mathematics

Article

Existence of Semi Linear Impulsive Neutral Evolution Inclusions with Infinite Delay in Frechet Spaces

Dimplekumar N. Chalishajar [1,*], Kulandhivel Karthikeyan [2,†] and Annamalai Anguraj [3,†]

1 Department of Applied Mathematics, Virginia Military Institute (VMI), 431 Mallory Hall, Lexington, VA 24450, USA
2 Department of Mathematics, KSR College of Technology, Tiruchengode 637215, India; karthi_phd2010@yahoo.co.in
3 Department of Mathematics, PSG College of Arts and Science, Coimbatore 641014 , India; angurajpsg@yahoo.com
* Correspondence: dipu17370@gmail.com; Tel.: +1-540-817-8105
† These authors contributed equally to this work.

Academic Editor: Hari M. Srivastava
Received: 25 January 2016; Accepted: 25 March 2016; Published: 6 April 2016

Abstract: In this paper, sufficient conditions are given to investigate the existence of mild solutions on a semi-infinite interval for first order semi linear impulsive neutral functional differential evolution inclusions with infinite delay using a recently developed nonlinear alternative for contractive multivalued maps in Frechet spaces due to Frigon combined with semigroup theory. The existence result has been proved without assumption of compactness of the semigroup. We introduced a new phase space for impulsive system with infinite delay and claim that the phase space considered by different authors are not correct.

Keywords: impulsive differential inclusions; fixed point; Frechet spaces; nonlinear alternative due to Frigon

Mathematics Subject Classification (2000): 34A37, 34G20, 47H20

1. Introduction

In recent years, impulsive differential and partial differential equations have become more important in some mathematical models of real phenomena, especially in control, biological and medical domains. In these models, the investigated simulating processes and phenomena usually are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously in the form of impulses. The theory of impulsive differential equations has seen considerable development, see the monographs of Bainov and Semeonov [1], Lakshimikantham [2] and Perestyuk [3]. Recently, several works reported existence results for mild solutions for impulsive neutral functional differential equations or inclusions, such as ([4–11]) and references therein. However, the results obtained there are only in connection with finite delay. Since many systems arising from realistic models heavily depend on histories (*i.e.*, there is an effect of infinite delay on state equations), there is a real need to discuss partial functional differential systems with infinite delay, where numerous properties of their solutions are studied and detailed bibliographies are given. The literature related to first and second order nonlinear non autonomous neutral impulsive systems with or without state dependent delay is not vast. To the best of our knowledge, this is almost an untreated article in a literature and is one of the main motivations of this paper.

When the delay is infinite, the notion of the phase space plays an important role in the study of both qualitative and quantitative theory. A usual choice is a seminormed space satisfying suitable axioms, introduced by Hale and Kato in [12]; see also Corduneanu and Lakshmikantham [13]; Graef [14] and Baghli and Benchohra [15,16]. Unfortunately, we can not find broad literature about the system involving infinite delay with impulse effects. Henderson and Ouahab [17] discussed existence results for nondensely defined semilinear functional differential inclusions in Frechet spaces. Hernández *et al.* [18] studied existence of solutuions for impulsive partial neutral functional differential equations for first and second order systems with infinite delay. Recently, Arthi and Balachandran [19] proved controllability of the second order impulsive functional differential equations with state dependent delay using fixed point approach and cosine operator theory. It has been observed that the existence or the controllability results proved by different authors are through an axiomatic defination of the phase space given by Hale and Kato [12]. However, as remarked by Hino, Murakami, and Naito [20], it has come to our attention that these axioms for the phase space are not correct for the impulsive system with infinite delay [21,22]. This motivated us to generate a new phase space for the existence of a nonautonomous impulsive neutral inclusion with infinite delay. This is another motivation of this paper. To the best of our knowledge, the result proved in this paper is new and are not available in the literature.

On the other hand, researchers have been proving the controllability results using compactness assumption of semigroups and the family of cosine operators. However, as remarked by Triggiani [23], if X is an infinite dimensional Banach space, then the linear control system is never exactly controllable on the given interval if either B is compact or associated semigroup is compact. According to Triggiani [23], this is a typical case for most control systems governed by parabolic partial differential equations and hence the concept of exact controllability is very limited for many parabolic partial differential equations. Nowadays, researchers are engaged to overcome this problem, refer to ([19,21,22]). Very recently, Chalishajar and Acharya [22] studied the controllability of second order neutral functional differential inclusion, with infinite delay and impulse effect on unbounded domain, without compactness of the family of cosine operators. Ntouyas and O'Regan [24] gave some remarks on controllability of evolution equations in Banach paces and proved a result without compactness assumption.

In the last few years, researchers have diverted to fractional impulsive equations due to their extensive applications in various fields. Fečkan *et al.* [25] have discussed the existence of PC-mild solutions for Cauchy problems and nonlocal problems for impulsive fractional evolution equations involving Caputo fractional derivative by utilizing the theory of operators semigroup, probability density functions via impulsive conditions, a new concept on a PC-mild solution is introduced in their paper. We refer to the readers the book of Zhou [26]. Recently, Fu *et al.* [27] studied the existence of PC-mild solutions for Cauchy and nonlocal problems of impulsive fractional evolution equations for which the impulses are not instantaneous, by using the theory of operator semigroups, probability density functions, and some suitable fixed point theorems.

The rest of this paper is organized as follows: In Section 2, we introduce the system, recall some basic definitions, and preliminary facts that will be used throughout this paper. The existence theorems for semi linear impulsive neutral evolution inclusions with infinite delay, and their proofs are arranged in Section 3. Finally, in Section 4, an example is presented to illustrate the applications of the obtained results.

2. Preliminaries

In this paper, we shall consider the existence of mild solutions for first order impulsive partial neutral functional evolution differential inclusions with infinite delay in a Banach space E

$$\frac{d}{dt}\Big[y(t) - g(t, y_t)\Big] \in A(t)y(t) + F(t, y_t) \tag{1}$$
$$t \in J = [0, +\infty), \quad t \neq t_k, \quad k = 1, 2, \ldots$$
$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, 2 \ldots \tag{2}$$
$$y_0 = \phi \in \mathcal{B}_h \tag{3}$$

where $F : J \times \mathcal{B}_h \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all subsets of E, $g : J \times \mathcal{B}_h \to E$ and $I_k : E \to E, k = 1, 2, \ldots$ are given functions, $\phi \in \mathcal{B}_h$ are given functions and $\{A(t)\}_{0 \leq t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s) \in J \times J}$ for $0 \leq s \leq t < +\infty$. $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+)$ and $y(t_k^-)$ represent right and left limits of $y(t)$ at $t = t_k$ respectively. For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}_h$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. We assume that the histories y_t belongs to some abstract phase space $\mathcal{B}_h$ to be specified below.

We present the abstract phase space $\mathcal{B}_h$. Assume that $h : (-\infty, 0) \to (0, \infty)$ be a continuous function with $l = \int_{-\infty}^{0} h(s)ds < +\infty$. Define,

$$\mathcal{B}_h := \{\phi : [-\infty, 0] \to X \text{ such that, for any } r > 0, \phi(\theta) \text{ is bounded and measurable function on } [-r, 0] \text{ and } \int_{-\infty}^{0} h(s) \sup\nolimits_{s \leq \theta \leq 0} |\phi(\theta)| ds < +\infty\}.$$

Here, $\mathcal{B}_h$ endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} |\phi(\theta)| ds, \ \ \forall \phi \in \mathcal{B}_h$$

Then, it is easy to show that $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space.

Lemma 1. *Suppose $y \in \mathcal{B}_h$; then, for each $t \in J, y_t \in \mathcal{B}_h$. Moreover,*

$$l|y(t)| \leq \|y_t\|_{\mathcal{B}_h} \leq l \sup_{s \leq \theta \leq 0} (|y(s)| + \|y_0\|_{\mathcal{B}_h})$$

where $l := \int_{-\infty}^{0} h(s)ds < +\infty$.

Proof. For any $t \in [0, a]$, it is easy to see that, y_t is bounded and measurable on $[-a, 0]$ for $a > 0$, and

$$\begin{aligned}
\|y_t\|_{\mathcal{B}_h} &= \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} |y_t(\theta)| ds \\
&= \int_{-\infty}^{-t} h(s) \sup_{\theta \in [s,0]} |y(t+\theta)| ds + \int_{-t}^{0} h(s) \sup_{\theta \in [s,0]} |y(t+\theta)| ds \\
&= \int_{-\infty}^{-t} h(s) \sup_{\theta_1 \in [t+s,t]} |y(\theta_1)| ds + \int_{-t}^{0} h(s) \sup_{\theta_1 \in [t+s,t]} |y(\theta_1)| ds \\
&\leq \int_{-\infty}^{-t} h(s) \left[\sup_{\theta_1 \in [t+s,0]} |y(\theta_1)| + \sup_{\theta_1 \in [0,t]} |y(\theta_1)| \right] ds + \int_{-t}^{0} h(s) \sup_{\theta_1 \in [0,t]} |y(\theta_1)| ds \\
&= \int_{-\infty}^{-t} h(s) \sup_{\theta_1 \in [t+s,0]} |y(\theta_1)| ds + \int_{-\infty}^{0} h(s) ds. \sup_{s \in [0,t]} |y(s)| \\
&\leq \int_{-\infty}^{-t} h(s) \sup_{\theta_1 \in [s,0]} |y(\theta_1)| ds + l. \sup_{s \in [0,t]} |y(s)| \\
&\leq \int_{-\infty}^{0} h(s) \sup_{\theta_1 \in [s,0]} |y(\theta_1)| ds + l \sup_{s \in [0,t]} |y(s)| \\
&= \int_{-\infty}^{0} h(s) \sup_{\theta_1 \in [s,0]} |y_0(\theta_1)| ds + l \sup_{s \in [0,t]} |y(s)| \\
&= l \sup_{s \in [0,t]} |y(s)| + \|y_0\|_{\mathcal{B}_h}
\end{aligned}$$

Since $\phi \in \mathcal{B}_h$, then $y_t \in \mathcal{B}_h$. Moreover,

$$\|y_t\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} |y_t(\theta)| ds \geq |y_t(\theta)| \int_{-\infty}^{0} h(s) ds = l|y(t)|$$

The proof is complete. □

Next, we introduce definitions, notation and preliminary facts from multi-valued analysis, which are useful for the development of this paper (see [28]).

Let $C([0, b], E)$ denote the Banach space of all continuous functions from $[0, b]$ into E with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : 0 \leq t \leq b\}$$

and let $L^1([0, \infty), E)$ be the Banach space of measurable functions $y : [0, \infty) \to E$, that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^\infty \|y(t)\| dt \quad \text{for all} \quad y \in L^1([0, \infty), E)$$

Let X be a Frechet space with a family of semi-norms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$. Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\bar{M}_n > 0$ such that

$$\|y\|_n \leq \bar{M}_n \quad \text{for all} \quad y \in Y$$

To X we associate a sequence of Banach spaces $\{(X^n, \| \cdot \|_n)\}$ as follows: for every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by :$x \sim_n y$ if and only if $\|x - y\|_n = 0$ for all $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \| \cdot \|_n)$ the quotient space, the completion of X^n with respect to $\| \cdot \|_n$. To every $Y \subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: for every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we define $Y^n = \{[x]_n : x \in Y\}$. We denote

$\bar{Y}_n$, $\text{int}(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior, and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \ldots \quad \text{for every} \quad x \in X$$

Let (X, d) be a metric space. We use the following notations:
$\mathcal{P}_{cl}(X) := \{Y \in \mathcal{P}(X) : Y\text{closed}\}, \mathcal{P}_b(X) := \{Y \in \mathcal{P}(X) : Y\text{bounded}\}$
$\mathcal{P}_{cv}(X) := \{Y \in \mathcal{P}(X) : Y\text{convex}\}, \mathcal{P}_{cp}(X) := \{Y \in \mathcal{P}(X) : Y\text{compact}\}$.
Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to R^+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) := \max\{\sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(\mathcal{A}, b)\}$$

where $d(\mathcal{A}, b) := \inf_{a \in \mathcal{A}} d(a, b), d(a, \mathcal{B}) := \inf_{b \in \mathcal{B}} d(a, b)$. Then, $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [29]).

Definition 1. *We say that a family $\{A(t)\}_{t \geq 0}$ generates a unique linear evolution system $\{U(t,s)\}_{(t,s) \in \Delta}$ for $\Delta_1 = \{(t,s) \in J \times J : 0 \leq s \leq t < +\infty\}$ satisfying the following properties:*

(1) $U(t,t) = I$ where I is the identity operator in E,
(2) $U(t,s)U(s,\tau) = U(t,\tau) \quad \text{for} \quad 0 \leq \tau \leq s \leq t < +\infty$,
(3) $U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(t,s) \in \Delta_1$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)y$ is continuous.

More details on evolution systems and their properties could be found in the books of Ahmed [30], Engel and Nagel [31], and Pazy [32].

Definition 2. *A multivalued map $G : J \to \mathcal{P}_{cl}(X)$ is said to be measurable if for each $x \in E$, the function $Y : J \to X$ defined by*

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable where d is the metric induced by the normed Banach space X.

Definition 3. *A function $F : J \times \mathcal{B}_h \to \mathcal{P}(X)$ is said to be an L^1_{loc}-Caratheodory multivalued map if it satisfies:*
(i) $x \to F(t,y)$ is continuous(with respect to the metric H_d) for almost all $t \in J$;
(ii) $t \to F(t,y)$ is measurable for each $y \in \mathcal{B}_h$;
(iii) for every positive constant k there exists $h_k \in L^1_{loc}(J; R^+)$ such that

$$\|F(t,y)\| \leq h_k(t) \quad \textit{for all} \quad \|y\|_{\mathcal{B}_h} \leq k \quad \textit{and for almost all} \quad t \in J$$

A multivalued map $G : X \to \mathcal{P}(X)$ has convex(closed) values if $G(x)$ is convex (closed) for all $x \in X$. We say that G is bounded on bounded sets if $G(B)$ is bounded in X for each bounded set B of X, i.e.,

$$\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$$

Finally, we say that G has fixed point if there exists $x \in X$ such that $x \in G(x)$.
For each $y \in B_*$, let the set $S_{F,y}$ known as the set of selectors from F defined by

$$S_{F,y} = \{v \in L^1(J; E) : v(t) \in F(t, y_t), \quad \text{a.e.} t \in J\}$$

For more details on multivalued maps, we refer to the books of Aubin and Cellina [33] and Deimling [34], Gorniewicz [35], Hu and Papageorgiou [36], and Tolstonogov [37].

Definition 4. *A multivalued map $F : X \to \mathcal{P}(X)$ is called an admissible contraction with constant $\{k_n\}_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that*
(i) $H_d(F(x), F(y)) \leq k_n\|x-y\|_n$ for all $x, y \in X$.
(ii) For every $x \in X$ and every $\epsilon \in (0,\infty)^n$, there exists $y \in F(x)$ such that

$$\|x-y\|_n \leq \|x-F(x)\|_n + \epsilon_n \quad \textit{for every} \quad n \in \mathbb{N}$$

The following nonlinear alternative will be used to prove our main results.
Theorem 2.1 (Nonlinear Alternative of Frigon, [38,39]). Let X be a Frechet space and U an open neighborhood of the origin in X and let $N : \bar{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then, one of the following statements holds:
(C1) N has a fixed point;
(C2) There exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.

3. Existence Results

We consider the space

$$\begin{aligned} PC = \{y : (-\infty,\infty) \to E | y(t_k^-) \quad \text{and} \quad y(t_k^-) \quad \text{exist with} \quad y(t_k) = y(t_k^-), \\ y(t) = \phi(t) \quad \text{for} \quad t \in (-\infty,\infty), \quad y_k \in C(J_k, E), k = 1,2,3,\dots\} \end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_k+1], k = 1,2,3,\dots$

Now, we set

$$B_* = \{y : (-\infty,\infty) \to E : y \in PC \cap \mathcal{B}_h\}$$
$$B_k = \{y \in B_* : \sup_{t\in J_k^*} |y(t)| < \infty\}, \quad \text{where} \quad J_k^* = (-\infty, t_k]$$

Let $\|\cdot\|_k$ be the semi-norm in B_k defined by

$$\|y\|_k = \|y_0\|_{\mathcal{B}_h} + \sup\{|y(s)| : 0 \leq s \leq t_k\}, \quad y \in B_k$$

To prove our existence results for the impulsive neutral functional differential evolution problem with infinite delay $(1)-(3)$. Firstly, we define the mild solution.

Definition 5. *We say that the function $y(\cdot) : (-\infty,+\infty) \to E$ is a mild solution of the evolution system $(1)-(3)$ if $y(t) = \phi(t)$ for all $t \in (-\infty, 0]$, $\Delta y|_{t=t_k} = I_k(y(t_k^-)), k = 1,2,\dots$ and the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot) \in L^1(J;E) : f(t) \in F(t, y_t)$ a.e in J such that y satisfies the following integral equation:*

$$\begin{aligned} y(t) \quad = \quad & U(t,0)\Big[\phi(0) - g(0,\phi)\Big] + g(t,y_t) + \int_0^t U(t,s)A(s)g(s,y_s)ds \\ & + \int_0^t U(t,s)f(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), \quad \textit{for each} \quad t \in [0,+\infty) \end{aligned} \tag{4}$$

We need to introduce the following hypotheses, which are assumed hereafter:
(H1) There exists a constant $M_1 \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq M_1 \quad \text{for every} \quad (t,s) \in \Delta_1$$

(H2) The multifunction $F : J \times \mathcal{B}_h \to \mathcal{P}(E)$ is L^1_{loc}-Caratheodory with compact and convex values for each $u \in \mathcal{B}_h$ and there exist a function $p \in L^1_{loc}(J, R^+)$ and a continuous nondecreasing function $\psi : J \to (0, \infty)$ such that

$$\|F(t,u)\|_{\mathcal{P}(E)} \leq p(t)\psi(\|u\|_{\mathcal{B}_h}) \quad \text{for a.e} \quad t \in J \quad \text{and each} \quad u \in \mathcal{B}_h$$

(H3) For all $r > 0$, there exists $l_r \in L^1_{loc}(J; R^+)$ such that

$$H_d(F(t,u) - F(t,v)) \leq l_r(t)\|u - v\|_{\mathcal{B}_h}$$

for each $t \in J$ and for all $u, v \in \mathcal{B}_h$ with $\|u\|_{\mathcal{B}_h} \leq r$ and $\|v\|_{\mathcal{B}_h} \leq r$ and

$$d(0, F(t,0)) \leq l_r(t) \quad \text{a.e} \quad t \in J$$

(H4) There exists a constant $M_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq M_0 \quad \text{for all} \quad t \in J$$

(H5) There exists a constant $d_k > 0, \quad k = 1, 2, \ldots$ such that

$$\|I_k(x) - I_k(\bar{x})\| \leq d_k\|x - \bar{x}\| \quad \text{for each} \quad k = 1, 2, \ldots \quad \text{for all} \quad x, \bar{x} \in E$$

(H6) There exists a constant $0 < L < \frac{1}{M_0 K_n}$ such that

$$\|A(t)g(t,\phi)\| \leq L(\|\phi\|_{\mathcal{B}_h} + 1) \quad \text{for all} \quad t \in J, \quad \phi \in \mathcal{B}_h$$

(H7) There exists a positive constant $c_k, \quad k = 1, 2, \ldots$ such that

$$\|I_k(z)\| \leq c_k \quad \text{for all} \quad z \in E, \quad \sum_{k=1}^{\infty} c_k < \infty$$

(H8) There exists a constant $L_* > 0$ such that

$$\|A(s)g(s,\phi) - A(\bar{s})g(\bar{s},\bar{\phi})\| \leq L_*(|s - \bar{s}| + \|\phi - \bar{\phi}\|_{\mathcal{B}_h}) \quad \text{for all} \quad s, \bar{s} \in J \quad \text{and} \quad \phi, \bar{\phi} \in \mathcal{B}_h.$$

For every $n \in \mathbb{N}$, let us take here $\bar{l_n}(t) = M_1 K_n \Big[L_* + l_n(t)\Big]$ for the family of semi-norm $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$. In what follows, we fix $\tau > 1$ and assume

$$\Big[M_0 L_* K_n + \frac{1}{\tau} + M_1 \sum_{k=1}^{m} d_k\Big] < 1$$

Theorem 3.1 Suppose that hypotheses (H1)-(H8) are satisfied. Moreover

$$\int_{\delta_n}^{+\infty} \frac{ds}{s + \psi(s)} > \frac{M_1 K_n}{1 - M_0 L K_n} \int_0^n \max(L, p(s))ds \quad \text{for each} \quad n \in \mathbb{N} \tag{5}$$

with

$$\begin{aligned} \delta_n \quad = \quad & (K_n M_1 H + M_n)\|\phi\|_{\mathcal{B}_h} + \frac{K_n}{1 - M_0 L K_n}\Big[(M_1 + 1)M_0 L + M_1 L n \\ & + M_0 L[M_1(K_n H + 1) + M_n]\|\phi\|_{\mathcal{B}_h} + M_1 \sum_{k=1}^{m} c_k\Big] \end{aligned}$$

Then, the impulsive neutral evolution problem $(1)-(3)$ has a mild solution.

Proof. We transform the Problem $(1)-(3)$ into a fixed point problem. Consider an operator $N : B_* \to B_*$ defined by

$$N(y) = h \in B_* : h(t) = \begin{cases} \phi(t) \quad \text{if} \quad t \leq 0 \\ U(t,0)[\phi(0) - g(0,\phi)] + g(t,y_t) + \int_0^t U(t,s)A(s)g(s,y_s)ds \\ + \int_0^t U(t,s)f(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), t \in J \end{cases}$$

where $f \in S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y_t) \quad \text{for a.e} \quad t \in J\}$. Clearly, the fixed points of the operator N are mild solutions of the Problem $(1)-(3)$. We remark also that, for each $y \in B_*$, the set $S_{F,y}$ is nonempty since F has a measurable selection by (H2) (see [40], Theorem III.6).

For $\phi \in \mathcal{B}_h$, we will define the function $x(\cdot) : (-\infty, +\infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), \quad \text{if} \quad t \in (-\infty, 0] \\ U(t,0)\phi(0), \quad \text{if} \quad t \in J \end{cases}$$

Then, $x_0 = \phi$. For each function, $z \in C([0,\infty), E)$. We can decompose y into $y(t) = z(t) + x(t)$. Let $B_*^k = \{z \in B_k : z_0 = 0\}$. Then, for any $z \in B_*^k$, we have

$$\|z\|_k = \|z_0\|_{\mathcal{B}_h} + \sup\{\|z(s)\| : 0 \leq s \leq t_k\| = \sup\{\|z(s)\| : 0 \leq s \leq t_k\}$$

Thus $(B_*^k, \|\cdot\|_k)$is a Banach space, if we set $C_1 = \{z \in B_*; z_0 = 0\}$ with the Bielecki-norm on B_*^k defined by

$$\|z\|_{B_*^k} = \max\{\|z(t)\|e^{-\tau L_n^*(t)} : t \in [0, t_k]\}$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds, \bar{l}_n(t) = M_1 K_n l_n(t)$ and $\tau > 0$ is a constant. Then C_1 is a Frechet space with family of seminorms $\|\cdot\|_{B_*^k}$. It is obvious that y satisfies (4) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) \quad &= \quad g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds + \int_0^t U(t,s)f(s)ds \\ &\quad + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^+) + x(t_k^-)) \end{aligned}$$

where $f(t) \in F(t, z_t + x_t)$ a.e $t \in J$.

Let us define a multivalued operator $\mathcal{F} : C_1 \to C_1$ by

$$\begin{aligned} \mathcal{F}(z) \quad &= \quad h \in B_*^k : h(t) = g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds \\ &\quad + \int_0^t U(t,s)f(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-), x(t_k^-)), \quad t \in J \end{aligned}$$

where $f \in S_{F,z} = \{v \in L^1(J,E) : v(t) \in F(t, z_t + x_t) \quad \text{for a.e } t \in J\}$. Obviously, the operator inclusion N has a fixed point is equivalent to the operator inclusion $\mathcal{F}$ has one, so it turns to prove that $\mathcal{F}$ has a fixed point.

Let $z \in B_*^k$ be a possible fixed point of the operator $\mathcal{F}$. Given $n \in \mathbb{N}$, then z should be solution of the inclusion $z \in \lambda\mathcal{F}(z)$ for some $\lambda \in (0,1)$ and there exists $f \in S_{F,z} \Leftrightarrow f(t) \in F(t, z_t + x_t)$ such that, for each $t \in [0,n]$, we have

$$\begin{aligned}
\|z(t)\| &= \|A^{-1}(t)\|_{B(E)}\|A(t)g(t,z_t+x_t)\| + \|U(t,0)\|_{B(E)}\|A^{-1}(0)\|_{B(E)}\|A(0)g(0,\phi)\| \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)}\|A(s)g(s,z_s+x_s)\|ds + \int_0^t \|U(t,s)\|_{B(E)}\|f(s)\|ds \\
&\quad + \sum_{0<t_k<t} \|U(t,t_k)\|_{B(E)}\|I_k(z(t_k^-)+x(t_k^-))\| \\
&\leq M_0L(\|z_t+x_t\|_{\mathcal{B}_h}+1) + M_1M_0L(\|\phi\|_{\mathcal{B}_h}+1) + M_1\int_0^t L(\|z_s+x_s\|_{\mathcal{B}_h}+1)ds \\
&\quad + M_1\int_0^t p(s)\psi(\|z_s+x_s\|_{\mathcal{B}_h})ds + M_1\sum_{k=1}^m c_k \\
&\leq M_0L\|z_t+x_t\|_{\mathcal{B}_h} + M_0L(1+M_1) + M_1Ln + M_1M_0L\|\phi\|_{\mathcal{B}_h} + M_1\int_0^t L\|z_s+x_s\|_{\mathcal{B}_h}ds \\
&\quad + M_1\int_0^t p(s)\psi(\|z_s+x_s\|_{\mathcal{B}_h})ds + M_1\sum_{k=1}^m c_k
\end{aligned}$$

Assumption $(A1)$ gives

$$\begin{aligned}
\|z_s+x_s\|_{\mathcal{B}_h} &\leq \|z_s\|_{\mathcal{B}_h} + \|x_s\|_{\mathcal{B}_h} \\
&\leq K(s)\|z(s)\| + M(s)\|z_0\|_{\mathcal{B}_h} + K(s)\|x(s)\| + M(s)\|x_0\|_{\mathcal{B}_h} \\
&\leq K_n\|z(s)\| + K_n\|U(s,0)\|_{B(E)}\|\phi(0)\| + M_n\|\phi\|_{\mathcal{B}_h} \\
&\leq K_n\|z(s)\| + K_nM_1\|\phi(0)\| + M_n\|\phi\|_{\mathcal{B}_h} \\
&\leq K_n\|z(s)\| + K_nM_1H\|\phi\|_{\mathcal{B}_h} + M_n\|\phi\|_{\mathcal{B}_h} \\
&\leq K_n\|z(s)\| + (K_nM_1H + M_n)\|\phi\|_{\mathcal{B}_h}
\end{aligned}$$

Set

$$C_n = (K_nM_1H + M_n)\|\phi\|_{\mathcal{B}_h}$$

then we have

$$\|x_s+z_s\|_{\mathcal{B}_h} \leq K_n\|z(s)\| + C_n \tag{6}$$

Using the inequality Equation (6) and the nondecreasing character of ψ, we obtain,

$$\begin{aligned}
\|z(t)\| &\leq M_0L(K_n\|z(t)\| + C_n) + M_0L(1+M_1) + M_1Ln + M_1M_0L\|\phi\|_{\mathcal{B}_h} \\
&\quad + M_1\int_0^t L(K_n\|z(s)\| + C_n)ds + M_1\int_0^t p(s)\psi(K_n\|z(s)\| + C_n)ds \\
&\quad + M_1\sum_{k=1}^m c_k \\
&\leq M_0LK_n\|z(t)\| + M_0L(1+M_1) + M_1Ln + M_0LC_n + M_1M_0L\|\phi\|_{\mathcal{B}_h} \\
&\quad + M_1\Big[\int_0^t L(K_n\|z(s)\| + C_n)ds + \int_0^t p(s)\psi(K_n\|z(s)\| + C_n)ds\Big] \\
&\quad + M_1\sum_{k=1}^m c_k
\end{aligned}$$

Then

$$\begin{aligned}(1 - M_0LK_n)\|z(t)\| &\leq (M_1+1)M_0L + MLn + M_0LC_n + M_1M_0L\|\phi\|_{\mathcal{B}_h} + M_1\sum_{k=1}^{m} c_k \\ &+ M_1\Big[\int_0^t L(K_n\|z(s)\| + C_n)ds + \int_0^t p(s)\psi(K_n\|z(s)\| + C_n)ds\Big]\end{aligned}$$

Set

$$\delta_n = C_n + \frac{K_n}{1 - M_0LK_n}\Big[(M_1+1)M_0L + MLn + M_0LC_n + M_1M_0L\|\phi\|_{\mathcal{B}_h} + M_1\sum_{k=1}^{m} c_k\Big]$$

Thus,

$$K_n\|z(t)\| + C_n \leq \delta_n + \frac{M_1K_n}{1 - M_0LK_n}\Big[\int_0^t L(K_n\|z(s)\| + C_n)ds + \int_0^t p(s)\psi(K_n\|z(s)\| + C_n)ds\Big]$$

We consider the function μ defined by

$$\mu(t) = \sup\{K_n\|z(s)\| + C_n : 0 \leq s \leq t\}, 0 \leq t < +\infty$$

Let $t^* \in [0,t]$ be such that $\mu(t) = K_n\|z(t^*)\| + C_n$. By the previous inequality, we have

$$\mu(t) \leq \delta_n + \frac{M_1K_n}{1 - M_0LK_n}\Big[\int_0^t L\mu(s)ds + \int_0^t p(s)\psi(\mu(s))ds\Big] \quad \text{for} \quad t \in [0,n]$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have $\mu(t) \leq v(t)$ for all $t \in [0,n]$. From the definition of v, we have $v(0) = \delta_n$ and

$$v'(t) = \frac{M_1K_n}{1 - M_0LK_n}[L\mu(t) + p(t)\psi(\mu(t))] \quad \text{a.e} \quad t \in [0,n]$$

Using the nondecreasing character of ψ, we get

$$v'(t) = \frac{M_1K_n}{1 - M_0LK_n}[Lv(t) + p(t)\psi(v(t))] \quad \text{a.e} \quad t \in [0,n]$$

This implies that for each $t \in [0,n]$and using the condition (5), we get

$$\begin{aligned}\int_{\delta_n}^{v(t)} \frac{ds}{s + \psi(s)} &\leq \frac{M_1K_n}{1 - M_0LK_n}\int_0^t \max(L, p(s))ds \\ &\leq \frac{M_1K_n}{1 - M_0LK_n}\int_0^n \max(L, p(s))ds \\ &< \int_{\delta_n}^{+\infty} \frac{ds}{s + \psi(s)}\end{aligned}$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_{B_*^k} \leq \mu(t)$, we have $\|z\|_{B_*^k} \leq \Lambda_n$.

Set $\Omega = \{z \in B_*^k : \sup\{\|z(t)\| : 0 \leq t \leq n\} < \Lambda_n + 1 \quad \text{for all} \quad n \in \mathbb{N}\}$.

Clearly, Ω is an open subset of B^k_*. We shall show that $\mathcal{F} : \bar{\Omega} \to \mathcal{P}(B^k_*)$ is a contraction and an admissible operator. First, we prove that $\mathcal{F}$ is a contraction. Let $z, \bar{z} \in B^k_*$ and $h \in \mathcal{F}(\bar{z})$. Then, there exists $f(t) \in F(t, z_t + x_t)$ such that for each $t \in [0, n]$,

$$\begin{aligned} h(t) &= g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds \\ &+ \int_0^t U(t,s)f(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-) + x(t_k^-)) \end{aligned}$$

From (H3) it follows that,

$$H_d(F(t, z_t + x_t), F(t, \bar{z}_t + x_t)) \le l_n(t)\|z_t - \bar{z}_t\|_{\mathcal{B}_h}$$

Hence, there is $\rho \in F(t, \bar{z}_t + x_t)$ such that

$$\|f(t) - \rho\| \le l_n(t)\|z_t - \bar{z}_t\|_{\mathcal{B}_h} \quad t \in [0, n]$$

Consider $\Omega_* : [0, n] \to \mathcal{P}(E)$, given by

$$\Omega_* = \{\rho \in E : \|f(t) - \rho\| \le l_n(t)\|z_t - \bar{z}_t\|\}$$

Since the multivalued operator $\mathcal{V}(t) = \Omega_*(t) \bigcap F(t, \bar{z}_t + x_t)$ is measurable (in [40], see proposition III, 4), there exists a function $\bar{f}(t)$, which is a measurable selection for $\mathcal{V}$. Thus, $\bar{f}(t) \in F(t, \bar{z}_t + x_t)$ and using (A1), we obtain for each $t \in [0, n]$

$$\begin{aligned} \|f(t) - \bar{f}(t)\| &\le l_n(t)\|z_t - \bar{z}_t\|_{\mathcal{B}_h} \\ &\le l_n(t)\Big[K(t)\|z(t) - \bar{z}(t)\| + M(t)\|z_0 - \bar{z}_0\|_{\mathcal{B}_h}\Big] \\ &\le l_n(t)K_n\|z(t) - \bar{z}(t)\| \end{aligned} \tag{7}$$

Let us define, for each $t \in [0, n]$

$$\begin{aligned} \bar{h}(t) &= g(t, \bar{z}_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, \bar{z}_s + x_s)ds \\ &+ \int_0^t U(t,s)\bar{f}(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(\bar{z}(t_k^-) + x(t_k^-)) \end{aligned}$$

Then, for each $t \in [0, n]$ and $n \in \mathbb{N}$ and using (H1) and (H3)-(H6) and (H8), we get

$$\begin{aligned}
\|h(t) - \bar{h}(t)\| &\leq \|g(t, z_t + x_t) - g(t, \bar{z}_t + x_t)\| \\
&\quad + \int_0^t \|U(t,s)A(s)\Big[g(s, z_s + x_s) + g(s, \bar{z}_s + x_s)\Big]\|ds \\
&\quad + \int_0^t \|U(t,s)\Big[f(s) - \bar{f}(s)\Big]\|ds \\
&\quad + \sum_{0<t_k<t} \|U(t, t_k)\Big[I_k(z(t_k^-) + x(t_k^-)) - I_k(\bar{z}(t_k^-) + x(t_k^-))\Big]\| \\
&\leq \|A^{-1}(t)\|_{B(E)}\|A(t)g(t, z_t + x_t) - A(t)g(t, \bar{z}_t + x_t)\| \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)}\|A(s)g(s, z_s + x_s) - A(s)g(s, \bar{z}_s + x_s)\|ds \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)}\|f(s) - \bar{f}(s)\|ds \\
&\quad + \sum_{0<t_k<t} \|U(t, t_k)\|_{B(E)}\|I_k(z(t_k^-) + x(t_k^-)) - I_k(\bar{z}(t_k^-) + x(t_k^-))\| \\
&\leq M_0 L_* \|z_t - \bar{z}_t\|_{\mathcal{B}_h} + \int_0^t M_1 L_* \|z_s - \bar{z}_s\|_{\mathcal{B}_h} ds \\
&\quad + \int_0^t M_1 \|f(s) - \bar{f}(s)\|ds \\
&\quad + M_1 \sum_{k=1}^{m} d_k \|(z(t_k^-) - \bar{z}(t_k^-))\|
\end{aligned}$$

Using (A1) and (7), we obtain

$$\begin{aligned}
\|h(t)-\bar{h}(t)\| &\leq M_0L_*K(t)\|z(t)-\bar{z}(t)\| + \int_0^t M_1L_*K(s)\|z(s)-\bar{z}(s)\|ds \\
&\quad + \int_0^t M_1 l_n(s)K_n\|z(s)-\bar{z}(s)\|ds \\
&\quad + M_1\sum_{k=1}^m d_k \sup_{s\in[0,t_k]}\|z(s)-\bar{z}(s)\| \\
&\leq M_0L_*K_n\|z(t)-\bar{z}(t)\| + \int_0^t M_1K_n\Big[L_* + l_n(s)\Big]\|z(s)-\bar{z}(s)\|ds \\
&\quad + M_1\sum_{k=1}^m d_k \sup_{s\in[0,t_k]}\|z(s)-\bar{z}(s)\| \\
&\leq M_0L_*K_n\|z(t)-\bar{z}(t)\| + \int_0^t \bar{l}_n(s)\|z(s)-\bar{z}(s)\|ds \\
&\quad + M_1\sum_{k=1}^m d_k \sup_{s\in[0,t_k]}\|z(s)-\bar{z}(s)\| \\
&\leq M_0L_*K_n\Big[e^{\tau L_n^*(t)}\Big]\Big[e^{-\tau L_n^*(t)}\|z(t)-\bar{z}(t)\|\Big] \\
&\quad + \int_0^t \Big[l_n(s)e^{\tau L_n^*(s)}\Big]\Big[e^{-\tau L_n^*(s)}\|z(s)-\bar{z}(s)\|\Big]ds \\
&\quad + M_1\sum_{k=1}^m d_k e^{\tau L_n^*(s)}e^{-\tau L_n^*(s)} \sup_{s\in[0,t_k]}\|z(s)-\bar{z}(s)\| \\
&\leq M_0L_*K_n e^{\tau L_n^*(t)}\|z-\bar{z}\|_{B_*^k} + \int_0^t \left[\frac{e^{\tau L_n^*(s)}}{\tau}\right]' ds\|z-\bar{z}\|_{B_*^k} \\
&\quad + M_1\sum_{k=1}^m d_k e^{\tau L_n^*(s)}\|z-\bar{z}\|_{B_*^k} \\
&\leq M_0L_*K_n e^{\tau L_n^*(t)}\|z-\bar{z}\|_{B_*^k} + \frac{1}{\tau}e^{\tau L_n^*(t)}\|z-\bar{z}\|_{B_*^k} \\
&\quad + M_1\sum_{k=1}^m d_k e^{\tau L_n^*(t)}\|z-\bar{z}\|_{B_*^k} \\
&\leq \Big[M_0L_*K_n + \frac{1}{\tau} + M_1\sum_{k=1}^m d_k\Big]e^{\tau L_n^*(t)}\|z-\bar{z}\|_{B_*^k}
\end{aligned}$$

Therefore,

$$\|h-\bar{h}\|_{B_*^k} \leq \Big[M_0L_*K_n + \frac{1}{\tau} + M_1\sum_{k=1}^m d_k\Big]\|z-\bar{z}\|_{B_*^k}$$

By an analogous relation, obtained by interchanging the roles of z and $\bar{z}$, it follows that

$$H_d(\mathcal{F}(z)-\mathcal{F}(\bar{z})) \leq \Big[M_0L_*K_n + \frac{1}{\tau} + M_1\sum_{k=1}^m d_k\Big]\|z-\bar{z}\|_{\bar{B}}$$

Thus, the operator $\mathcal{F}$ is a contraction for all $n \in \mathbb{N}$.

Now, we shall show that $\mathcal{F}$ is an admissible operator. Let $z \in B_*^k$. Set, for every $n \in \mathbb{N}$, the space,

$$B_{**}^k = \{y : (-\infty, n] \to E : y|_{[0,n]} \in PC([0,n], E), y_0 \in \mathcal{B}_h\}$$

and let us consider the multivalued operator $\mathcal{F} : B^k_{**} \to \mathcal{P}_{cl}(B^k_{**})$ defined by

$$\begin{aligned}\mathcal{F}(z) = h \in B^k_{**} : h(t) &= g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds \\ &\quad + \int_0^t U(t,s)f(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-) + x(t_k^-)), \quad t \in [0,n]\end{aligned}$$

where $f \in S^n_{F,y} = \{v \in L^1([0,n], E) : v(t) \in F(t, y_t) \quad \text{for a.e} \quad t \in [0,n]\}$.
From (H1) to (H7), and since F is multivalued map with compact values, we can prove that for every $z \in B^k_{**}, \quad \mathcal{F}(z) \in \mathcal{P}_{cl}(B^k_{**})$ and there exists $z_* \in B^k_{**}$ such that $z_* \in \mathcal{F}(z_*)$. Let $h \in B^k_{**}, \quad \bar{y} \in \bar{\mu}$ and $\epsilon > 0$. Assume that $z_* \in \mathcal{F}(\bar{z})$, then we have

$$\begin{aligned}\|\bar{z}(t) - z_*(t)\| &\leq \|\bar{z}(t) - h(t)\| + \|z_*(t) - h(t)\| \\ &\leq e^{\tau L^*_n(t)} \|\bar{z}(t) - \mathcal{F}(\bar{z})\|_{B^k_*} + \|z_* - h\|\end{aligned}$$

Since h is arbitrary, we may suppose that

$$h \in B(z_*, \epsilon) = \{h \in B^k_{**} : \|h - z_*\|_{B^k_*} \leq \epsilon\}$$

Therefore,

$$\|\bar{z} - z_*\|_n \leq \|\bar{z} - \mathcal{F}(\bar{z})\|_{B^k_*} + \epsilon$$

If z is not in $\mathcal{F}(\bar{z})$, then $\|z_* - \mathcal{F}(\bar{z})\| \neq 0$. Since $\mathcal{F}(\bar{z})$ is compact, there exists $x \in \mathcal{F}(\bar{z})$ such that

$$\|z_* - \mathcal{F}(\bar{z})\| = \|z_* - x\|$$

Then, we have

$$\begin{aligned}\|\bar{z}(t) - z_*(t)\| &< \|\bar{z}(t) - h(t)\| + \|x(t) - h(t)\| \\ &\leq e^{\tau L^*_n} \|\bar{z} - \mathcal{F}(\bar{z})\|_{B^k_*} + \|x(t) - h(t)\|\end{aligned}$$

Thus, $\|\bar{z} - x\|_{B^k_*} \leq \|\bar{z} - \mathcal{F}(\bar{z})\|_{B^k_*} + \epsilon$.

Therefore, $\mathcal{F}$ is an admissible operator contraction.

From the choice of Ω, there is no $z \in \partial\Omega$ such that $z = \lambda\mathcal{F}(z)$ for some $\lambda \in (0,1)$. Then, the statement $(C2)$ in Theorem (2.1) does not hold. This implies that the operator $\mathcal{F}$ has a fixed point z^*. Then $y^*(t) = z^*(t) + x(t), \quad t \in (-\infty, +\infty)$ is a fixed point of the operator $\mathbb{N}$, which is a mild solution of the Problem $(1) - (3)$.
Hence the proof. □

4. Example

As an application of Theorem (3.1), we study the following impulsive neutral differential system:

$$\begin{aligned}\frac{\partial}{\partial t}\Big[v(t,\xi) - \int_{-\infty}^0 T(\theta)u(t, v(t+\theta, \xi))d\theta\Big] &\in a(t,\xi)\frac{\partial^2 v}{\partial \xi^2}(t,\xi) \\ &\quad + \int_{-\infty}^0 P(\theta)Q(t, v(t+\theta,\xi))d\theta \\ t \in [0, +\infty), \quad \xi \in [0,\pi] & \\ v(t,0) = v(t,\pi) = 0 & \\ z(t_k^+) - z(t_k^-) = I_k(z(t_k^-)), \quad k = 1,2,\ldots & \\ v(\theta,\xi) = v_0(\theta,\xi) \qquad -\infty < \theta \leq 0, \xi \in [0,\pi] &\end{aligned} \tag{8}$$

where $a(t,\xi)$ is a continuous function and is uniformly Holder continuous in t; $T, P : (-\infty, 0] \to R$; $u : (-\infty, 0] \times R \to R$ and $v_0 : (-\infty, 0] \times [0, \pi] \to R$ are continuous functions and $Q : [0, +\infty) \times R \to \mathcal{P}(R)$ is a multivalued map with compact convex values.

Consider $E = L^2([0,\pi], R)$ and define $A(t)$ by $A(t)w = a(t,\xi)w''$ with domain $D(A) = \{w \in E : w, w'$ are absolutely continuous, $w'' \in E, w(0) = w(\pi) = 0\}$.

Then, $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption (H1)(see [41]). We can define respectively that

$$g(t,\phi)(\xi) = \int_{-\infty}^{0} T(\theta)u(t,\phi(\theta)(\xi)d\theta, \qquad -\infty < \theta \leq 0, \quad \xi \in [0,\pi]$$

and

$$F(t,\phi)(\xi) = \int_{-\infty}^{0} P(\theta)Q(t,\phi(\theta)(\xi)d\theta, \qquad -\infty < \theta \leq 0, \quad \xi \in [0,\pi]$$

Then, in order to prove the existence of mild solutions of the System (8), we suppose the following assumptions:

(i) u is Lipschitz with respect to its second argument. Let lip(u) denotes the Lipschitz constant of u.
(ii) There exist $p \in L^1(J, R^+)$ and a nondecreasing continuous function $\psi : [0,+\infty) \to (0,+\infty)$ such that

$$|Q(t,x)| \leq p(t)\psi(|x|), \quad \text{for} \quad t \in J, \quad \text{and} \quad x \in R$$

(iii) T, P are integrable on $(-\infty, 0]$.
(iv) There exist positive constants c_k and $d_k, k = 1, 2, \ldots$ such that

$$\|I_k(x) - I_k(\bar{x})\| \leq d_k\|x - \bar{x}\|,$$
$$\|I_k(z)\| \leq c_k, \quad \text{for all} \quad x, \bar{x}, z \in E$$

By the dominated convergence theorem, one can show that $f \in S_{F,y}$ is a continuous function from $\mathcal{B}_h$ to E. Moreover, the mapping g is Lipschitz continuous in its argument. In fact, we have

$$\|g(t,\phi_1) - g(t,\phi_2)\| \leq M_0 L_* lip(u) \int_{-\infty}^{0} \|T(\theta)\|d\theta\|\phi_1 - \phi_2\|, \quad \text{for} \quad \phi_1, \phi_2 \in \mathcal{B}_h$$

On the other hand, we have for $\phi \in \mathcal{B}_h$ and $\xi \in [0,\pi]$

$$\|F(t,\phi)(\xi)\| \leq \int_{-\infty}^{0} \|p(t)P(\theta)\|\xi(\|(\phi(\theta))(\xi)\|)d\theta$$

Since the function ξ is nondecreasing, it follows that

$$\|F(t,\phi)\|_{\mathcal{P}(E)} \leq p(t)\int_{-\infty}^{0} \|P(\theta)\|d\theta\xi(\|\phi\|), \quad \text{for} \quad \phi \in \mathcal{B}_h$$

Proposition: Under the above assumptions, if we assume that condition (5) in Theorem (3.1) is true, $\phi \in \mathcal{B}_h$, then the System (8) has a mild solution which is defined in $(-\infty, +\infty)$.

5. Conclusions

In this manuscript, we have proved the existence result of first order impulsive neutral evolution inclusion with infinite delay in Frechet spaces. Here, we defined a new notion of phase space and proved the result without compactness of an evolution operator, using a recently developed nonlinear alternative for contractive multivalued maps due to Frigon. The same result can be generalized for controllability of an impulsive neutral evolution inclusion with infinite delay of the form [19,21]

$$\begin{aligned} \frac{d}{dt}\Big[y(t) - g(t, y_t)\Big] &\in A(t)y(t) + Bu(t) + F(t, y_t) \\ t \in J = [0, +\infty), \quad t &\neq t_k, \quad k = 1, 2, \ldots \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, 2, \ldots \\ y_0 &= \phi \in \mathcal{B}_h \end{aligned}$$

Acknowledgments: Authors wish to express their gratitude to the anonymous referees for their valuable suggestions and comments for improving this manuscript. This work is supported by Grant In Aid research fund of Virginia Military Instittue, USA.

Author Contributions: Dimplekumar N. Chalishajar defined the new phase space and compared with the phase space given by Hall and Kato. Dimplekumar N. Chalishajar and Kulandhivel Karthikeyan conceived and designed the control problem experiment and study the existence result of first order impulsive neutral evolution inclusion with infinite delay in Frechet spaces without compactness of an evolution operator; Kulandhivel Karthikeyan performed the analysis and wrote the paper; Annamalai Anguraj analyzed and contributed materials/analysis tools.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bainov, D.D.; Simeonov, P.S. *Systems with Impulsive Effect*; Ellis Horwood: Chichester, UK, 1989.
2. Lakshimikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Science: Singapore, 1989.
3. Samoilenko, A.M.; Perestyuk, N.A. *Impulsive Differential Equations*; World Scientific Publishing: Singapore, 1995; Volume 14.
4. Anguraj, A.; Arjunan, M.M.; Hernández, M.E. Existence results for an impulsive neutral functional differential equation with state-dependent delay. *Appl. Anal.* **2007**, *86*, 861–872.
5. Chang, Y.K.; Anguraj, A.; Karthikeyan, K. Existence for impulsive neutral integrodifferential inclusions with nonlocal initial conditions via fractional operators. *Nonlin. Anal. TMA* **2009**, *71*, 4377–4386.
6. Benchohra, M.; Henderson, J.; Ntouyas, S.K. *Impulsive Differential Equations and Inclusions*; Hindawi Publishing Corporation: New York, NY, USA, 2006; Volume 2.
7. Nieto, J.J.; O'Regan, D. Variational approach to impulsive differential equations. *Nonlinear Anal. Real World Appl.* **2009**, *10*, 680–690.
8. Zavalishchin, S.T.; Sesekin, A.N. *Dynamic Impulse Systems: Theory and Applications*; Kluwer Academic Publishers: New York, NY, USA, 1997.
9. Wang, L.; Chen, L.; Nieto, J.J. The dynamics of an epidemic model for pest control with impulsive effect. *Nonlinear Anal. Real World Appl.* **2010**, *11*, 1374–1386.
10. Tai , Z.; Lun, S. On controllability of fractional impulsive neutral infinite delay evolution integrodifferential systems in Banach spaces. *Appl. Math. Lett.* **2012**, *25*, 104–110.
11. Gao, S.; Chen, L.; Nieto, J.J.; Torres, A. Analysis of a delayed epidemic model with pulse vaccination and saturation incidence. *Vaccine* **2006**, *24*, 6037–6045.
12. Hale, J.; Kato, J. Phase space for retarded equations with infinite delay. *Funkc. Ekvacio.* **1978**, *21*, 11–41.
13. Corduneanu, C.; Lakshmikantham, V. Equations with unbounded delay. *Nonlinear Anal.* **1980**, *4*, 831–877.
14. Graef, J.R.; Ouahab, A. Some existence and uniqueness results for first order boundary value problems for impulsive functional differential equations with infinite delay in Frechet spaces. *Int. J. Math Math Sci.* **2006**, *2006*, 1–16.
15. Baghli, S.; Benchohra, M. Uniqueness results for partial functional differential equations in Frechet spaces. *Fixed Point Theory* **2008**, *9*, 395–406.
16. Baghli, S.; Benchohra, M. Multivalued evolution equations with infinite delay in Frechet spaces. *Electron. J. Qual. Theory Differ. Equ.* **2008**, *33*, 1–24.
17. Henderson, J.; Ouahab, A. Existence results for nondensely defined semilinear functional differential inclusions in Frechet spaces. *Electron. J. Qual. Theory Differ. Equ.* **2005**, *17*, 1–17.

18. Hernández, E.M.; Rabello, M.; Henriquez, H. Existence of solution of impulsive partial neutral functional differential equations. *J. Math. Anal. Appl.* **2007**, *331*, 1135–1158.
19. Arthi, G.; Balachandran, K. Controllability of second order impulsive functional differential equations with state dependent delay. *Bull. Korean Math. Soc.* **2011**, *48*, 1271–1290.
20. Hino, Y.; Murakami, S.; Naito, T. *Functional Differential Equations with Infinite Delay*; Springer-Verlag: Berlin, Germany, 1991; Volume 1473.
21. Chalishajar, D.N. Controllability of second order impulsive neutral functional differential inclusions with infinite delay. *J. Optim. Theory. Appl.* **2012**, doi:10.1007/s10957-012-0025-6.
22. Chalishajar, D.N.; Acharya, F.S. Controllability of second order semilinear neutral functional impulsive differential inclusion with infinite delay in Banach spaces. *Bull. Korean Math. Soc.* **2011**, *48*, 813–838.
23. Triggiani, R. Addendum : A note on lack of exact controllability for mild solution in Banach spaces. *SIAM J. Control Optim.* **1977**, *15*, 407–411.
24. Ntouyas, S.; O'Regan, D. Some remarks on controllability of evolution equations in Banach spaces. *Electron. J. Differ. Equ.* **2009**, *2009*, 1–6.
25. Fečkan, M.; Wang, J.-R.; Zhou, Y. On the new concept of solutions and existence results for impulsive fractional evolution equations. *Dyn. Part. Differ. Equ.* **2011**, *8*, 345–361.
26. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.
27. Fu, X.; Liu, X.; Lu, B. On a new class of impulsive fractional evolution equations. *Adv. Differ. Equ.* **2015**, *2015*, 1–16.
28. Yosida, K. *Functional Analysis*, 6th ed.; Springer-Verlag: Berlin, Dermany, 1980.
29. Kisielewicz, M. *Differential Inclusions and Optimal Control*; Kluwer: Dordrecht, The Netherlands, 1991.
30. Ahmed, N.U. *Semigroup Theory with Applications to Systems and Control*; John Wiley & Sons: New York, NY, USA, 1991.
31. Engel, K.J.; Nagel, R. *One-parameter Semigroups for Linear Evolution Equations*. Springer-Verlag: New York, NY, USA, 2000.
32. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Springer-Verlag: New York, NY, USA, 1988.
33. Aubin, J.P.; Cellina, A. *Differential Inclusions*; Berlin Heidelberg: Tokyo, Japan, 1984.
34. Deimling, K. *Multivalued Differential Equations*; Walter de Gruyter: Berlin, Germany, 1992.
35. Gorniewicz, L. *Topological Fixed Point Theory of Multivalued Mappings*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1999; Volume 495.
36. Hu, S.; Papageorgiou, N. *Handbook of Multivalued Analysis*; Wolters Kluwer: Dordrecht, The Netherlands, 1997.
37. Tolstonogov, A.A. *Differential Inclusions in a Banach Space*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2000.
38. Frigon, M. Fixed point results for multivalued contractions on gauge spaces. In *Set Valued Mapping Applications in Nonlinear Analysis*; Taylor & Francis: London, UK, 2002; Volume 4, pp. 175–181.
39. Frigon, M. Fixed point and continuation results for contractions in metric and gauge spaces. In *Fixed Point Theory & Applications*; Banach Center Publications: Warsaw, Poland, 2007; Volume 77, pp. 89–114.
40. Castaing, C.; Valadier, M. *Convex Analysis and Measurable Multifunctions*; Springer-Verlag: New York, NY, 1977; Volume 580.
41. Freidman, A. *Partial Differential Equations*; Holt McDougal: New York, NY, USA, 1969.