A}_+^{\perp}(t) \cdot \left(\frac{\partial}{\partial t} + \mathrm{ad}(B)\right) \hat{A}_-^{\perp}(t)\right] dx_+ dx_-\ dt.$$

Proof. Because $m(B) := \partial_t + \mathrm{ad}(B)$ is an anti-symmetric operator, we have

$$\begin{aligned}
&\int_0^1 \left\langle \hat{A}^\perp(t)), m(B)\cdot \hat{A}^\perp(t) \right\rangle_{\mathbb{R}^2,\mathfrak{g}} \, dt \\
&= \frac{\kappa}{2\pi}\int_0^1 \int_{\mathbb{R}^2} \mathrm{Tr}\left[\hat{A}_+^\perp \cdot m(B)\hat{A}_-^\perp dx_+ \wedge dx_- + \hat{A}_-^\perp \cdot m(B)\hat{A}_+^\perp dx_- \wedge dx_+\right](t)\, dt \\
&= \frac{\kappa}{2\pi}\mathrm{Tr}\int_0^1 \int_{\mathbb{R}^2} \left[\hat{A}_+^\perp \cdot m(B)\hat{A}_-^\perp\right] dx_+ \wedge dx_- + \left[\hat{A}_+^\perp \cdot m(B)\hat{A}_-^\perp\right](t)\, dx_+ \wedge dx_- \, dt \\
&= \frac{\kappa}{\pi}\int_0^1 \int_{\mathbb{R}^2} \mathrm{Tr}\left[\hat{A}_+^\perp \cdot \left(\frac{\partial}{\partial t} + \mathrm{ad}(B)\right)\hat{A}_-^\perp\right](t)\, dx_+ \wedge dx_- \, dt.
\end{aligned}$$

□

Thus, from Equation (8), we can write

$$d\mu_B^\perp(\hat{A}^\perp) = e^{-\frac{i}{2}\int_0^1 \langle \hat{A}_+^\perp, m(B)\hat{A}_-^\perp\rangle_{g,\mathfrak{g}}(t)dt} D\hat{A}_+^\perp D\hat{A}_-^\perp \equiv e^{i\langle \hat{A}_+^\perp, m(B)\hat{A}_-^\perp\rangle} D\hat{A}_+^\perp D\hat{A}_-^\perp. \tag{12}$$

Here and what follows, $\langle\cdot,\cdot\rangle$ will always denote an inner product in a Hilbert space.

Definition 5. *(Orthonormal basis $\{E_i\}_{i=1}^N$)*
The orthonormal basis in $\mathfrak{g}$, $\{E_i\}_{i=1}^N$ will be fixed throughout this article. Let $\sum_{i=1}^N \gamma_i \otimes E_i \in H \otimes \mathfrak{g}$, $\sum_{i=1}^N \delta_i \otimes E_i \in H \otimes \mathfrak{g}$ and $\sum_{i=1}^N u_i \otimes E_i \in H \otimes \mathfrak{g}$. We let $\langle\cdot,\cdot\rangle_\flat$ denote a $\mathfrak{g}$-valued inner product, i.e.,

$$\left\langle \sum_{i=1}^N \gamma_i \otimes E_i, \sum_{j=1}^N \delta_j \otimes E_j \right\rangle_\flat = \sum_{i=1}^N \langle \gamma_i, \delta_i\rangle E_i.$$

Refer to Definition 4. We will now give a heuristic argument for Expression 10. Let δ denote the Dirac delta function and for $E, F \in \mathfrak{g}$, we write

$$\langle \delta(x)\otimes E, f\otimes F\rangle = f(x)\mathrm{Tr}[-EF], \ \ \langle \delta(x)\otimes\delta(t)\otimes E, g\otimes F\rangle = g(x,t)\mathrm{Tr}[-EF].$$

Let $x_i \in \mathbb{R}^2$, $t_i \in [0,1)$ and $c_\pm^i, d_\pm^i, d_0^i \in \mathbb{R}$, with $d^i = (d_+^i, d_-^i)$. Define

$$\alpha_{i,\pm} = \sum_{j=1}^N c_\pm^i \delta_{x_i}\otimes\delta_{t_i}\otimes E_j, \ \ \beta_{i,\pm} = \sum_{j=1}^N d_\pm^i \delta_{x_i}\otimes E_j, \ \ \beta_{i,0} = \sum_{j=1}^N \delta_{x_i}\otimes E_j.$$

We will also write $\beta_i = \beta_{i,+}\otimes dx_+ + \beta_{i,-}\otimes dx_-$. Denote

$$\begin{aligned}
W_1(\hat{A}^\perp) &= \exp\left(\sum_{i=1}^R \left[\langle \hat{A}_+^\perp, \alpha_{i,+}\rangle + \langle \hat{A}_-^\perp, \alpha_{i,-}\rangle\right]\right), \\
W_2(A_c^\perp, B) &= \exp\left(\sum_{i=1}^R \langle A_{c,+}^\perp, \beta_{i,+}\otimes dx_+\rangle + \langle A_{c,-}^\perp, \beta_{i,-}\otimes dx_-\rangle + \langle B, d_0^i\beta_{i,0}\rangle\right) \\
&:= \exp\left(\sum_{i=1}^R \langle A_c^\perp, \beta_i\rangle + \langle B, d_0^i\beta_{i,0}\rangle\right).
\end{aligned}$$

For simplicity, we want to make sense of

$$\frac{1}{Z}\int_{\mathscr{A}^{||}}\left[\int_{\mathscr{A}_c^\perp}\left[\int_{\hat{\mathscr{A}}^\perp} W_1(\hat{A}^\perp)W_2(A_c^\perp, B)d\mu_B^\perp(\hat{A}^\perp)\exp\left(i\langle A_c^\perp, dB\rangle_{\mathbb{R}^2,\mathfrak{g}}\right)\right] DA_c^\perp\right]\tilde{\Delta}[B]\, DB, \tag{13}$$

with Z is a normalizing constant. Note that $d\mu_B(\hat{A}^{\perp})$ is defined by Equation (12). As discussed in Subsection 2.2, we drop the term Y in Equation (10).

Write $m(B) = \dfrac{\partial}{\partial t} + \mathrm{ad}(B)$. Then $[m(B)\hat{A}_-^{\perp}](x,\cdot) = m(B(x))\hat{A}_-^{\perp}(x,\cdot)$, so

$$\begin{aligned}\langle \hat{A}_-^{\perp}, \delta_{x_i}\otimes\delta_{t_i}\otimes E_j\rangle =&\langle m(B(x_i))^{-1}m(B)\hat{A}_-^{\perp}, \delta_{x_i}\otimes\delta_{t_i}\otimes E_j\rangle \\ =&\langle m(B)\hat{A}_-^{\perp}, -m(B(x_i))^{-1}[\delta_{x_i}\otimes\delta_{t_i}]\otimes E_j\rangle.\end{aligned}$$

Note that we make use of the fact that $m(B)$ is a skew symmetric operator, so $\langle m(B)\cdot,\cdot\rangle = \langle\cdot, -m(B)\cdot\rangle$.

Now we make the following substitution $m(B)\hat{A}_-^{\perp} \mapsto \hat{A}_-^{\perp}$. The Jacobian factor is $\tilde{\Delta}[B]^{-1} = \det[m(B)]^{-1}$, thus Expression 13 becomes,

$$\begin{aligned}\frac{1}{\bar{Z}}\int_{\mathcal{A}^{||}}\left[\int_{\mathcal{A}_c^{\perp}}\left[\int_{\hat{\mathcal{A}}^{\perp}}\exp\left[\sum_{i=1}^{R}\left[\left\langle \hat{A}_+^{\perp}, \alpha_{i,+}\right\rangle + \left\langle \hat{A}_-^{\perp}, -m(B(x_i))^{-1}\alpha_{i,-}\right\rangle\right]\right]\right.\right.\\ \left.\left.\times\, e^{i\langle \hat{A}_+^{\perp},\hat{A}_-^{\perp}\rangle} D\hat{A}_+^{\perp} D\hat{A}_-^{\perp}\right] W_2 \exp\left(i\langle A_c^{\perp}, dB\rangle\right) DA_c^{\perp}\right] DB,\end{aligned}$$

where

$$\begin{aligned}\bar{Z} :=& \int_{\hat{\mathcal{A}}^{\perp}} e^{i\langle \hat{A}_+^{\perp},\hat{A}_-^{\perp}\rangle} D\hat{A}_+^{\perp} D\hat{A}_-^{\perp}\cdot\int_{\mathcal{A}^{||}}\int_{\mathcal{A}_c^{\perp}}\exp\left(i\langle A_c^{\perp}, dB\rangle_{\mathbb{R}^2,\mathfrak{t}}\right) DA_c^{\perp}\ DB\\ :=&\bar{Z}_1\cdot\bar{Z}_2.\end{aligned}$$

Now, $B \in \mathcal{S}(\mathbb{R}^2)$ and $A_c^{\perp} \in \Omega^1(\mathbb{R}^2)$. With this new notation, we can write

$$\begin{aligned}\langle A_c^{\perp}, dB\rangle_{\mathbb{R}^2,\mathfrak{g}} =& -\langle A_c^{\perp}, **dB\rangle_{\mathbb{R}^2,\mathfrak{g}} = \langle A_c^{\perp}, *dB\rangle_{g,\mathfrak{g}},\\ \langle \xi, B\rangle_{g,\mathfrak{g}} =& -\frac{\kappa}{2\pi}\mathrm{Tr}\int_{\mathbb{R}^2} B\wedge dd^{-1}*\xi\\ =&\frac{\kappa}{2\pi}\mathrm{Tr}\int_{\mathbb{R}^2} dB\wedge d^{-1}*\xi = -\langle d^{-1}*\xi, *dB\rangle_{g,\mathfrak{g}},\end{aligned}\tag{14}$$

using Stokes' Theorem and Equation (11). So if we make the substitution $\tilde{A}_c^{\perp} = *dB$, then

$$B(x_j) = \sum_{i=1}^{N}\langle B, \delta_{x_j}\otimes E_i\rangle E_i = \sum_{i=1}^{N}\langle \tilde{A}_c^{\perp}, -d^{-1}*\delta_{x_j}\otimes E_i\rangle E_i = \langle \tilde{A}_c^{\perp}, -d^{-1}*\beta_{j,0}\rangle_{\flat},$$

and

$$\begin{aligned}&\exp\left(\sum_{i=1}^{R}\left\langle B, \sum_{j=1}^{N}\delta_{x_i}\otimes E_j\right\rangle d_0^i\right)\exp\left(i\langle A_c^{\perp}, dB\rangle_{\mathbb{R}^2,\mathfrak{g}}\right)\ DA_c^{\perp}\ \ DB\\ &\qquad\sim\exp\left(-\sum_{i=1}^{R}\langle \tilde{A}_c^{\perp}, d_0^i d^{-1}*\beta_{i,0}\rangle\right)\exp\left(i\langle A_c^{\perp}, \tilde{A}_c^{\perp}\rangle_{g,\mathfrak{g}}\right)\ DA_c^{\perp} D\tilde{A}_c^{\perp},\end{aligned}$$

up to some constant.

Thus, the path integral, up to a constant, can written in the form

$$\frac{1}{\bar{Z}}\int_{\mathcal{A}_c^{\perp}\times\tilde{\mathcal{A}}_c^{\perp}} \exp\left(\sum_{i=1}^{R}\langle A_c^{\perp},\beta_i\rangle\right)\exp\left(-\sum_{i=1}^{R}\left\langle \tilde{A}_c^{\perp}, d_0^i\, d^{-1}*\beta_{i,0}\right\rangle\right)$$
$$\left\{\int_{\hat{\mathcal{A}}^{\perp}}\exp\left(\sum_{i=1}^{R}\left[\left\langle \hat{A}_+^{\perp},\alpha_{i,+}\right\rangle+\left\langle \hat{A}_-^{\perp}, -m\left(\left\langle \tilde{A}_c^{\perp}, -d^{-1}*\beta_{i,0}\right\rangle_{\flat}\right)^{-1}\alpha_{i,-}\right\rangle\right]\right)\right.$$
$$\left.\times\exp\left[i\langle \hat{A}_+^{\perp},\hat{A}_-^{\perp}\rangle\right]\ D\hat{A}_+^{\perp}D\hat{A}_-^{\perp}\right\}\ \exp\left(i\langle A_c^{\perp},\tilde{A}_c^{\perp}\rangle\right)\ DA_c^{\perp}D\tilde{A}_c^{\perp}. \tag{15}$$

We wish to point out that this integral in Expression 15 is of the form

$$\int_{H_2^{\times 2}} e^{\langle v_+,\beta\rangle}e^{\langle v_-,\hat{\beta}_0\rangle}\int_{H_1^{\times 2}} e^{\langle u_+,\alpha_+\rangle+\langle u_-,\sum_i T(\langle v_-,\hat{\beta}_{i,0}\rangle_{\flat})\alpha_{i,-}\rangle}e^{i\langle u_+,u_-\rangle}\ Du_+Du_-e^{i\langle v_+,v_-\rangle}\ Dv_+Dv_-, \tag{16}$$

where $T(\langle v_-,\hat{\beta}_{i,0}\rangle_{\flat})$ is a linear operator that maps $H_1\rightarrow H_1$,

$$\alpha_{\pm}=\sum_i\alpha_{i,\pm},\ \beta_{\pm}=\sum_i\beta_{i,\pm},\ \beta_0=\sum_i d_0^i\beta_{i,0},\ \hat{\beta}_0=\sum_i d_0^i\hat{\beta}_{i,0},$$

and $\hat{\beta}_{i,0}=-d^{-1}*\beta_{i,0}$, $\beta=\beta_+\otimes dx_+ + \beta_-\otimes dx_-$.

Thus, our goal is to give a sensible definition for Expression 16. From Expression 15, we also need to define

$$\left(\frac{\partial}{\partial t}+\mathrm{ad}\,(\lambda)\right)^{-1}\delta_y\otimes\delta_s\otimes E_j,\ d^{-1}*\delta_w,\ \lambda\in\mathfrak{g}.$$

Unfortunately, the Dirac delta function δ is not inside $\overline{\mathbb{S}}_{\kappa}(\mathbb{R}^2\times[0,1))$. Therefore, the term $\langle \hat{A}_+^{\perp},\delta_x\rangle$ is ill defined. Furthermore, the operators $\left(\frac{\partial}{\partial t}+\lambda\right)^{-1}$ and $d^{-1}*$ will be shown later, to be only defined on a dense subspace of $\overline{\mathbb{S}}_{\kappa}(\mathbb{R}^2\times[0,1))\otimes\mathfrak{g}$ and $\overline{\mathbb{S}}_{\kappa}(\mathbb{R}^2)$ respectively. Hence these operators do not operate on the Dirac delta function.

We would like to end this section by saying that to define the path integral, we will need the following inputs, namely $\alpha_{i,\pm}\in H_1$, $\beta_{i,+}\otimes dx_+ + \beta_{i,-}\otimes dx_-\in H_2$ and $\beta_{i,0}\in\overline{\mathbb{S}}_{\kappa}(\mathbb{R}^2)\otimes\mathfrak{g}$. And $\hat{\beta}_{i,0}$ in Expression refe.x.1 is given by $-d^{-1}*\beta_{i,0}\in\Omega^1(\mathbb{R}^2)\otimes\mathfrak{g}$, which will be defined later.

The reader should think of $\alpha\equiv(\alpha_+,\alpha_-)\in H_1^{\times 2}$, and similarly,

$$\begin{pmatrix}\beta\\ \hat{\beta}_0\end{pmatrix}\equiv\begin{pmatrix}\beta_+\otimes dx_+ + \beta_-\otimes dx_-\\ -d^{-1}*\beta_0\end{pmatrix}\in\left(\Omega^1(\mathbb{R}^2)\otimes\mathfrak{g}\right)^{\times 2}=H_2^{\times 2}.$$

The path integral is simply an integral over the product space $H_1^{\times 2}\times H_2^{\times 2}$, which we will define in the next section.

4. Functional Integral

Consider the real Hilbert space spanned by $\{z^n:\ z\in\mathbb{C}\}_{n=0}^{\infty}$, integrable with respect to the Gaussian measure, equipped with a sesquilinear complex inner product, given by

$$\langle z^r, z^{r'} \rangle = \frac{1}{\pi} \int_{\mathbb{C}} z^r \cdot \overline{z^{r'}} e^{-|z^2|} \, dx \, dp, \ z = x + \sqrt{-1}p. \tag{17}$$

Note that $\overline{z_j}$ means complex conjugate. Denote this Hilbert space over $\mathbb{R}$, by $\mathcal{H}^2(\mathbb{C})$. An orthonormal basis is given by

$$\left\{ \frac{z^n}{\sqrt{n!}} : \ n \geq 0 \right\}.$$

Let $\mathcal{H}^2(\mathbb{C}^3)$ be the smallest Hilbert space containing $\mathcal{H}^2(\mathbb{C}) \otimes \mathcal{H}^2(\mathbb{C}) \otimes \mathcal{H}^2(\mathbb{C})$.

It is well-known that there is no sensible notion of Lebesgue measure on an infinite dimensional space. Our next strategy will be to define a Gaussian type of measure on $\mathcal{H}^2(\mathbb{C}^3)$. Unfortunately, this space is too small to support a Gaussian measure.

Let $x \in \mathcal{H}^2(\mathbb{C}^3)$, $x = \sum_{i_1,i_2,i_3 \geq 0} c_{i_1,i_2,i_3} \dfrac{z_1^{i_1} z_2^{i_2} z_3^{i_3}}{\sqrt{i_1! i_2! i_3!}}$. Introduce a norm by setting

$$\parallel x \parallel = \sup_{z \in B(0,1/2)} \sum_{i_1,i_2,i_3 \geq 0} |c_{i_1,i_2,i_3}| |z_1^{i_1} z_2^{i_2} z_3^{i_3}|. \tag{18}$$

Here, $B(0,1/2)$ is the ball with radius $1/2$, center 0 in $\mathbb{C}^3$. Note this norm is weaker than the L^2 norm in $\mathcal{H}^2(\mathbb{C}^3)$.

Using this weaker norm, complete $\mathcal{H}^2(\mathbb{C}^3)$ into a Banach space B. In [1], it was shown that one can equip B with a Gauss measure $\tilde{\mu}_\kappa$, with variance $1/\kappa$. Identify $y \in B^* \subset \mathcal{H}^2(\mathbb{C}^3) \subset B$ and denote the pairing $(x, y)_\sharp = y(x)$.

The space B can be described explicitly. Let $\mathcal{H}^2(\mathbb{C}^3)_{\mathbb{C}} = \mathcal{H}^2(\mathbb{C}^3) \otimes_{\mathbb{R}} \mathbb{C}$ and $B^*_{\mathbb{C}} = B^* \otimes_{\mathbb{R}} \mathbb{C}$. In [1], it was shown that

Proposition 2.
1. *The support of $\tilde{\mu}_\kappa$ in the Banach space B is the space of holomorphic $\mathbb{C}$-valued functions on $\mathbb{C}^3$.*
2. *Let $w \in \mathbb{C}^3$ and define an evaluation map, $\chi_w : x \in B \mapsto x(w)$. Then χ_w is in $B^*_{\mathbb{C}}$.*

Remark 3. *Note that χ will play the role of the Dirac delta function discussed earlier. The advantage of this is that now $\chi \in B^* \subset \mathcal{H}^2(\mathbb{C}^3)$.*

Notation 3. *We denote the Abstract Wiener space containing H by B, with Gauss measure $\tilde{\mu}_\theta$, variance $1/\theta$. If $H^{\times 2} \subset (B \times B, \tilde{\nu}_\theta)$ is an Abstract Wiener space, then $\tilde{\nu}_\theta = \tilde{\mu}_\theta \times \tilde{\mu}_\theta$.*

Definition 6. *Recall in Section 3, we said that there are 2 Hilbert spaces that need to be considered for the path integral. Instead of considering the space of Schwartz functions, we will replace it by considering the Hilbert space $\mathcal{H}^2(\mathbb{C}^3)$ and complete it into an Abstract Wiener space, denoted by $B(\mathbb{R}^2 \times [0,1))$, with Wiener measure $\tilde{\mu}_\theta$. Consider the Hilbert space $\mathcal{H}^2(\mathbb{C}^2)$, which is the smallest Hilbert space containing $\mathcal{H}^2(\mathbb{C}) \otimes \mathcal{H}^2(\mathbb{C})$. In a similar way, we can construct an Abstract Wiener space containing it, denoted by $B(\mathbb{R}^2)$, with Wiener measure $\tilde{\nu}_\theta$. There are two Abstract Wiener spaces that we will consider in this article, necessary for the definition of the path integral;*

1. *Consider the tensor product $\mathcal{H}^2(\mathbb{C}^3) \otimes \mathfrak{g}$ and complete it into an Abstract Wiener space, denoted by $B(\mathbb{R}^2 \times [0,1)) \otimes \mathfrak{g}$. The Abstract Wiener measure will be the product measure $\tilde{\mu}_\theta \times \cdots \times \tilde{\mu}_\theta$, N copies in total, N is the dimension of $\mathfrak{g}$.*

2. *Consider the direct product* $(\mathcal{H}^2(\mathbb{C}^2)\otimes\mathfrak{g})^{\times 2}$*, and complete it into an Abstract Wiener space, denoted by* $(B(\mathbb{R}^2)\otimes\mathfrak{g})^{\times 2}$*. The Abstract Wiener measure will be the product measure* $\tilde{\nu}_\theta\times\cdots\times\tilde{\nu}_\theta$*,* $2N$ *copies in total.*

Let H be any Hilbert space and $u=(u_+,u_-),\ \alpha=(\alpha_+,\alpha_-)\in H^{\times 2}$. The following expression,

$$\frac{1}{Z_1}\int_{u\in H^{\times 2}} e^{\langle u,\alpha\rangle}e^{i\langle u_+,u_-\rangle}Du_+Du_-, \tag{19}$$

with

$$Z_1=\int_{u\in H^{\times 2}} e^{i\langle u_+,u_-\rangle}Du_+Du_-$$

is the basis whereby the Chern-Simons path integral is build upon. We define Expression 19 as

$$\lim_{\theta\to i}\frac{1}{Z_\theta}\int_{u\in H^{\times 2}} e^{\langle u,\alpha\rangle}e^{i\langle u_+,u_-\rangle}e^{i|u|^2/2}e^{-\theta|u|^2/2}Du_+Du_-,$$

with

$$Z_\theta=\int_{u\in H^{\times 2}} e^{i\langle u_+,u_-\rangle}e^{i|u|^2/2}e^{-\theta|u|^2/2}Du_+Du_-.$$

Suppose B is an Abstract Wiener space containing H. Now, one can show that there exists a complex measure ν_θ on $B^{\times 2}\equiv B\times B$, such that $|\nu_\theta|$ is a probability measure on $B^{\times 2}$ and we can define for $\alpha\in B^{\times 2,*}\subset H^{\times 2}$,

$$\int_{u\in H^{\times 2}} e^{\langle u,\alpha\rangle}e^{i\langle u_+,u_-\rangle}e^{i|u|^2/2}e^{-\theta|u|^2/2}Du_+Du_- := \int_{u\in B^{\times 2}} e^{(u,\alpha)_\sharp}d\nu_\theta(u).$$

Furthermore, it can be shown directly that

$$\int_{u\in B^{\times 2}} e^{(u,\alpha)_\sharp}d\nu_\theta(u)=\exp\left(\frac{i(|\alpha_+|^2+|\alpha_-|^2+2\langle\alpha_+,\alpha_-\rangle)/2\theta^2}{1-(2i/\theta)}\right)e^{\frac{1}{2\theta}(|\alpha_+|^2+|\alpha_-|^2)}.$$

Thus, using analytic continuation, we will define

$$\begin{aligned}\lim_{\theta\to i}&\int_{u\in H^{\times 2}} e^{\langle u,\alpha\rangle}e^{i\langle u_+,u_-\rangle}e^{i|u|^2/2}e^{-\theta|u|^2/2}Du_+Du_-\\ &:=\lim_{\theta\to i}\exp\left(\frac{i(|\alpha_+|^2+|\alpha_-|^2+2\langle\alpha_+,\alpha_-\rangle)/2\theta^2}{1-(2i/\theta)}\right)e^{\frac{1}{2\theta}(|\alpha_+|^2+|\alpha_-|^2)}\\ &=e^{i\langle\alpha_+,\alpha_-\rangle}.\end{aligned}$$

The reader may refer to [1] for details.

We can now give a definition to the heuristic Expression 19.

Definition 7. *Let* $\alpha=(\alpha_+,\alpha_-)\in B^{\times 2,*}\subset H^{\times 2}\subset B^{\times 2}$ *and* $u=(u_+,u_-)\in H^{\times 2}$*. Then we define*

$$\frac{1}{Z_1}\int_{H^{\times 2}} e^{\langle u,\alpha\rangle}e^{i\langle u_+,u_-\rangle}Du_+Du_- := \mathbb{E}_{B^{\times 2}}\left[e^{(\cdot,\alpha)_\sharp}\right]=e^{i\langle\alpha_+,\alpha_-\rangle}.$$

We remark that $\mathbb{E}_{B^{\times 2}}$ is not taking expectation, but rather it should be viewed as a linear functional acting on functions of the form $\exp[(\cdot, \alpha^{\perp})_{\sharp}]$.

Let $\beta, \alpha_j \in B^*$. One can show that for any polynomials $p_1, \ldots, p_m$, we have

$$\begin{aligned}&\lim_{\theta \to i} \int_{B^{\times 2}} \prod_{i=1}^{m} p((u, (0, \alpha_i))_{\sharp}) e^{(u, (\beta, 0))_{\sharp}} d\nu_{\theta}(u) \\ =&\lim_{\theta \to i} \int_{B^{\times 2}} \prod_{i=1}^{m} p_i(d/ds_i) e^{(u, (\beta, \sum_{i=1}^{m} s_i \alpha_i))_{\sharp}} \Big|_{s_i=0} d\nu_{\theta}(u) = \prod_{j=1}^{m} p_j(i\langle \beta, \alpha_j \rangle).\end{aligned}$$

Thus, we can extend Definition 7 to include polynomials.

However, given a general (continuous and bounded) function F on one variable, then it is not clear that

$$\int_{B^{\times 2}} F((u_-, \alpha_-)_{\sharp}) e^{(u_+, \alpha_+)_{\sharp}} d\nu_{\theta}(u)$$

admits an analytic continuation. However, from the above calculations, it is possible to extend Definition 7 to include F.

Definition 8. *Let $\beta = (\beta_+, 0)$, $\alpha^j = (0, \alpha_-^j)$ with $\beta_+, \alpha_-^j \in B^*$ for $j = 1, \ldots, m$. Let F_j be continuous functions on $\mathbb{R}$ and Y be continuous on $B^{\times 2} \otimes \mathbb{C}$, with $Y(u) = Y(u_-)$ for any $u \in B^{\times 2}$, such that $Y(0) \neq 0$. Then we define*

$$\begin{aligned}&\frac{1}{Z_1} \int_{H^{\times 2}} \prod_{j=1}^{m} F^j(\langle u, \alpha^j \rangle) e^{\langle u, \beta \rangle} e^{i\langle u_+, u_- \rangle} Y(u_-)\ Du_+ Du_- \\ :=&\frac{1}{Y(0)} \mathbb{E}_{B^{\times 2}} \left[\prod_{j=1}^{m} F^j((\cdot, \alpha^j)_{\sharp}) e^{(\cdot, \beta)_{\sharp}} Y \right] = \frac{Y(i\beta_+)}{Y(0)} \prod_{j=1}^{m} F^j(i\langle \beta_+, \alpha_-^j \rangle),\end{aligned}$$

if

$$Z_1 := \int_{H^{\times 2}} e^{i\langle u_+, u_- \rangle} Y(u_-)\ Du^+ Du^-.$$

Remark 4. *Recall that Y in Subsection 2.2 is defined as an infinite dimensional determinant. If we use Definition 8, notice that $Y(0) := 0$. Hence the normalizing constant is 0. As such, we have to remove the term Y in order to obtain non trivial results for the path integral.*

Notation 4. *Let B be an Abstract Wiener space containing the Hilbert space H, equipped with inner product $\langle \cdot, \cdot \rangle$. We will also write*

$$\left(\sum_{i=1}^{N} u_i \otimes E_i, \sum_{j=1}^{N} \delta_j \otimes E_j \right)_{\flat} = \sum_{i=1}^{N} (u_i, \delta_i)_{\sharp} E_i,$$

for $u_i \in B, \delta_i \in B^*$ and $\{E_i\}_{i=1}^{N}$ is an orthonormal basis in $\mathfrak{g}$.

Let H_1, H_2 be 2 Hilbert spaces and B_1, B_2 be Abstract Wiener spaces containing them respectively. For any $\lambda \in \mathfrak{g}$, let $T(\lambda)$ be a linear operator that maps $B_1^* \otimes \mathfrak{g}$ to $B_1^* \otimes \mathfrak{g}$. Recall the path integral we want to make sense of is given by Expression 16.

Proposition 3. *Refer to Notation 4. Let* $\alpha_+, \alpha_{i,-} \in (B_1 \otimes \mathfrak{g})^*$, $\beta, \hat{\beta}_{i,0} \in (B_2 \otimes \mathfrak{g})^{\times 2,*}$. *For any* $\lambda \in \mathfrak{g}$, *let* $T(\lambda) : (B_1 \otimes \mathfrak{g})^* \rightarrow (B_1 \otimes \mathfrak{g})^*$ *be a bounded linear operator. Using Definition 8, we define Expression 16 as*

$$\mathbb{E}_{(B_2 \otimes \mathfrak{g})^{\times 2}} \left\{ e^{(*, (\beta, \hat{\beta}_0))_\sharp} \mathbb{E}_{(B_1 \otimes \mathfrak{g})^{\times 2}} \left[e^{(\cdot, (\alpha_+, \sum_i T[(*, (0, \hat{\beta}_{i,0}))_\flat] \alpha_{i,-}))_\sharp} \right] \right\} = e^{i\langle \beta, \hat{\beta}_0 \rangle} e^{i\langle \alpha_+, \sum_j T(i\langle \beta, \hat{\beta}_{j,0} \rangle_\flat) \alpha_{j,-} \rangle}, \quad (20)$$

with $\hat{\beta}_0 = \sum_i \delta_0^i \hat{\beta}_{i,0}$, $\delta_0^i \in \mathbb{R}$.

Proof. By Definition 8,

$$\mathbb{E}_{(B_1 \otimes \mathfrak{g})^{\times 2}} \left[e^{(\cdot, (\alpha_+, \sum_j T((*, (0, \hat{\beta}_{j,0}))_\flat) \alpha_{j,-}))_\sharp} \right] = e^{i\langle \alpha_+, \sum_j T((*, (0, \hat{\beta}_{j,0}))_\flat) \alpha_{j,-} \rangle}.$$

Using the definition again, we have

$$\mathbb{E}_{(B_2 \otimes \mathfrak{g})^{\times 2}} \left[e^{(*, (\beta, \hat{\beta}_0))_\sharp} e^{i\langle \alpha_+, \sum_j T((*, (0, \hat{\beta}_{j,0}))_\flat) \alpha_{j,-} \rangle} \right] = e^{i\langle \beta, \hat{\beta}_0 \rangle} e^{i\langle \alpha_+, \sum_j T(i\langle \beta, \hat{\beta}_{j,0} \rangle_\flat) \alpha_{j,-} \rangle}.$$

□

The 2 Abstract Wiener spaces, B_1 and B_2 we have in mind, are defined as follows:

$$H_1 = \mathcal{H}^2(\mathbb{C}^3) \subset B_1 = B(\mathbb{R}^2 \times [0, 1)),$$
$$H_2 = \mathcal{H}^2(\mathbb{C}^2)^{\times 2} \subset B_2 = B(\mathbb{R}^2)^{\times 2}.$$

5. Linear Operators

Refer back to Expression 15. If we wish to apply Proposition 3, then we have to define linear operators $m(\lambda)^{-1}$ and $d^{-1}*$, $\lambda \in \mathfrak{g}$. But $m(\lambda)^{-1}$ does not map $\overline{\mathcal{S}}_\kappa(\mathbb{R}^2 \times [0, 1)) \otimes \mathfrak{g}$ into $\overline{\mathcal{S}}_\kappa(\mathbb{R}^2 \times [0, 1)) \otimes \mathfrak{g}$. And for any $\gamma \in \mathcal{S}_\kappa(\mathbb{R}^2) \subset \Gamma^0(\mathbb{R}^2)$, $d^{-1} * \gamma \notin \mathcal{S}_\kappa(\mathbb{R}^2)$. Thus it seems that we are not able to apply Proposition 3.

However, as long as we can make sense of the RHS of Equation (20), we can define the Chern-Simons path integral. If one goes back to Expression 15 and compare with Expression 16, what we really need to define are the terms

$$\langle \alpha_+, \sum_j m(\lambda_j)^{-1} \alpha_{j,-} \rangle, \ \lambda_j = \langle \beta, d^{-1} * \beta_{j,0} \rangle_\flat \text{ and } \langle \beta, d^{-1} * \beta_0 \rangle.$$

Once we can define these terms, we can proceed to define our Chern-Simons path integral.

Now, we define the path integral as a linear functional, on the direct product of 2 Abstract Wiener spaces, $(B(\mathbb{R}^2 \times [0, 1)) \otimes \mathfrak{g})^{\times 2} \times (B(\mathbb{R}^2) \otimes \mathfrak{g})^{\times 4}$. The operators $\left(\frac{\partial}{\partial t} + \lambda \right)^{-1}$ and $d^{-1}*$ act on a dense subspace in $\overline{\mathcal{S}}_\kappa(\mathbb{R}^2 \times [0, 1)) \otimes \mathfrak{g}$ and $\mathcal{S}_\kappa(\mathbb{R}^2)$ respectively. We need to transfer these operators to act on $\mathcal{H}^2(\mathbb{C}^3) \otimes \mathfrak{g}$ and $\mathcal{H}^2(\mathbb{C}^2)$. To do this, we need to construct an isometry between these Hilbert spaces.

Fortunately, there is a natural map, the Segal Bargmann transform Ψ_κ, that sends

$$\Psi_\kappa : \frac{1}{\sqrt{n!}} h_n(\kappa s) \otimes \frac{\sqrt{\kappa}}{(2\pi)^{1/4}} e^{-\kappa^2 |s|^2/4} \longmapsto \frac{1}{\sqrt{n!}} z^n.$$

In the sequel, we will extend this definition Ψ_κ to tensor products or direct products of hermite polynomials.

For example, on the tensor product space $\overline{\mathcal{S}}_\kappa(\mathbb{R}^2) \otimes \overline{\mathcal{S}}_\kappa([0,1))$, we have $\Psi_\kappa : \overline{\mathcal{S}}_\kappa(\mathbb{R}^2) \otimes \overline{\mathcal{S}}_\kappa([0,1)) \rightarrow \mathcal{H}^2(\mathbb{C}^3)$, by

$$\Psi_\kappa : \left(\frac{\kappa}{2\pi}\right)^{-1/2} H_{p_r}(\kappa \cdot)\frac{\sqrt{\phi_\kappa}}{\sqrt{p_r!}} \otimes \frac{1}{\sqrt{n!}}\tilde{h}_{n,\kappa} \longmapsto \frac{z^{p_r}}{\sqrt{p_r!}} \otimes \frac{w^n}{\sqrt{n!}},\ (z,w) \in \mathbb{C}^3.$$

Similarly,

$$\Psi_\kappa : \left(\frac{\kappa}{2\pi}\right)^{-1/2} H_{p_r}(\kappa \cdot)\frac{\sqrt{\phi_\kappa}}{\sqrt{p_r!}} \longmapsto \frac{z^{p_r}}{\sqrt{p_r!}},\ z \in \mathbb{C}^2.$$

Recall $\Omega^1(\mathbb{R}^2) \cong \overline{\mathcal{S}}_\kappa(\mathbb{R}^2)^{\times 2}$, so we have $\Psi_\kappa^{-1}(\beta_+, \beta_-) = \Psi_\kappa^{-1}(\beta_+) \otimes dx_+ + \Psi_\kappa^{-1}(\beta_-) \otimes dx_- \in \Omega^1(\mathbb{R}^2)$ for $\beta_\pm \in \mathcal{H}^2(\mathbb{C}^2)$.

Definition 9. *1. Let $\lambda \in \mathfrak{g}$. We define an operator $m(\lambda)^{-1} = \partial_\lambda^{-1} \equiv \left(\frac{\partial}{\partial t} + \mathrm{ad}(\lambda)\right)$ acting on $\overline{\mathcal{S}}_\kappa(\mathbb{R}^2 \times [0,1)) \otimes \mathfrak{g}$ by*

$$(\partial_\lambda^{-1} h)(x,u) := \frac{1}{2}\left[\int_0^u - \int_u^1\right] e^{(s-u)\mathrm{ad}(\lambda)} h(x,s)\, ds,\ u \in [0,1),\ x \in \mathbb{R}^2.$$

We leave to the reader to check that

$$\left(\frac{\partial}{\partial u} + \mathrm{ad}(\lambda)\right)\partial_\lambda^{-1} h = h.$$

2. *For $\alpha \in B(\mathbb{R}^2 \times [0,1)) \otimes \mathfrak{g}$, we define an operator $\widetilde{m(\lambda)^{-1}}$ by*

$$\widetilde{m(\lambda)^{-1}}\alpha = m(\lambda)^{-1}\Psi_\kappa^{-1}\alpha.$$

3. *Recall we have the exterior derivative d and the Hodge star operator acting on $\Gamma^0(\mathbb{R}^2) \oplus \Gamma^1(\mathbb{R}^2) \oplus \Gamma^2(\mathbb{R}^2)$. We define a linear operator $d^{-1}* : \mathcal{S}_\kappa(\mathbb{R}^2) \rightarrow \Gamma^1(\mathbb{R}^2)$ by*

$$\begin{aligned}
&(d^{-1}*\beta_0)(x_+, x_-)\\
&= \frac{1}{2}\left[\left(\int_{-\infty}^{x_+} - \int_{x_+}^{\infty}\right)\beta_0(\tau, x_-)d\tau\right] \otimes dx_- - \frac{1}{2}\left[\left(\int_{-\infty}^{x_-} - \int_{x_-}^{\infty}\right)\beta_0(x_+, \tau)d\tau\right] \otimes dx_+\\
&:= (\partial_{x_+}^{-1}\beta_0)(x_+, x_-) \otimes dx_- - (\partial_{x_-}^{-1}\beta_0)(x_+, x_-) \otimes dx_+.
\end{aligned}$$

4. *For $\beta_0 \in B(\mathbb{R}^2)^*$, we define an operator $\widetilde{d^{-1}*}$ by*

$$\widetilde{(d^{-1}*)}\beta_0 = d^{-1} * \Psi_\kappa^{-1}\beta_0.$$

Remark 5.
When $\lambda = 0$, then $\partial_0^{-1} = \left(\frac{\partial}{\partial t}\right)^{-1} \equiv \partial_t^{-1}$. The operator ∂_0^{-1} and the operator ∂_2^{-1} which appeared in [1] differs by a factor 2, i.e., $\partial_2^{-1} = 2\partial_t^{-1}$.

By their definitions, it is clear that $\widetilde{m(\lambda)^{-1}}\alpha_{j,-} \notin \mathcal{S}_\kappa(\mathbb{R}^2 \times [0,1)) \otimes \mathfrak{g}$ and $\widetilde{(d^{-1}*)}\beta_{j,0} \notin \Omega^1(\mathbb{R}^2) \otimes \mathfrak{g}$. However, to define the path integral in Proposition 3, what we really need to define is

$$\langle \alpha_+, \widetilde{m(\lambda)^{-1}}\alpha_{j,-}\rangle \ \text{ and } \ \langle \beta, \widetilde{(d^{-1}*)}\beta_{j,0}\rangle,$$

where $\beta = (\beta_+, \beta_-)$.

Definition 10. *We define for* $\alpha_\pm \in \mathcal{H}^2(\mathbb{C}^3) \otimes \mathfrak{g}$ *and* $\beta_\pm, \beta_0 \in \mathcal{H}^2(\mathbb{C}^2) \otimes \mathfrak{g}$,

$$\langle \alpha_+, \widetilde{m(\lambda)^{-1}}\alpha_- \rangle := -\frac{\kappa}{2\pi}\int_0^1 dt \int_{\mathbb{R}^2} \mathrm{Tr}\left[(\Psi_\kappa^{-1}\alpha_+) \cdot \partial_\lambda^{-1}(\Psi_\kappa^{-1}\alpha_-)\right] dx_+ dx_-$$
$$\langle (\beta_+, \beta_-), \widetilde{(d^{-1}*)}\beta_0 \rangle := -\frac{\kappa}{2\pi}\mathrm{Tr}\int_{\mathbb{R}^2}\left[\Psi_\kappa^{-1}(\beta_+) \otimes dx_+ + \Psi_\kappa^{-1}(\beta_-) \otimes dx_-\right] \wedge *d^{-1} * \Psi_\kappa^{-1}(\beta_0).$$

Remark 6. *It is possible that the integrals might not be defined. However, as we will show later, for our choice of* α *and* β*, the integrals are well-defined.*

Recall we define the evaluation map χ_w in Proposition 2. The linear functionals $\alpha_\pm = \sum_s \alpha_{s,\pm}$, $\beta_\pm = \sum_s \beta_{s,\pm}$ and $\beta_0 = \sum_s \beta_{s,0}\Delta_{s,0}$, we have in mind are of the form

$$\alpha_{s,\pm} = \sum_j \chi_{\mathbf{a}_s} \otimes \chi_{t_s}\Delta_{s,\pm}^j \otimes E_j,$$
$$\beta_{s,\pm} = \sum_j \chi_{\mathbf{a}_s}\Delta_{s,\pm}^j \otimes E_j,\ \beta_{s,0} = \sum_j \chi_{\mathbf{a}_s} \otimes E_j,$$

where $\Delta_{s,\pm}^j, \Delta_{s,0}^j, \Delta_{s,0} \in \mathbb{R}$, $\mathbf{a}_s \in \mathbb{R}^2 \subset \mathbb{C}^2$ and $t_s \in \mathbb{R} \in \mathbb{C}$. Next, we need to know how to compute $\Psi_\kappa^{-1}(\chi_{\mathbf{a}} \otimes \chi_t)$.

Proposition 4. *For each* $(\mathbf{t}, s) \in \mathbb{R}^2 \times \mathbb{R} \subseteq \mathbb{C}^3$,

$$\Psi_\kappa : \varsigma\sqrt{\phi_\kappa}(\cdot - \mathbf{t})e^{\kappa^2(\mathbf{t}^2)/8} \otimes \frac{\sqrt{\kappa}}{(2\pi)^{1/4}}e^{-\kappa^2(\eta_\kappa - s)^2/4}e^{\kappa^2 s^2/8}\sqrt{\eta_\kappa'}$$
$$\longmapsto \sum_{n=0}^\infty\sum_{r=0}^\infty\sum_{p_r} z^{p_r}w^n\frac{\kappa^r\mathbf{t}^{p_r}}{2^r \cdot p_r!}\frac{\kappa^n s^n}{2^n \cdot n!} = \chi_{\kappa(\mathbf{t},s)/2} \in B(\mathbb{R}^2 \times [0,1))^*.$$

Proof. We will leave to the reader to check that

$$\chi_{\kappa(\mathbf{t},s)/2}(z,w) = \sum_{n=0}^\infty\sum_r\sum_{p_r}\frac{(\kappa\mathbf{t}/2)^{p_r}}{\sqrt{p_r!}}\frac{(\kappa s/2)^n}{\sqrt{n!}}\frac{z^{p_r}}{\sqrt{p_r!}}\frac{w^n}{\sqrt{n!}},\ z \in \mathbb{C}^2,\ w \in \mathbb{C}.$$

Now Ψ_κ^{-1} maps $\chi_{\kappa(\mathbf{t},s)/2}$ to

$$\varsigma\sqrt{\phi_\kappa}(\mathbf{x})\sum_{r=0}^\infty\sum_{p_r}H_{p_r}(\kappa x)\frac{(\kappa\mathbf{t}/2)^{p_r}}{p_r!} \otimes \sum_{n=0}^\infty\frac{(\kappa s/2)^n}{n!}\tilde{h}_{n,\kappa}(t)$$
$$= \varsigma\sqrt{\phi_\kappa}(\mathbf{x})e^{\kappa^2(2\mathbf{x}\cdot\mathbf{t} - |\mathbf{t}|^2)/4}e^{\kappa^2|\mathbf{t}|^2/8} \otimes e^{\kappa^2(2s\eta_\kappa(t) - s^2)/4}\frac{\sqrt{\kappa}}{(2\pi)^{1/4}}e^{-\kappa^2\eta_\kappa(t)^2/4}e^{\kappa^2 s^2/8}\sqrt{\eta_\kappa'(t)},$$

for $\mathbf{t}$, s real, which upon simplification gives

$$\varsigma\sqrt{\phi_\kappa}(\cdot - \mathbf{t})e^{\kappa^2|\mathbf{t}|^2/8} \otimes \frac{\sqrt{\kappa}}{(2\pi)^{1/4}}e^{-\kappa^2(\eta_\kappa(t) - s)^2/4}e^{\kappa^2 s^2/8}\sqrt{\eta_\kappa'(t)}.$$

Here, $\mathbf{x} \cdot \mathbf{t}$ is the usual scalar product in $\mathbb{R}^2$. $\square$

Notation 5. *Define for* $s \in [0, 1)$,

$$\tilde{q}^s_\kappa(t) = \frac{\sqrt{\kappa}}{(2\pi)^{1/4}} e^{-\kappa^2(\eta_\kappa(t)-s)^2/4} \sqrt{\eta'_\kappa(t)}. \tag{21}$$

Note that $\eta_\kappa(0) = -\infty$ *and* $\lim_{s\to 1^-} \eta_\kappa(s) = 0$, *so we define* $\tilde{q}^0_\kappa(t) = 0 = \lim_{s\to 1^-} \tilde{q}^s_\kappa(t)$. *We will also write*

$$p^{\mathbf{a}}_\kappa = \frac{\varsigma\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot-\mathbf{a}|^2/4}, \ \varsigma = (\kappa/2\pi)^{-1/2}.$$

Finally, for $\mathbf{x} \in \mathbb{R}^2$, $y \in \mathbb{R}$, *define*

$$\psi(\mathbf{x}) = e^{-|\mathbf{x}|^2/2}, \ \hat{\psi}(y) = e^{-y^2/2}.$$

Corollary 1. *Under the isometry* Ψ_κ,

$$\Psi_\kappa : p^{\mathbf{x}}_\kappa \otimes \tilde{q}^s_\kappa \longmapsto \psi(\kappa\mathbf{x}/2)\chi_{\kappa\mathbf{x}/2} \otimes \hat{\psi}(\kappa s/2)\chi_{\kappa s/2},$$

and

$$\Psi_\kappa : p^{\mathbf{x}}_\kappa \longmapsto \psi(\kappa\mathbf{x}/2)\chi_{\kappa\mathbf{x}/2},$$

whereby $\mathbf{x} \in \mathbb{R}^2$ *and* $s \in [0, 1)$.

Lemma 2. *Suppose*

$$\hat{\alpha}^\perp_\pm = \sum_i \chi_{\kappa\mathbf{a}_\pm/2}\psi(\kappa\mathbf{a}_\pm/2) \otimes \chi_{\kappa t_\pm/2}\hat{\psi}(\kappa t_\pm/2)\Delta^i_\pm \otimes E_i \in (B(\mathbb{R}^2 \times [0,1)) \otimes \mathfrak{g})^*,$$

whereby $\mathbf{a}_\pm \in \mathbb{R}^2 \subset \mathbb{C}^2$, $t_\pm \in [0, 1) \subset \mathbb{C}$ *and* $\Delta^i_\pm \in \mathbb{R}$. *For* $\lambda \in \mathfrak{g}$, *we have*

$$\left\langle \hat{\alpha}^\perp_+, \widetilde{m(\lambda)^{-1}}\hat{\alpha}^\perp_- \right\rangle = \sum_{i,j=1}^N \langle p^{\mathbf{a}_+}_\kappa, p^{\mathbf{a}_-}_\kappa\rangle \left\langle \tilde{q}^{t_+}_\kappa \otimes E_i, \partial^{-1}_\lambda \tilde{q}^{t_-}_\kappa \otimes E_j \right\rangle \Delta^i_+\Delta^j_-. \tag{22}$$

Proof. Using Definition 10 and Corollary 1,

$$\begin{aligned}
&\left\langle \hat{\alpha}^\perp_+, \widetilde{m(\lambda)^{-1}}\hat{\alpha}^\perp_- \right\rangle \\
&= -\frac{1}{2}\mathrm{Tr}\sum_{i,j}\int_0^1 \left[\left\langle \int_0^\tau ds\, p^{\mathbf{a}_+}_\kappa \otimes \tilde{q}^{t_+}_\kappa(s)\Delta^i_+ \otimes E_i, e^{(s-\tau)\mathrm{ad}(\lambda)} p^{\mathbf{a}_-}_\kappa \otimes \tilde{q}^{t_-}_\kappa((\tau))\Delta^j_- \otimes E_j \right\rangle \right. \\
&\quad \left. - \left\langle \int_\tau^1 ds\, p^{\mathbf{a}_+}_\kappa \otimes \tilde{q}^{t_+}_\kappa(s)\Delta^i_+ \otimes E_i, e^{(s-\tau)\mathrm{ad}(\lambda)} p^{\mathbf{a}_-}_\kappa \tilde{q}^{t_-}_\kappa(\tau)\Delta^j_- \otimes E_j \right\rangle \right] d\tau \\
&= -\frac{1}{2}\mathrm{Tr}\sum_{i,j=1}^N \langle p^{\mathbf{a}_+}_\kappa, p^{\mathbf{a}_-}_\kappa\rangle \int_0^1\int_0^\tau \tilde{q}^{t_+}_\kappa(s) \otimes E_i \cdot e^{(s-\tau)\mathrm{ad}(\lambda)}\tilde{q}^{t_-}_\kappa(\tau) \otimes E_j\, ds d\tau \cdot \Delta^i_+\Delta^j_- \\
&\quad + \frac{1}{2}\mathrm{Tr}\sum_{i,j=1}^N \langle p^{\mathbf{a}_+}_\kappa, p^{\mathbf{a}_-}_\kappa\rangle \int_0^1\int_\tau^1 \tilde{q}^{t_+}_\kappa(s) \otimes E_i \cdot e^{(s-\tau)\mathrm{ad}(\lambda)}\tilde{q}^{t_-}_\kappa(\tau) \otimes E_j\, ds d\tau \cdot \Delta^i_+\Delta^j_- \\
&:= \sum_{i,j=1}^N \langle p^{\mathbf{a}_+}_\kappa, p^{\mathbf{a}_-}_\kappa\rangle \left\langle \tilde{q}^{t_+}_\kappa \otimes E_i, \partial^{-1}_\lambda \tilde{q}^{t_-}_\kappa \otimes E_j \right\rangle \Delta^i_+\Delta^j_-.
\end{aligned}$$

□

Lemma 3. *Suppose for* $r = \pm, 0$,

$$\beta_r = \sum_i \chi_{\kappa \mathbf{a}_r/2}\psi(\kappa \mathbf{a}_r/2)\Delta_r^i \otimes E_i \in (B(\mathbb{R}^2)\otimes \mathfrak{g})^*,$$

whereby $\mathbf{a}_\pm \in \mathbb{R}^2 \subset \mathbb{C}^2$ *and* $\Delta_r^i \in \mathbb{R}$. *Then, we have for* $\beta = (\beta_+, \beta_-)$,

$$\left\langle \beta, \widetilde{(d^{-1}*)}\beta_0 \right\rangle = \sum_{i=1}^N \left\langle p_\kappa^{\mathbf{a}_+}\Delta_+^i \otimes dx_+ + p_\kappa^{\mathbf{a}_-}\Delta_-^i \otimes dx_-, d^{-1} * p_\kappa^{\mathbf{a}_0}\Delta_0^i \right\rangle. \tag{23}$$

Remark 7. *We also have*

$$\left\langle \beta, \widetilde{(d^{-1}*)} \sum_i \chi_{\kappa \mathbf{a}_0/2}\psi(\kappa \mathbf{a}_0/2) \otimes E_i \right\rangle_\flat = \sum_{i=1}^N \left\langle p_\kappa^{\mathbf{a}_+}\Delta_+^i \otimes dx_+ + p_\kappa^{\mathbf{a}_-}\Delta_-^i \otimes dx_-, d^{-1} * p_\kappa^{\mathbf{a}_0} \right\rangle E_i.$$

Proof. Using Definition 10 and Corollary 1,

$$\begin{aligned}
&\left\langle \beta, \widetilde{(d^{-1}*)}\beta_0 \right\rangle \\
=& \sum_{i,j} \left\langle \left(\chi_{\kappa \mathbf{a}_+/2}\psi(\kappa \mathbf{a}_+/2)\Delta_+^i, \chi_{\kappa \mathbf{a}_-/2}\psi(\kappa \mathbf{a}_-/2)\Delta_-^i\right) \otimes E_i, \widetilde{(d^{-1}*)}\chi_{\kappa \mathbf{a}_0/2}\psi(\kappa \mathbf{a}_0/2) \otimes E_j \Delta_0^j \right\rangle \\
=& \sum_i \left\langle p_\kappa^{\mathbf{a}_+}\Delta_+^i \otimes dx_+ + p_\kappa^{\mathbf{a}_-}\Delta_-^i \otimes dx_-, d^{-1} * p_\kappa^{\mathbf{a}_0}\Delta_0^i \right\rangle \\
:=& \sum_i \frac{\kappa}{2\pi} \int_{\mathbb{R}^2} \left[p_\kappa^{\mathbf{a}_+}\Delta_+^i \otimes dx_+ + p_\kappa^{\mathbf{a}_-}\Delta_-^i \otimes dx_-\right] \wedge *d^{-1} * p_\kappa^{\mathbf{a}_0}\Delta_0^i.
\end{aligned}$$

□

6. Definition of the Chern-Simons Path Integral

Consider

$$\begin{aligned}
W_1(\hat{A}^\perp) &= \exp\left[\langle \hat{A}_+^\perp, \Psi_\kappa^{-1}\alpha_+\rangle + \langle \hat{A}_-^\perp, \Psi_\kappa^{-1}\alpha_-\rangle\right], \\
W_2(A_c^\perp, B) &= \exp\left[\langle A_c^\perp, \Psi_\kappa^{-1}\beta\rangle + \langle B, \Psi_\kappa^{-1}\beta_0\rangle\right].
\end{aligned}$$

Here, we have $\alpha_\pm = \sum_s \alpha_{s,\pm}$, $\beta_\pm = \sum_s \beta_{s,\pm}$, $\beta_0 = \sum_s \beta_{s,0}\Delta_{s,0}$, where

$$\begin{aligned}
\alpha_{s,\pm} &= \sum_i \chi_{\kappa \mathbf{a}_{s,\pm}/2}\psi(\kappa \mathbf{a}_{s,\pm}/2) \otimes \chi_{\kappa t_{s,\pm}/2}\hat{\psi}(\kappa t_{s,\pm}/2)\Delta_{s,\pm}^i \otimes E_i, \\
\beta_{s,\pm} &= \sum_i \chi_{\kappa \mathbf{a}_{s,\pm}/2}\psi(\kappa \mathbf{a}_{s,\pm}/2)\Delta_{s,\pm}^i \otimes E_i,\ \beta_{s,0} = \sum_i \chi_{\kappa \mathbf{a}_{s,0}/2}\psi(\kappa \mathbf{a}_{s,0}/2) \otimes E_i,
\end{aligned}$$

for $\mathbf{a}_{s,\pm}, \mathbf{a}_{s,0} \in \mathbb{R}^2$, $t_{s,\pm} \in [0,1)$. We want to give a definition for Expression 13, with W_1 and W_2 defined above. In Subsection 3.1, we also showed that Expression 13 can be written heuristically as Expression 15. By replacing the Dirac delta function δ_x by $\chi(\kappa x/2)\psi(\kappa x/2)$ in Expression 15 and applying Proposition 3, we have the following definition.

Definition 11. *(Chern-Simons Path Integral)*
Refer to Definition 10. Write $\beta = (\beta_+, \beta_-)$. Applying Proposition 3 to Expression 15, we define Expression 13, with W_1 and W_2 defined as above, as

$$\begin{aligned}
&\mathbb{E}^{\kappa}_{CS}\left[\exp\left[(\cdot,\alpha)_\sharp + (\cdot,\beta)_\sharp + (\cdot,\beta_0)_\sharp\right]\right] \\
&:= \exp\left[-i\left(\left\langle \alpha_+, \sum_r m\left(i\left\langle \beta, -\widetilde{(d^{-1}*)}\beta_{r,0}\right\rangle_\flat\right)^{-1}\alpha_{r,-}\right\rangle + \left\langle \beta, \widetilde{(d^{-1}*)}\beta_0\right\rangle\right)\right].
\end{aligned}$$

Using Lemmas 2 and 3, the exponent can be explicitly computed as

$$\begin{aligned}
&-i\sum_{s,r}\sum_{i,l=1}^{N}\left\langle p_\kappa^{\mathbf{a}_{s,+}}, p_\kappa^{\mathbf{a}_{r,-}}\right\rangle \left\langle \tilde{q}_\kappa^{t_{s,+}} \otimes E_i, \partial_{\lambda_r}^{-1}\tilde{q}_\kappa^{t_{r,-}} \otimes E_l\right\rangle \Delta^i_{s,+}\Delta^l_{r,-} \\
&\quad - i\sum_{s,r}\sum_{i=1}^{N}\left\langle p_\kappa^{\mathbf{a}_{s,+}}\Delta^i_{s,+} \otimes dx_+ + p_\kappa^{\mathbf{a}_{s,-}}\Delta^i_{s,-} \otimes dx_-, d^{-1} * p_\kappa^{\mathbf{a}_{r,0}}\Delta_{r,0}\right\rangle,
\end{aligned}$$

with

$$\begin{aligned}
\lambda_r &= i\left\langle \beta, -\widetilde{(d^{-1}*)}\beta_{r,0}\right\rangle_\flat \\
&= -i\sum_{s}\sum_{i=1}^{N}\left\langle p_\kappa^{\mathbf{a}_{s,+}}\Delta^i_{r,+} \otimes dx_+ + p_\kappa^{\mathbf{a}_{s,-}}\Delta^i_{r,-} \otimes dx_-, d^{-1} * p_\kappa^{\mathbf{a}_{r,0}}\right\rangle E_i.
\end{aligned}$$

Definition 12. *(Time ordering operator)*
For any permutation $\sigma \in S_r$,

$$\mathcal{T}(A(s_{\sigma(1)})\cdots A(s_{\sigma(r)})) = A(s_1)\cdots A(s_r),\ s_1 > s_2 > \ldots > s_r.$$

Suppose now our matrices $A^k(s)$ are indexed by the curves k and time s. Extend the definition of the time ordering operator, first ordering in decreasing values of k, followed by the time s.

Definition 13. *($\widetilde{\mathrm{Tr}}$)*
Define a linear functional $\widetilde{\mathrm{Tr}}$ as follows. Suppose a matrix A is index by time s and representation $\rho(i)$, $i = 1, \ldots, r$. In other words, $A \equiv A(\rho(i), s)$. Let $\{A(\pi_1, s_1), \ldots, A(\pi_n, s_n)\}$ be a finite set of matrices. Let $S_i = \{j \in \{1, \ldots, n\} : \pi_j = \rho(i)\}$ and write $m_i := |S_i|$. For any $n \geq 1$, define a linear operator,

$$\begin{aligned}
&\widetilde{\mathrm{Tr}} : A(\pi_1, s_1) \otimes \cdots \otimes A(\pi_n, s_n) \\
&\quad \mapsto \mathrm{Tr}_{\rho(1)}[A(\rho(1), s_{\beta_1(1)}) \cdots \otimes A(\rho(1), s_{\beta_1(m_1)})] \cdots \mathrm{Tr}_{\rho(r)}[A(\rho(r), s_{\beta_r(1)}) \cdots A(\rho(r), s_{\beta_r(m_r)})],
\end{aligned}$$

such that for each $i = 1, \ldots, r$, $s_{\beta_i(1)} > s_{\beta_i(2)} > \ldots > s_{\beta_i(m_i)}$ and $\beta_i(j) \in S_i$ for $j = 1, \ldots, m_i$.

Let us apply Definition 11 to the Wilson Loop observable, given by Equation (2).

Notation 6. *Suppose $L = \{l^k\}_{k=1}^n$ is a link in $S^2 \times S^1$ such that the projected link on S^2 does not pass through $\mathbf{n}$. Using local coordinates $\mathcal{Y} \equiv (X, i_{S^1}^{-1})$, we map L into $\mathbb{R}^2 \times [0,1)$. Let $y^k : [0,1] \to \mathbb{R}^2 \times [0,1)$ be a parametrization for $(X, i_{S^1}^{-1}) \circ l^k \equiv \tilde{l}^k$, such that $|y^{j,\prime}| \neq 0$, hence giving an orientation to each curve. In components with respect to the local coordinates (x_+, x_-, t), we have $y^k = (y^k_+, y^k_-, y^k_0) \equiv (\mathbf{y}^k, y^k_0)$.*

We will also write $y_s^k = (y_{+,s}^k, y_{-,s}^k, y_{0,s}^k) = (y_+^k(s), y_-^k(s), y_0^k(s))$. *Without loss of generality, we also assume that* $y_0^j(s) = 0$ *for only finite number of points* s *in* $[0,1]$.

Next, we map $\mathbb{R}^2 \times [0,1)$ *inside* $\mathbb{C}^3$, *by*

$$(\mathbf{x}, s) \longmapsto \frac{\kappa}{2}(\mathbf{x}, s) \in \mathbb{R}^3 \subset \mathbb{C}^3.$$

As a result of this scaling, we represent our original link $L \subset S^2 \times S^1$ *as a set of (possibly open ended) curves.*

For each curve l^k, *let* ρ_k *be a representation for* $\mathfrak{g}$.

Remark 8. *The scaling of the curves by* $\hbar := \kappa/2$ *was carried out in [1].*

We will now define the Wilson Loop observable on the set of curves $\{\tilde{l}^k\}_{k=1}^n$, in $\mathbb{C}^3$. We will scale the integrand by $\psi/(8\pi)^{1/4}$, which was also carried out in the case of $\mathbb{R}^3$ and hence interpret the line integral in Equation (3) as (See also Subsection 10.1.)

$$\begin{aligned}
&\sum_{k=1}^n \int_{\kappa\hat{y}^k/2} \frac{\psi}{(8\pi)^{1/4}} \left[\kappa\hat{\psi}\hat{A}_+^\perp \otimes dx_+ + \kappa\hat{\psi}\hat{A}_-^\perp \otimes dx_- + A_{c,+}^\perp \otimes dx_+ + A_{c,-}^\perp \otimes dx_- + B \otimes dt\right] \\
&= \frac{\hbar}{(8\pi)^{1/4}} \sum_k \int_0^1 ds\ \psi(\hbar\mathbf{y}_s^k)\hat{\psi}(\hbar y_{0,s}^k)\left[\hat{A}_+^\perp(\hbar\mathbf{y}_s^k, \hbar y_{0,s}^k) y_{+,s}^{k,\prime} + \hat{A}_-^\perp(\hbar\mathbf{y}_s^k, \hbar y_{0,s}^k) y_{-,s}^{k,\prime}\right] \\
&\qquad + \psi(\hbar\mathbf{y}_s^k)\left[A_{c,+}^\perp(\hbar\mathbf{y}_s^k) y_{+,s}^{k,\prime} + A_{c,-}^\perp(\hbar\mathbf{y}_s^k) y_{-,s}^{k,\prime} + B(\hbar\mathbf{y}_s^k) y_{0,s}^{k,\prime}\right] \\
&= \left(\hat{A}^\perp, \frac{\kappa\hbar}{(8\pi)^{1/4}} \sum_{k=1}^n \int_0^1 ds\ \hat{\alpha}_{k,s}^\perp\right)_\flat + \left(A_c^\perp, \frac{\hbar}{(8\pi)^{1/4}} \sum_{k=1}^n \int_0^1 ds\ \beta_{k,s}^\perp\right)_\flat \\
&\qquad + \left(B, \frac{\hbar}{(8\pi)^{1/4}} \sum_{k=1}^n \int_0^1 ds\ \beta_{k,s,0} y_{0,s}^{k,\prime}\right)_\flat,
\end{aligned}$$

whereby $\hat{\alpha}_{k,s}^\perp = (\hat{\alpha}_{k,s,+}^\perp, \hat{\alpha}_{k,s,-}^\perp)$, $\beta_{k,s}^\perp = (\beta_{k,s,+}^\perp, \beta_{k,s,-}^\perp)$, and

$$\begin{aligned}
\hat{\alpha}_{k,s,\pm}^\perp &= \sum_i \chi_{\hbar\mathbf{y}_s^k}\psi(\hbar\mathbf{y}_s^k) \otimes \chi_{\hbar y_{0,s}^k}\hat{\psi}(\hbar y_{0,s}^k) y_{\pm,s}^{k,\prime} \otimes \rho_k(E_i^s), \\
\beta_{k,s,\pm}^\perp &= \sum_i \chi_{\hbar\mathbf{y}_s^k}\psi(\hbar\mathbf{y}_s^k) y_{\pm,s}^{k,\prime} \otimes \rho_k(E_i^s),\ \beta_{k,s,0} = \sum_i \chi_{\hbar\mathbf{y}_s^k}\psi(\hbar\mathbf{y}_s^k) \otimes \rho_k(E_i^s).
\end{aligned}$$

Note that k tracks the representation used and s tracks the ordering of $\rho_k(E_i^s)$.

Corollary 2. *Refer to Notation 6. Consider the 3 manifold* $M = S^2 \times S^1$. *Let* $I_n = \{1, 2, \ldots, n\} \times [0,1]$, $I_n^2 = (\{1, 2, \ldots, n\} \times [0,1])^{\times 2}$ *and denote*

$$\int_{I_n^2} \equiv \sum_{j,k=1}^n \iint_{[0,1]^2}.$$

Also denote

$$y_{\pm,s}^{j,\prime} = y_\pm^{j,\prime}(s),\ y_{0,s}^{j,\prime} = y_0^{j,\prime}(s).$$

Apply Definition 11, the Wilson Loop observable, with a charge q, is defined as

$$\begin{aligned}
&Z(M,\kappa,q;l^i,\rho_i)\\
:=&\mathbb{E}^{\kappa}_{CS}\left[\widetilde{\mathrm{Tr}}\exp\left[\frac{q}{(8\pi)^{1/4}}\sum_{k=1}^{n}\int_{\kappa\hat{y}^k/2}\psi\left[\kappa\hat{\psi}\hat{A}^{\perp}_{+}\otimes dx_+ + \kappa\hat{\psi}\hat{A}^{\perp}_{-}\otimes dx_- + A^{\perp}_{c,+}\otimes dx_+ + A^{\perp}_{c,-}\otimes dx_- + B\otimes dt\right]\right]\right]\\
=&\widetilde{\mathrm{Tr}}\Bigg[\exp\left(-\frac{iq^2\kappa^2}{4}\frac{\kappa^2}{\sqrt{8\pi}}\int_{I_n^2}\sum_{i,l=1}^{N}\left\langle p_\kappa^{\mathbf{y}_s^j}\tilde{q}_\kappa^{y^j_{0,s}}\otimes E_i, p_\kappa^{\mathbf{y}_t^k}\partial^{-1}_{\lambda_{k,t}}\tilde{q}_\kappa^{y^k_{0,t}}\otimes E_l\right\rangle y^{j,\prime}_{+,s}y^{k,\prime}_{-,t}dsdt\otimes\rho_j(E^s_i)\otimes\rho_k(E^t_l)\right)\otimes\\
&\exp\left(-\frac{iq^2\kappa^2}{4\sqrt{8\pi}}\sum_{i=1}^{N}\int_{I_n^2}\left\langle p_\kappa^{\mathbf{y}_s^j}y^{j,\prime}_{+,s}\otimes dx_+ + p_\kappa^{\mathbf{y}_s^j}y^{j,\prime}_{-,s}\otimes dx_-, d^{-1}*p_\kappa^{\mathbf{y}_t^k}y^{k,\prime}_{0,t}\right\rangle dsdt\otimes\rho_j(E^s_i)\otimes\rho_k(E^t_i)\right)\Bigg].
\end{aligned}\tag{24}$$

where

$$\lambda_{k,t} = -i\sum_{i=1}^{N}\sum_{j=1}^{n}\int_0^1\frac{\kappa}{2(8\pi)^{1/4}}\left\langle p_\kappa^{\mathbf{y}_s^j}y^{j,\prime}_{+,s}\otimes dx_+ + p_\kappa^{\mathbf{y}_s^j}y^{j,\prime}_{-,s}\otimes dx_-, d^{-1}*p_\kappa^{\mathbf{y}_t^k}\right\rangle ds\otimes E_i. \tag{25}$$

Note that $\lambda_{k,t}$ is dependent on κ, but we omit κ to ease the notation.

Proof. Observe that we can commute $\mathbb{E}^{\kappa}_{CS}$ and $\widetilde{\mathrm{Tr}}$ because the time ordering only acts on the matrices $\rho_j(E^s_i)$ and $\rho_j(F^s_i)$. Note that the time ordering operator $\mathcal{T}$ arranges the matrices according to j, followed by s. Now apply Definition 11, by replacing the finite sum in the definition by an integral, using a Riemannian sum type of argument. See [1] for such an argument. To obtain the RHS of Equation (24), we apply Lemmas 2 and 3. For more details, we refer the reader to [2]. □

Equation (24) will not give us the link invariants we desire, as the path integral depends on the parametrization used. And the path integral depends on the parameter κ as we used the parameter κ in constructing the isometry Ψ_κ.

To obtain the link invariants, the rest of this article will focus on computing the limit as κ goes to infinity, of the RHS of Equation (24). It is only by taking the limit as κ goes to infinity that we will obtain the desired link invariants, independent of the parametrization used.

7. Planar Diagrams

Definition 14. *(Framed link)*
Let $L = \{l^k\}_{k=1}^n$ be a link in $S^2\times S^1$. Define a continuous normal vector v^k along each closed curve l^k such that v^k is nowhere tangent to l^k. Let $\hat{l}^k$ be a new closed curve obtained by shifting l^k in the direction ϵv^k, $\epsilon > 0$ is some small number. Now, $l^k\cup\hat{l}^k$ forms a closed thin band or ribbon, whereby a finite number of twists can be added. We will write (L,v) to denote a framed link, $v\equiv\{v^j\}_{j=1}^n$.

The Wilson Loop observable can be computed from a link diagram in $\mathbb{R}^2$. Up to isotopy, we insist that the truncated link, $\mathcal{Y}(L)$ is embedded "nicely" inside $\mathbb{R}^2\times[0,1)$ and thus projected "nicely" onto $\mathbb{R}^2$, so that we get a nice planar diagram . The following definition makes this 'nice' embedding and projection more precise.

Definition 15. *(Planar Diagrams)*

Assume that a link $L = \{l^1, l^2, \ldots l^n\} \in S^2 \times S^1$ is made up of individual closed curves that do not intersect one another, i.e., $l^j \cap l^k = \emptyset$ for any j, k and when projected onto S^2, the curve does not pass through the north pole $\mathbf{n}$. Using local coordinates $\mathcal{Y} = (X, i_{S^1}^{-1})$, we map the link into $\mathbb{R}^2 \times [0,1)$, denoted by $\mathcal{Y}(L)$. We will refer it as a truncated link. Project the truncated link $\mathcal{Y}(L)$ onto the $\mathbb{R}^2$ plane using the projection map $P_0 : \mathbb{R}^2 \times [0,1) \to \mathbb{R}^2$.

Parametrise each curve $\mathcal{Y}(l^j)$ by $y^j = (y_+^j, y_-^j, y_0^j) : [0,1] \to \mathbb{R}^2 \times [0,1)$ such that $|y^{j,\prime}| \neq 0$, hence giving an orientation to each curve. Without loss of generality, we also assume that $y_0^j(s) = 0$ for only finite number of points s in $[0,1]$.

Note that in the following definitions, it applies if L is just a knot, i.e., $n = 1$. We define a truncated link diagram for $\{y^j\}_{j=1}^n$ on the plane $\mathbb{R}^2$ if the following conditions are met.

1. *Define a standard projection of the truncated link $\mathcal{Y}(L)$ onto $\mathbb{R}^2$ if the following conditions are satisfied:*

   a *for any $p \in \mathbb{R}^2$, $P_0^{-1}(p)$ intersects at most 2 distinct arcs in L. We say p is called a crossing if $P_a^{-1}(p)$ intersects exactly 2 distinct arcs in $\mathcal{Y}(L)_a$.*

   b *at each crossing $p = P_0(y^j(s_0)) = P_0(y^k(t_0))$, there exists an $\epsilon > 0$ such that for all $|s - s_0| < \epsilon$ and $|t - t_0| < \epsilon$, the tangent vectors $(P_0(y^j))'(s)$ and $P_0(y^k)'(t)$ are linearly independent at p. Furthermore, we also assume that $0 < y_0^j(s) < 1$ and $0 < y_0^k(t) < 1$ in a small neighborhood containing s_0 and t_0 respectively.*

   Denote the set of crossings between curves y^j and y^k by $\mathrm{DP}(y^j, y^k)$. And $\mathrm{DP}(y^j) \equiv \mathrm{DP}(y^j, y^j)$ will denote the set of crossings in y^j. We will write $\mathrm{DP}(L)$ to denote the set of crossings of the standard projection of the truncated link $\mathcal{Y}(L)$ onto $\mathbb{R}^2$.

2. *For each curve y^j, write the interval $[0,1)$ as a union of intervals $\bigcup_{i=1}^{n(y^j)} A(y^j)^i$, where in each interval $A(y^j)^i$, $s \in A(y^j)^i \mapsto (y_+^j(s), y_-^j(s))$ is a bijection. Write $C(y^j)^i := (y_+^j(A(y^j)^i), y_-^j(A(y^j)^i)) \in \mathbb{R}^2$ be the image of the interval $A(y^j)^i$ under y^j. Without loss of generality, further assume the image $C(y^j)^i$ contains at most one crossing which is an interior point in $C(y^j)^i$.*

3. *Given 2 arcs $C(y^j)^i, C(y^k)^{\hat{i}}$ which intersect at p, define $\mathrm{sgn}(J(C(y^j)^i, C(y^k)^{\hat{i}}))$ to be the sign of the determinant of the Jacobian $J(C(y^j)^i, C(y^k)^{\hat{i}}) = y_+^{j,\prime}(s) y_-^{k,\prime}(t) - y_+^{k,\prime}(s) y_-^{j,\prime}(t)$ at the crossing $p = (y_+^j(s), y_-^j(s)) = (y_+^k(t), y_-^k(t))$. Otherwise, define it to be zero if the 2 arcs do not intersect at all. We will also write $\mathrm{sgn}(p; y^j : y^k) \equiv \mathrm{sgn}(J(C(y^j)^i, C(y^k)^{\hat{i}}))$, $p = C(y^j)^i \cap C(y^k)^{\hat{i}}$ and call this the orientation of p.*

4. *Using the same notation as the previous item, for each crossing $p \in C(y^j)^i \cap C(y^k)^{\hat{i}}$, define*

$$\mathrm{sgn}(C(y^j)^i : C(y^k)^{\hat{i}}) = \begin{cases} 1, & y_0^j > y_0^k; \\ -1, & y_0^j < y_0^k. \end{cases}$$

   If the 2 arcs do not intersect, set it to be 0. We will also write $\mathrm{sgn}(p; y_0^j : y_0^k) \equiv \mathrm{sgn}(C(y^j)^i : C(y^k)^{\hat{i}})$ and call this the height of p.

5. *For each crossing $p \in \mathrm{DP}(y^j, y^k)$, the algebraic crossing number is defined by*

$$\varepsilon(p) = \mathrm{sgn}(J(C(y^j)^i, C(y^k)^{\hat{i}})) \cdot \mathrm{sgn}(C(y^j)^i : C(y^k)^{\hat{i}}) \in \{\pm 1\}.$$

This is actually well defined on an oriented truncated link diagram, independent of the parametrization used.

Remark 9. *The sets* $\mathrm{DP}(L)$ *and* $\mathrm{QP}(L)$ *only make sense for a truncated link diagram in* $\mathbb{R}^2$*. Different link diagrams on* $\mathbb{R}^2$ *will give different sets of crossings.*

We can also represent a truncated link diagram with a graph, which would be more convenient to use in computing the Wilson Loop observables in the next section. The vertices will represent crossings on a link diagram.

Definition 16. *(Edges.)*
Let $L = \{l^k\}_{k=1}^n$ *be a link in* $S^2 \times S^1$*. Let* $\{y^k\}_{k=1}^n$ *be a parametrization for* $(X, i_{S^1}^{-1}) \circ l^k \subset \mathbb{R}^2 \times [0, 1)$ *and project it down onto* $\mathbb{R}^2$*, forming a planar graph. Refer to Definition 15.*

1. *The vertex set* $V(L)$ *will be the set of crossings in* $\mathrm{DP}(L)$*. The terms vertices and crossings are used interchangeably. The set of edges* $E(L)$ *is simply the set of lines in the planar diagram of* L *joining each vertex. Each edge* $e : [\epsilon_1, \epsilon_2] \rightarrow \mathbb{R}^2$*,* $0 \leq \epsilon_1 < \epsilon_2 \leq 1$*. The end points* $e(\epsilon_1), e(\epsilon_2)$ *will be a vertex or crossing in* $\mathrm{DP}(L)$*. Each vertex has 4 edges incident onto it.*
2. *Fix a* j*. For each crossing* p *in* $\mathrm{DP}(y^j, y^k)$*,* $k \neq j$*, let* $V_j(y^k)$ *be the set of all such points* p*.*
3. *Suppose* $p \in \mathrm{DP}(l^j)$*. Let* $V_j(y^j)$ *be the set of all such* p*'s on a planar diagram of* y^j*.*
4. *Define* $V(y^j) = \bigcup_k V_j(y^k)$*, which defines the vertex set of the graph* y^j*. The set of edges* $E(y^j)$*, is a subset of* $E(\mathcal{Y}(L))$*, joining only vertices in* $V(y^j)$*. Note that* $E(\mathcal{Y}(L)) = \bigcup_k E(y^k)$*.*
5. *Suppose* e *and* $\hat{e}$ *belong to* $E(y^j)$*. We say that an edge* $e : [\epsilon_1, \epsilon_2] \rightarrow \mathbb{R}^2$ *precedes another edge* $\hat{e} : [\hat{\epsilon}_1, \hat{\epsilon}_2] \rightarrow \mathbb{R}^2$ *if* $\epsilon_2 \leq \hat{\epsilon}_1$*.*
6. *Each crossing* $p \in \mathrm{DP}(\mathcal{Y}(L))$ *is denoted by 4 edges, labeled by* $(e^+(p), e^-(p), \bar{e}^+(p), \bar{e}^-(p))$*, whereby* e^+ *and* e^- *are edges belonging to* $E(C^j)$ *with the bigger index* j *and* $e^-(\bar{e}^-)$ *is the edge that precedes* $e^+(\bar{e}^+)$ *at the vertex* p*. When all 4 edges belong to the same curve, then* $\bar{e}^+$ *and* $\bar{e}^-$ *are the edges that precede* e^+ *and* e^-*.*
7. *Now suppose we define a frame on* $\mathcal{Y}(L)$ *and project the framed oriented truncated link onto* $\mathbb{R}^2$*. The crossings in the planar diagram will define the set of vertices as in the case of an oriented link. A half twist* q *will be represented by a vertex with only 2 edges incident onto it, labeled* $(e^+(q), e^-(q))$*. Thus, a full twist, given by 2 consecutive half twists, twisted in the same direction, will be represented in the planar graph of the curve* y^j *by 2 vertices, joined together by a common edge.*

Remark 10. *For a half twist* q*, we can define an algebraic number* $\varepsilon(q)$ *associated to it. A positive half twist is given an algebraic number +1; a negative half twist is given an algebraic number –1. We refer the reader to [2], whereby there is a discussion on how to define the algebraic number of a half twist in a framed link.*

In the next section, we will show how to calculate the Wilson Loop observable using the graph of a framed truncated link diagram.

8. Wilson Loop Observables

Let L be a link in $S^2 \times S^1$ and using local coordinates, we represent each component of the link L by $y^j \equiv (\mathbf{y}^j, y_0^j) : [0,1] \rightarrow \mathbb{R}^2 \times [0,1)$.

Recall from Corollary 2, we have the double sum in the exponent,

$$\sum_{i,l=1}^{N} \left\langle p_\kappa^{\mathbf{y}_s^j} \tilde{q}_\kappa^{y_{0,s}^j} \otimes E_i, p_\kappa^{\mathbf{y}_t^k} \partial_{\lambda_{k,t}}^{-1} \tilde{q}_\kappa^{y_{0,t}^k} \otimes E_l \right\rangle = \sum_{i,l=1}^{N} \left\langle p_\kappa^{\mathbf{y}_s^j}, p_\kappa^{\mathbf{y}_t^k} \right\rangle \left\langle \tilde{q}_\kappa^{y_{0,s}^j} \otimes E_i, \partial_{\lambda_{k,t}}^{-1} \tilde{q}_\kappa^{y_{0,t}^k} \otimes E_l \right\rangle,$$

where $\lambda_{k,t}$ was defined in Equation (25).

Lemma 4. *We have $\lambda_{k,t} \rightarrow 0$ as $\kappa \rightarrow \infty$. Furthermore,*

$$\kappa^2 \sum_{i,l=1}^{N} \left\langle \tilde{q}_\kappa^{t_s} \otimes E_i, \partial_{\lambda_{k,t}}^{-1} \tilde{q}_\kappa^{t_r} \otimes E_l \right\rangle - \kappa^2 \sum_{i=1}^{N} \left\langle \tilde{q}_\kappa^{t_s} \otimes E_i, \partial_0^{-1} \tilde{q}_\kappa^{t_r} \otimes E_i \right\rangle \longrightarrow 0.$$

Proof. Using Item 2 from Lemma 5,

$$\begin{aligned}\Gamma_\kappa(t) &:= \sum_{i=1}^{N} \sum_{j=1}^{n} \int_0^1 \frac{\kappa}{2} \left\langle p_\kappa^{\mathbf{y}_s^j} y_{+,s}^{j,\prime} \otimes dx_+ + p_\kappa^{\mathbf{y}_s^j} y_{-,s}^{j,\prime} \otimes dx_-, d^{-1} * p_\kappa^{\mathbf{y}_t^k} \right\rangle ds \otimes E_i \\ &= \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\kappa}{2} \int_0^1 \left\langle p_\kappa^{\mathbf{y}_s^j}, \partial_{x_+}^{-1} p_\kappa^{\mathbf{y}_t^k} \right\rangle y_{-,s}^{j,\prime} \otimes E_i - \left\langle p_\kappa^{\mathbf{y}_s^j}, \partial_{x_-}^{-1} p_\kappa^{\mathbf{y}_t^k} \right\rangle y_{+,s}^{j,\prime} \otimes E_i \, ds \\ &\longrightarrow 0\end{aligned}$$

as $\kappa \rightarrow \infty$. From Definition 9,

$$\sum_{i,l=1}^{N} \langle \tilde{q}_\kappa^{t_s} \otimes E_i, \partial_0^{-1} \tilde{q}_\kappa^{t_r} \otimes E_l \rangle = \sum_{i=1}^{N} \langle \tilde{q}_\kappa^{t_s} \otimes E_i, \partial_0^{-1} \tilde{q}_\kappa^{t_r} \otimes E_i \rangle .$$

By Item 1 of Lemma 5, we have

$$\begin{aligned}&\kappa^2 \sum_{i,l=1}^{N} \left\langle \tilde{q}_\kappa^{t_s} \otimes E_i, \left(\partial_{\Gamma_\kappa(r)}^{-1} \tilde{q}_\kappa^{t_r}\right) \otimes E_l - \left(\partial_0^{-1} \tilde{q}_\kappa^{t_r}\right) \otimes E_l \right\rangle \\ &= \kappa^2 \sum_{i,l=1}^{N} \frac{1}{2} \left\langle \tilde{q}_\kappa^{t_s} \otimes E_i, \int_0^{\cdot} e^{(u-\cdot)\Gamma_k(r)} \tilde{q}_\kappa^{t_r}(u) \otimes E_l du - \int_{\cdot}^1 e^{(u-\cdot)\Gamma_k(r)} \tilde{q}_\kappa^{t_r}(u) \otimes E_l du - 2\left(\partial_0^{-1} \tilde{q}_\kappa^{t_r}(s)\right) \otimes E_l \right\rangle \\ &= \kappa^2 \sum_{i,l=1}^{N} \frac{1}{2} \left\langle \tilde{q}_\kappa^{t_s} \otimes E_i, \int_0^{\cdot} (e^{(u-\cdot)\Gamma_k(r)} - 1) \tilde{q}_\kappa^{t_r}(u) \otimes E_l du - \int_{\cdot}^1 (e^{(u-\cdot)\Gamma_k(r)} - 1) \tilde{q}_\kappa^{t_r}(u) \otimes E_l du \right\rangle \longrightarrow 0\end{aligned}$$

as $\kappa \rightarrow \infty$ and this completes the proof. □

As a result of Lemma 4, the limit of the RHS of Equation (24) is equivalent to compute the limit of

$$\begin{aligned}&\widetilde{\mathrm{Tr}}\Bigg[\exp\left(-\frac{iq^2\kappa^2}{4}\frac{\kappa^2}{\sqrt{8\pi}} \int_{I_n^2} \sum_{i=1}^{N} \left\langle p_\kappa^{\mathbf{y}_s^j} \tilde{q}_\kappa^{y_{0,s}^j}, p_\kappa^{\mathbf{y}_t^k} \partial_0^{-1} \tilde{q}_\kappa^{y_{0,t}^k} \right\rangle y_{+,s}^{j,\prime} y_{-,t}^{k,\prime} dsdt \otimes \rho_j(E_i^s) \otimes \rho_k(E_i^t)\right) \otimes \\ &\exp\left(-\frac{iq^2\kappa^2}{4\sqrt{8\pi}} \sum_{i=1}^{N} \int_{I_n^2} \left\langle p_\kappa^{\mathbf{y}_s^j} y_{+,s}^{j,\prime} \otimes dx_+ + p_\kappa^{\mathbf{y}_s^j} y_{-,s}^{j,\prime} \otimes dx_-, d^{-1} * p_\kappa^{\mathbf{y}_t^k} y_{0,t}^{k,\prime} \right\rangle dsdt \otimes \rho_j(E_i^s) \otimes \rho_k(E_i^t)\right)\Bigg].\end{aligned}$$

Notation 7. *For $x = (x_+, x_-, x_0), y = (y_+, y_-, y_0) \in \mathbb{R}^3$, let $\times$ denote the cross product $x \times y$ and let $[x \times y]_a$ denote the a-th component of $[x \times y]$, $a = \pm, 0$.*

Using the fact that ∂_0^{-1}, $\partial_{x_a}^{-1}$ are skew symmetric operators, it is straightforward to show that

$$\begin{aligned}\frac{\kappa^2}{4}\frac{\kappa^2}{\sqrt{8\pi}} & \int_{I_n^2} \langle p_\kappa^{\mathbf{y}_s^j}, p_\kappa^{\mathbf{y}_t^k}\rangle \cdot \langle \tilde{q}_\kappa^{y_{0,s}^j}, \partial_0^{-1} \tilde{q}_\kappa^{y_{0,t}^k}\rangle y_{+,s}^{j,\prime} y_{-,t}^{k,\prime} ds dt \\ &= \frac{\kappa^2}{4}\frac{\kappa^2}{\sqrt{8\pi}} \sum_{j\geq k} \int_{[0,1]^2} \delta_j^k \langle p_\kappa^{\mathbf{y}_s^j} \tilde{q}_\kappa^{y_{0,s}^j}, p_\kappa^{\mathbf{y}_t^k} \tilde{q}_\kappa^{y_{0,t}^k}\rangle_0 \left[y_s^j \times y_t^k\right]_0 ds dt, \end{aligned} \tag{26}$$

and

$$\begin{aligned}\frac{\kappa^2}{4\sqrt{8\pi}} & \int_{I_n^2} \left\langle p_\kappa^{\mathbf{y}_s^j} y_{+,s}^{j,\prime} \otimes dx_+ + p_\kappa^{\mathbf{y}_s^j} y_{-,s}^{j,\prime} \otimes dx_-, d^{-1} * p_\kappa^{\mathbf{y}_t^k}\ y_{0,t}^{k,\prime}\right\rangle ds dt \\ &= \sum_{a=\pm}\sum_{j\geq k} \frac{\kappa^2}{4\sqrt{8\pi}} \int_{[0,1]^2} \delta_j^k \langle p_\kappa^{\mathbf{y}_s^j}, \partial_{x_a}^{-1} p_\kappa^{\mathbf{y}_t^k}\rangle \left[y_s^{j,\prime} \times y_t^{k,\prime}\right]_a ds dt \end{aligned} \tag{27}$$

whereby $\delta_j^k = \delta_k^j := 1 - \delta_{jk}/2$, δ_{jk} is the Kronecker delta function.

Our next task is to compute the limit of Equations (26) and (27), as κ goes to infinity. We break up the computations into 2 simple lemmas.

Lemma 5. *We have*

1. $$\lim_{\kappa\to\infty} \frac{\kappa^2}{\sqrt{2\pi}} \langle \tilde{q}_\kappa^s, \partial_0^{-1} \tilde{q}_\kappa^t\rangle = \begin{cases} 1, & s > t; \\ -1, & s < t. \end{cases}$$
2. *Let $s = (s_+, s_-),\ t = (t_+, t_-) \in \mathbb{R}^2$. Then,*
$$\frac{\kappa^2}{4}\langle p_\kappa^s, \partial_{x_\pm}^{-1} p_\kappa^t\rangle = \frac{\kappa\sqrt{2\pi}}{4} e^{-\kappa^2|s_\mp - t_\mp|^2/8} \left\langle \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - s_\pm|^2/4}, \partial_{x_\pm}^{-1} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - t_\pm|^2/4}\right\rangle$$
and
$$\lim_{\kappa\to\infty} \left\langle \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - s_\pm|^2/4}, \partial_{x_\pm}^{-1} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - t_\pm|^2/4}\right\rangle = \begin{cases} 1, & s_\pm > t_\pm; \\ -1, & s_\pm < t_\pm. \end{cases}$$
3. $$\langle \tilde{q}_\kappa^s, \tilde{q}_\kappa^t\rangle = e^{-\kappa^2(s-t)^2/8}.$$
4. $$\langle p_\kappa^y, p_\kappa^z\rangle = e^{-\kappa^2|y-z|^2/8}.$$

Proof. We will prove (1) first. Make a substitution

$$\begin{aligned} y =& \eta_\kappa(r) - s \ \Rightarrow\ r = \eta_\kappa^{-1}(y + s) \equiv \zeta_\kappa(y), \\ z =& \eta_\kappa(r) - t \ \Rightarrow\ r = \eta_\kappa^{-1}(z + t) \equiv \theta_\kappa(z). \end{aligned}$$

Note that for any $y \in \mathbb{R}$,

$$\eta_\kappa^{-1}(y) \to 1/2$$

as $\kappa \to \infty$. By definition of $\tilde{q}_\kappa^s$ in Equation (21),

$$\frac{\kappa}{(2\pi)^{1/4}} \int_0^1 \frac{\sqrt{\kappa}}{(2\pi)^{1/4}} e^{-\kappa^2(\eta_\kappa(r)-s)^2/4} \sqrt{\eta_\kappa'(r)}\, dr = \frac{\kappa}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\kappa^2 y^2/4} \frac{1}{\sqrt{\eta'(\zeta_\kappa(y))}}\, dy$$
$$\longrightarrow \frac{\sqrt{2}}{\sqrt{\eta'(1/2)}} = \sqrt{2}.$$

And by definition of ∂_0^{-1} in Definition 9,

$$\begin{aligned}
&\frac{\kappa^2}{\sqrt{2\pi}} \langle \tilde{q}_\kappa^s, \partial_0^{-1} \tilde{q}_\kappa^t \rangle \\
=&\frac{\kappa}{2} \int_0^1 \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2(\eta_\kappa(r)-s)^2/4} \sqrt{\eta_\kappa'(r)} \cdot \int_0^r \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2(\eta_\kappa(\tau)-t)^2/4} \sqrt{\eta_\kappa'(\tau)}\, d\tau\, dr \\
&- \frac{\kappa}{2} \int_0^1 \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2(\eta_\kappa(r)-s)^2/4} \sqrt{\eta_\kappa'(r)} \cdot \int_r^1 \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2(\eta_\kappa(\tau)-t)^2/4} \sqrt{\eta_\kappa'(\tau)}\, d\tau\, dr \\
=&\frac{1}{2} \int_{-\infty}^{\infty} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2 y^2/4} \frac{1}{\sqrt{\eta'(\zeta_\kappa(y))}} \cdot \int_{-\infty}^{y+s-t} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2 z^2/4} \frac{1}{\sqrt{\eta'(\theta_\kappa(z))}}\, dz\, dy \\
&- \frac{1}{2} \int_{-\infty}^{\infty} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2 y^2/4} \frac{1}{\sqrt{\eta'(\zeta_\kappa(y))}} \cdot \int_{y+s-t}^{\infty} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2 z^2/4} \frac{1}{\sqrt{\eta'(\theta_\kappa(z))}}\, dz\, dy \\
\longrightarrow& \begin{cases} 1, & s > t; \\ -1, & s < t. \end{cases}
\end{aligned}$$

The last step requires the following explanation. Note that $1/\sqrt{\eta'}$ is bounded and there exists a small neighborhood $|z| < \delta$, δ small enough, such that for all $\kappa > 1$,

$$\left| \frac{1}{\sqrt{\eta'(\theta_\kappa(z))}} - \sqrt{2} \right| < \epsilon$$

for any given ϵ.

Using Notation 5, we have $p_\kappa^s = \varsigma \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - s|^2/4}$. We will only prove (2) for $+$, the other case is similar. Note that

$$\begin{aligned}
&\frac{\kappa^2}{4} \langle p_\kappa^s, \partial_{x_+}^{-1} p_\kappa^t \rangle \\
=&\frac{\kappa^2}{4} \int_{x\in\mathbb{R}^2} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|x_+ - s_+|^2/4} e^{-\kappa^2|x_- - s_-|^2/4} \cdot \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|x_- - t_-|^2/4} \left[\partial_{x_+}^{-1} e^{-\kappa^2|\cdot - t_+|^2/4}\right] dx_+ dx_- \\
=&\frac{\kappa^2}{4} \left\langle e^{-\kappa^2|\cdot - s_-|^2/4}, e^{-\kappa^2|\cdot - t_-|^2/4} \right\rangle \left\langle \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - s_+|^2/4}, \partial_{x_+}^{-1} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - t_+|^2/4} \right\rangle.
\end{aligned}$$

A direct computation will give

$$\frac{\kappa^2}{4} \left\langle e^{-\kappa^2|\cdot - s_-|^2/4}, e^{-\kappa^2|\cdot - t_-|^2/4} \right\rangle = \frac{\kappa\sqrt{2\pi}}{4} e^{-\kappa^2|s_- - t_-|^2/8}$$

and

$$\begin{aligned}
&\left\langle \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - s_+|^2/4}, \partial_{x_+}^{-1} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|\cdot - t_+|^2/4} \right\rangle \\
&= \frac{1}{2} \int_{\mathbb{R}} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|x_+ - s_+|^2/4} \left[\int_{-\infty}^{x_+} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|y_+ - t_+|^2/4}\, dy_+ - \int_{x_+}^{\infty} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2|y_+ - t_+|^2/4}\, dy_+ \right] dx_+ \\
&\longrightarrow \begin{cases} 1, & s_+ > t_+; \\ -1, & s_+ < t_+. \end{cases}
\end{aligned}$$

To prove (3) and (4), a direct computation gives

$$\begin{aligned}
\langle \tilde{q}_\kappa^s, \tilde{q}_\kappa^t \rangle &= \int_0^1 \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2(\eta_\kappa(r)-s)^2/4} e^{-\kappa^2(\eta_\kappa(r)-t)^2/4} \cdot \eta_\kappa'(r) dr \\
&= \int_{-\infty}^{\infty} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa^2(y-s)^2/4} e^{-\kappa^2(y-t)^2/4} dy \\
&= e^{-\kappa^2(s-t)^2/8},
\end{aligned}$$

and

$$\langle p_\kappa^y, p_\kappa^z \rangle = \frac{\kappa^2}{2\pi} \int_{\mathbb{R}^2} e^{-\kappa^2|x-y|^2/4} e^{-\kappa^2|x-z|^2/4} dx = e^{-\kappa^2|y-z|^2/8}.$$

□

The following lemma is is similar to Lemma 4.5 found in [1]. Note that there should not be a negative sign in Lemma 4.5.

Lemma 6. *Refer to Definition 15. For $j \neq k$,*

$$\begin{aligned}
&\lim_{\kappa\to\infty} \frac{\kappa^2}{\sqrt{8\pi}} \frac{\kappa^2}{4} \int_{A(y^j)^i} ds \int_{A(y^k)^{\hat{i}}} dt\, \langle p_\kappa^{\mathbf{y}_s^j} \tilde{q}_\kappa^{y_{0,s}^j}, p_\kappa^{\mathbf{y}_t^k} \partial_0^{-1} \tilde{q}_\kappa^{y_{0,t}^k} \rangle \left[y_s^{j,\prime} \times y_t^{k,\prime} \right]_0 dsdt \\
&= \pi \cdot \operatorname{sgn}(J(C(y^j)^i, C(y^k)^{\hat{i}})) \cdot \operatorname{sgn}(C(y^j)^i : C(y^k)^{\hat{i}}).
\end{aligned}$$

The proof is similar to the proof for Lemma 4.5 in [2], so the proof is omitted.

When $j = k$, we have a problem with the following expression, *i.e.*,

$$\lim_{\kappa\to\infty} \int_{[0,1]^2} \langle p_\kappa^{\mathbf{y}_s^j} \tilde{q}_\kappa^{y_{0,s}^j}, p_\kappa^{\mathbf{y}_t^j} \tilde{q}_\kappa^{y_{0,t}^j} \rangle_a \left[y_s^{j,\prime} \times y_t^{j,\prime} \right]_a dsdt \tag{28}$$

do not exist.

The solution as explained in [3], would be to consider a framing v^j whereby $v^j(\cdot) \in \mathbb{R}^3$ is a normal vector field along the curve y^j that is nowhere tangent to y^j. Define $z^{j,\epsilon} := y^j + \epsilon v^j$, ϵ is some small number, *i.e.*, $z^{j,\epsilon}$ is a parametrization of the shifted curve $y^{j,\epsilon}$ in the direction v^j. The limit in Expression 28 is now defined as

$$\lim_{\epsilon\to 0} \lim_{\kappa\to\infty} \frac{\kappa^2}{4} \frac{\kappa^2}{\sqrt{8\pi}} \int_{[0,1]^2} \langle p_\kappa^{\mathbf{y}_s^j} \tilde{q}_\kappa^{y_{0,s}^j}, p_\kappa^{\mathbf{z}_t^{j,\epsilon}} \tilde{q}_\kappa^{z_{0,t}^{j,\epsilon}} \rangle_a \left[y_s^{j,\prime} \times z_t^{j,\epsilon,\prime} \right]_0 dsdt. \tag{29}$$

The framing on the curve y^j will give rise to half twists. Using Lemma 6, one can show that the limit of Expression 29 can be written as a sum of the algebraic numbers of crossings and half twists, which form on the curve y^j. We refer the reader to [2] for the details.

We now focus on Expression 27. Unfortunately, the limit

$$\frac{\kappa^2}{4\sqrt{8}\pi}\int_{I_n^2}\left\langle p_\kappa^{\mathbf{y}_s^j} y_{+,s}^{j,\prime}\otimes dx_+ + p_\kappa^{\mathbf{y}_s^j} y_{-,s}^{j,\prime}\otimes dx_-, d^{-1}*p_\kappa^{\mathbf{y}_t^k} y_{0,t}^{k,\prime}\right\rangle dsdt$$

is not well-defined as κ goes to infinity. The limit, if it exists, will depend on the ambient isotopy of the link. This is similar to the self-linking problem.

Definition 17. *We define*

$$\begin{aligned}
&\lim_{\kappa\to\infty}\frac{\kappa^2}{4\sqrt{8}\pi}\int_{I_n^2}\left\langle p_\kappa^{\mathbf{y}_s^j} y_{+,s}^{j,\prime}\otimes dx_+ + p_\kappa^{\mathbf{y}_s^j} y_{-,s}^{j,\prime}\otimes dx_-, d^{-1}*p_\kappa^{\mathbf{y}_t^k} y_{0,t}^{k,\prime}\right\rangle dsdt\\
&= \lim_{\kappa\to\infty}\sum_{a=\pm}\sum_{j\geq k}\frac{\kappa^2}{4\sqrt{8}\pi}\int_{[0,1]^2}\delta_j^k\langle p_\kappa^{\mathbf{y}_s^j},\partial_{x_a}^{-1}p_\kappa^{\mathbf{y}_t^k}\rangle\left[y_s^{j,\prime}\times y_t^{k,\prime}\right]_a dsdt\\
&:= \lim_{\kappa\to\infty}\sum_{a=\pm}\sum_{j\geq k}\frac{\kappa^2}{4\sqrt{8}\pi}\int_{[0,1]^2}\delta_j^k\langle p_\kappa^{\mathbf{y}_s^j},\partial_{x_a}^{-1}p_\kappa^{\mathbf{y}_t^k}\rangle\left[y_s^{j,\prime}\times y_t^{k,\prime}\right]_a\cdot\langle\tilde{q}_\kappa^{y_{0,s}^j},\tilde{q}_\kappa^{y_{0,t}^k}\rangle\, dsdt.
\end{aligned}$$

Using Lemma 5, it is straightforward to show that this limit is equal to 0. Thus, the Expression 27 is defined as 0.

Definition 18. *Given $L=\{l^k\}_{k=1}^n$, an oriented framed link in $S^2\times S^1$, map it into $\mathbb{R}^2\times[0,1)$ using $\mathcal{Y}$. Let $y^k:[0,1]\to\mathbb{R}^2\times[0,1)$ be any parametrization of $\mathcal{Y}(l^k)$, whose image is then projected down onto the plane $\mathbb{R}^2$ to define a graph as in Definition 16. And let n_k be the dimension of each representation ρ_k and $\hat{n}$ be the maximum of all the n_k's.*

1. *Let N, $\tilde{N}$ be any positive integers. Define $F_{ac}\in M_N(\mathbb{C})$ with $[F_{ac}]_{ij}:=\delta_{ia}\delta_{jc}$, $\tilde{F}_{ac}\in M_{\tilde{N}}(\mathbb{C})$ with $[\tilde{F}_{ac}]_{ij}:=\delta_{ia}\delta_{jc}$. For $A\in M_N(\mathbb{C})\otimes M_{\tilde{N}}(\mathbb{C})$, the components are given by $[A]_{cd}^{ab}$ with respect to the basis $\{F_{ac}\otimes\tilde{F}_{bd}\}_{a,b,c,d}$ of $M_N(\mathbb{C})\otimes M_{\tilde{N}}(\mathbb{C})$.*
2. *Denote a map $g:E(L)\to\{1,2,\ldots,\hat{n}\}$ such that for each k,*

$$g|_{E(y^k)}:E(y^k)\to\{1,2,\ldots,n_k\}.$$

Let $S(L)$ denote all such mappings.

We are now ready to state our formula for the Wilson Loop observable in Equation (24), in the limit as κ goes to infinity.

Theorem 1. *Let $L=\{l^k\}_{k=1}^n$ be an oriented link in $S^2\times S^1$ which when projected down on S^2, does not pass through* $\mathbf{n}$. *Choose a framing for L. Map it into $\mathbb{R}^2\times[0,1)$ using $\mathcal{Y}=(X,i_{S^1}^{-1})$ and project it onto $\mathbb{R}^2$.*

Suppose for each curve l^j, we assign a representation $\rho_j:\mathfrak{g}\to End(V^j)$ to it. Refer to Definitions 5, 15, 16 and 18. For $A,B\in M_m(\mathbb{C})$, $\mu(A\otimes B)=A\cdot B$, the usual matrix multiplication.

Given any gauge group G with its complex Lie algebra $\mathfrak{g}$, the Wilson Loop observable in Equation (2), as κ goes to infinity, is given by

$$\lim_{\kappa\to\infty} Z(S^2\times S^1,\kappa,q;l^i,\rho_i) \\ = \sum_{g\in S(L)} \prod_{p\in \mathrm{DP}(L)} R(p)^{g(e^+(p)),g(\bar{e}^+(p))}_{g(e^-(p)),g(\bar{e}^-(p))} \prod_{p\in \mathrm{TDP}(L)} T(p)^{g(e^+(p))}_{g(e^-(p))}. \tag{30}$$

If $p\equiv(e^+,e^-,\bar{e}^+,\bar{e}^-)$, with $\{e^+,e^-\}\subseteq E(y^j)$ and $\{\bar{e}^+,\bar{e}^-\}\subseteq E(y^k)$, then

$$R(p) := \exp\left[-\varepsilon(p)\pi i q^2\sum_{i=1}^N \rho_j(E_i)\otimes\rho_k(E_i)\right] \in End(V^j)\otimes End(V^k).$$

If $p\in\mathrm{TDP}(L)$, with $p\equiv(e^+,e^-)\subseteq E(y^j)$, then

$$T(p) := \exp\left[-\frac{\varepsilon(p)\pi i q^2}{2}\sum_{i=1}^N \mu\left(\rho_j(E_i)\otimes\rho_j(E_i)\right)\right] \in End(V^j).$$

Note that ε is the algebraic crossing number and was defined in Definition 15. See also Remark 10.

Notation 8. *Suppose for all l, $\rho_l=\rho$ for some representation. Denote*

$$R^\pm \equiv R^\pm_\rho := \exp\left[\mp\pi i q^2\sum_{i=1}^N \rho(E_i)\otimes\rho(E_i)\right],$$
$$T^\pm \equiv T^\pm_\rho := \exp\left[\frac{\mp\pi i q^2}{2}\sum_{i=1}^N \mu(\rho(E_i)\otimes\rho(E_i))\right].$$

When the representation is clear, we will drop the subscript ρ.

Proof. Because of Lemma 4 and because Expression 27 is defined as 0, it suffices to compute the limit of

$$\exp\left(-\frac{iq^2\kappa^2}{4}\frac{\kappa^2}{\sqrt{8\pi}}\int_{I_n^2}\langle p_\kappa^{\mathbf{y}_s^j}\tilde{q}_\kappa^{y^j_{0,s}}, p_\kappa^{\mathbf{y}_t^k}\partial_0^{-1}\tilde{q}_\kappa^{y^k_{0,t}}\rangle y^{j,\prime}_{+,s}y^{k,\prime}_{-,t}dsdt\otimes\rho_j(E_i^s)\otimes\rho_k(E_i^t)\right). \tag{31}$$

To compute Expression 31, we note that it suffices to consider a framed truncated link diagram which is projected on $\mathbb{R}^2$ plane. Using Lemma 6, the exponent will be given by a sum of terms, each involving a crossing or half-twists. We also note that we will have a problem when $j=k$, whereby we have to consider Expressions 29 instead.

The rest of the proof now follows similarly to the argument used in Section 2.1 in [2]. □

9. Σ-Model

Equation (30) defines a $\mathbb{C}$-valued map on a framed link diagram $\Gamma(L)$. From the definition of $R(p)$, it is clear that it is invariant under ambient isotopies.

Notation 9. *Let $\lambda=q^2\geq 0$. Fix a N. Recall that given $A\otimes B\in M_N(\mathbb{C})\otimes M_N(\mathbb{C})$, the components $[A\otimes B]^{ab}_{cd}=A^a_c\otimes B^b_d\equiv A_{ac}\otimes B_{bd}$. The upper indices a and b refer to the rows, the lower indices c and d refer to the columns.*

Definition 19. *(State model for framed truncated links.)*
Fix a natural number N. A state model of type $(N, \mathbb{C})$ is given by $\mathcal{R} \equiv (R^{\pm}, T^{\pm})$, with $R^{\pm} \in \bigotimes^2 M_N(\mathbb{C})$, $T^{\pm} \in M_N(\mathbb{C})$. For every state model $(R^{\pm}, T^{\pm})$ of type $(N, \mathbb{C})$, there is a unique $\mathbb{C}$-valued mapping $\Sigma_{\mathcal{R}}$ on the set of framed truncated link diagrams $\{\Gamma(L)\}$ in $\mathbb{R}^2 \times [0, 1)$ for every framed link L in $S^2 \times S^1$, such that

$$\begin{aligned}&\Sigma_{\mathcal{R}}(\Gamma(L))\\ &:= \sum_{g \in S(L)} \prod_{p \in \mathrm{DP}(L)} [R^{\varepsilon(p)}]^{g(e^+(p)), g(\bar{e}^+(p))}_{g(e^-(p)), g(\bar{e}^-(p))} \prod_{p \in \mathrm{TDP}(L)} [T^{\varepsilon(p)}]^{g(e^+(p))}_{g(e^-(p))}\end{aligned}$$

holds.

When $\rho_l \equiv \rho$ for some representation ρ, Theorem 1 says that the Wilson Loop observable defines a state model on the set of framed link diagrams.

Two framed truncated link diagrams in $\mathbb{R}^2$ are equivalent if they can be obtained from the other by the Reidemeister Moves. Fortunately, there are algebraic conditions on $R^{\pm}$ and $T^{\pm}$ which tells us when $\Sigma_{\mathcal{R}}$ is invariant under the 3 Reidemeister moves.

Proposition 5. *(Reidemeister Move I'.)*
$\Sigma_{\mathcal{R}}$ is invariant under Reidemeister Move I' if

$$\sum_b [T^+]^a_b [T^-]^b_c = \sum_b [T^-]^a_b [T^+]^b_c = \delta^a_c \tag{32}$$

for all a, c.

Proposition 6. *(Reidemeister Move II.)*
$\Sigma_{\mathcal{R}}$ is invariant under Reidemeister Move II if

$$\sum_{i,j} [R^+]^{ab}_{ij} [R^-]^{ij}_{cd} = \delta^a_c \delta^b_d, \tag{33}$$

for all a, b, c, d. There are 2 Reidemeister Moves II, the other obtained by reversing the orientation of one strand, keeping the orientation of the other fixed. In this case, the equation becomes $\sum_{i,j} [R^-]^{ab}_{ij} [R^+]^{ij}_{cd} = \delta^a_c \delta^b_d$.

Proposition 7. *(Reidemeister Move III.)*
$\Sigma_{\mathcal{R}}$ is invariant under Reidemeister Move III if

$$\sum_{p,q,y} [A]^{pq}_{ij} [B]^{ly}_{pk} [C]^{nm}_{qy} = \sum_{p,q,y} [A]^{ln}_{pq} [B]^{pm}_{iy} [C]^{qy}_{jk}, \ A, B, C = R^{\pm} \tag{34}$$

for all i, j, k, l, m, n.

Finally, $\Sigma_{\mathcal{R}}$ satisfies the skein relation with parameters α, β and γ,

$$\alpha \Sigma_{\mathcal{R}}(\Gamma(L_+)) + \beta \Sigma_{\mathcal{R}}(\Gamma(L_-)) = \gamma \Sigma_{\mathcal{R}}(\Gamma(L_0)) \tag{35}$$

for all L_+, L_- and L_0 as in [6] if

$$\alpha [R^+]^{ab}_{cd} + \beta [R^-]^{ab}_{cd} = \gamma \delta^a_d \delta^b_c \tag{36}$$

for all a, b, c, d. Note that this is a correction to Equation 14 in [2].

Definition 20. *(Special elements in $\bigotimes^2 M_N(\mathbb{C})$.) Define I, J, K in $\bigotimes^2 M_N(\mathbb{C})$,*

$$I^{ab}_{cd} = \delta^a_c \delta^b_d,\ J^{ab}_{cd} := \delta^a_d \delta^b_{c,}\ K^{ab}_{cd} = \delta^a_b \delta^c_d. \tag{37}$$

K commutes with J. Note that if $\mathbb{I}$ is the identity matrix in $M_N(\mathbb{C})$, then $I = \mathbb{I} \otimes \mathbb{I}$. Furthermore, $J \cdot J = I$, $K \cdot K = NK$ and $K \cdot J = K = J \cdot K$.

We will now present the corrected version of 2 examples taken from [2].

Example 1 ($SU(N)$)

Suppose our Lie group is $G = SU(N)$. Considering its standard representation, one shows that $\sum_i E_i \otimes E_i = \frac{1}{N} I - J$. Hence,

$$R^{\pm} = \exp\left(\mp \pi i q^2/N\right)\left(\cos(q^2\pi) I \pm i \sin(q^2\pi) J\right), T^{\pm} = \exp\left(\mp \pi i (1 - N^2) q^2/2N\right).$$

Let $\lambda = q^2$, thus $R^{\pm}$ satisfy the Reidemeister Equations (33) and $T^{\pm}$ satisfy Reidemeister Equation (32) for any values of λ. It will satisfy Equation (34) if λ is an integer or half integer.

If we solve Equation (36), we get

$$\beta = -\alpha \exp(-2\pi i \lambda/N),\ 2i\alpha \exp(-\pi i \lambda/N) \sin(\pi\lambda) = \gamma$$

and hence

$$\Sigma_{\mathfrak{R}}(\Gamma(L_+)) - \exp(-2\pi i \lambda/N) \Sigma_{\mathfrak{R}}(\Gamma(L_-)) = 2i \exp(-\pi i \lambda/N) \sin(\pi\lambda) \Sigma_{\mathfrak{R}}(\Gamma(L_+)).$$

Therefore, $\Sigma_{\mathfrak{R}}$ satisfy a Homfly polynomial skein relation

$$l^{-1} \Sigma_{\mathfrak{R}}(\Gamma(L_+)) - l \Sigma_{\mathfrak{R}}(\Gamma(L_-)) - m \Sigma_{\mathfrak{R}}(\Gamma(L_0)) = 0, \tag{38}$$

with parameters $l = \exp(-\pi i \lambda/N)$ and $m = 2i \sin(\pi\lambda) = (l^{-N} - l^N)$. Compare with the Jones polynomial skein relation, given by

$$l^{-1} \Sigma_{\mathfrak{R}}(\Gamma(L_+)) - l \Sigma_{\mathfrak{R}}(\Gamma(L_-)) - (l^{1/2} - l^{-1/2}) \Sigma_{\mathfrak{R}}(\Gamma(L_0)) = 0.$$

Let us summarize the result as a theorem.

Theorem 2. *Consider the standard representation of $SU(N)$ and let q be the charge of the link. Then the Wilson Loop observable in Equation (30) can be written as a state model $\Sigma_{\mathfrak{R}}$ of a framed link. If q^2 is an integer or half integer, then $\Sigma_{\mathfrak{R}}$ defines a framed link invariant. Furthermore, $\Sigma_{\mathfrak{R}}$ satisfy a skein relation Equation (38).*

Example 2 ($SO(N)$)

Now consider $G = SO(N)$. Considering its standard representation, then $\sum_i E_i \otimes E_i = (K - J)/2$. Hence,

$$R^{\pm} = \exp(\pm \pi i q^2 (J - K)/2) = \cos(\pi q^2/2) I \pm i \sin(\pi q^2/2) J + \exp(\pm \pi q^2 i/2) \frac{\exp(\mp \pi i q^2 N/2) - 1}{N} K,$$

$$T^{\pm} = \exp(\pm \pi i q^2 (N - 1)/4).$$

Write $\lambda = q^2$. Note that $R^{\pm}$ satisfy the Reidemeister Equation (33) and $T^{\pm}$ satisfy Reidemeister Equation (32) for any values of λ . Equation (34) will be satisfied in any of the following 3 cases:

$$\begin{cases} \lambda \equiv 0 \pmod 4, & N \text{ is odd;} \\ \lambda \equiv 0 \pmod 2, & N/2 \text{ is odd;} \\ \lambda \in \mathbb{N} \cup \{0\}, & N/2 \text{ is even.} \end{cases}$$

Now, solve Equation (36), we get

$$\beta = -\alpha,\ 2i\alpha \sin(\pi\lambda/2) = \gamma.$$

Then,

$$\Sigma_{\mathfrak{R}}(\Gamma(L_+)) - \Sigma_{\mathfrak{R}}(\Gamma(L_-)) = 2i \sin(\pi\lambda/2)\Sigma_{\mathfrak{R}}(\Gamma(L_0)).$$

Compare this with the skein relation for the Conway polynomial,

$$\Sigma_{\mathfrak{R}}(\Gamma(L_+)) - \Sigma_{\mathfrak{R}}(\Gamma(L_-)) = z\Sigma_{\mathfrak{R}}(\Gamma(L_0)). \tag{39}$$

The only interesting case would be when N is a multiple of 4 and q^2 is an odd integer. Then the state model $\Sigma_{\mathfrak{R}}$ would satisfy the skein relation Equation (39) for the Conway polynomial, with $z = 2i$.

10. Final Comments

We would like to end this article with a few comments.

10.1. Normalizing Constants

In the definition of the line integral in the Wilson Loop observable, we scale $\hat{A}^{\perp}_{\pm}$ by $\kappa\psi\hat{\psi}/(8\pi)^{1/4}$ and $A^{\perp}_{c,\pm}$, B with $\psi/(8\pi)^{1/4}$. The factor ψ is required to obtain non trivial results when we take the limit as κ goes to infinity and this scaling was also done in [1]. But in that article, we notice that the constant was $\sqrt{\kappa}/(32\pi)^{1/4}$.

Now, the factor κ is put there for technical reasons. The thing we want to address is the discrepancy in the constants 8π and 32π. In fact, there is no discrepancy as the operator ∂_2^{-1} used in [1] is actually twice of the operator ∂_0^{-1} defined in this article. If we had used the operator ∂_2^{-1} instead in this article, then we would use the normalizing constant $\kappa/(32\pi)^{1/4}$, instead of $\kappa/(8\pi)^{1/4}$.

Finally, we would like to point out that the normalizing constants were specially chosen so that the R-matrices obtained in Theorem 1 will be consistent with the R-matrices obtained in [2].

10.2. The Solid Torus

Consider the solid torus $T \cong \overline{D_2} \times S^1$, where D_2 is the open disc of radius 1 in $\mathbb{R}^2$. Given any link embedded inside T or on the surface of a torus, we may as well assume that it is embedded inside $D_2 \times S^1$. Now the open disc is homeomorphic to $\mathbb{R}^2$, so we can map the link into $\mathbb{R}^2 \times [0, 1)$.

However, we wish to point out that quasi axial gauge or torus gauge fixing may not apply to $D_2 \times S^1$. We will not address this issue here. Instead we will use the RHS of Equation (10) as the heuristic expression for the Wilson Loop observable, for a manifold of the form $\Sigma \times S^1$, where Σ is any simply connected Riemann surface.

Hence we can apply the results in this article, define and compute the Wilson Loop observable and obtain link invariants for a link embedded inside $\Sigma \times S^1$. In particular, we can define the link invariants for a link embedded inside the solid torus T.

10.3. The W Polynomial

The Wilson Loop observable in the case of $SU(N)$, was meant to give us the Jones Polynomial of a link. However, the Wilson Loop observable gives us a number. So how does one even obtain a polynomial invariant out of it?

Firstly, we have shown that the Wilson Loop observable is an invariant for a framed link. Secondly, the Wilson Loop observable yields N^n for the unlink with n number of components.

Let us go back to our $SU(N)$ example. Now, when q^2 is an integer,

$$R^{\pm} = e^{\mp \pi i q^2/N} \cos(q^2 \pi) I,$$

and thus the state model just yield us

$$\Sigma_{\mathfrak{R}}(\Gamma(L)) = [(-1)^{q^2} l]^{(1-N^2)t(L)/2} l^{c(L)} N,\ l = (-1)^{q^2} e^{-\pi i q^2/N}.$$

Here, $c(L)$ is the sum of the algebraic numbers of all the crossings, and $t(L)$ is the sum of the algebraic numbers of all the half twists in L.

The more interesting case is when q^2 is a half integer. In this case,

$$R^{\pm} = \pm i e^{\mp \pi i q^2/N} \sin(q^2 \pi) J,$$

and the state model will yield a polynomial $W_{N,L}(l, l^{-1})$, whereby

$$\Sigma_{\mathfrak{R}}(\Gamma(L)) = W_{N,L}(l, l^{-1}),\ l = (-1)^{q^2 - 1/2} i e^{-\pi i q^2/N}.$$

Thus, the Wilson Loop observable defines a polynomial $W_{N,L}(l, l^{-1})$ for a framed link L.

Conflicts of Interest

The authors declare no conflict of interest.

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