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Methods for Multiple Attribute Decision Making with Interval-Valued Pythagorean Fuzzy Information

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Abstract: Interval-valued Pythagorean fuzzy numbers (IVPFNs) can easily describe the incomplete and indeterminate information by degrees of membership and non-membership, and the Hamy mean (HM) operator and dual HM (DHM) operators are a good tool for dealing with multiple attribute decision making (MADM) problems because it can capture the interrelationship among the multi-input arguments. Motivated by the studies regarding the HM operator and dual HM operator, we expand the HM operator and dual HM (DMM) operator to process the interval-valued Pythagorean fuzzy numbers (IVPFNs) and then to solve the MADM problems. Firstly, we propose some HM and DHM operators with IVPFNs. Moreover, we present some new methods to solve MADM problems with the IVPFNs. Finally, an applicable example is given.

Keywords: multiple attribute decision making (MADM); Hamy mean (HM) operator; dual Hamy mean (DHM) operator; interval-valued Pythagorean fuzzy numbers (IVPFNs)

1. Introduction

Atanassov [1] gave the intuitionistic fuzzy set (IFS) based on the fuzzy set [2] such that their sum is not greater than one. After their existence, researchers have applied these theories in different disciplines [3–17] and found that they are more profitable to handle the uncertainties during the analysis. The above theories have been successfully defined, but in some cases, it is not possible to handle the situation by IFS. For instance, if a decision maker (DM) may take the membership degrees of any element as 0.8 and 0.5, then, clearly their sum is not less than one. Hence, under such cases, IFS can have some deficiencies. In order to resolve this, Pythagorean fuzzy set (PFS) [18,19], an extension of IFSs, has emerged as a good tool for describing the indeterminacy in uncertain multiple attribute decision making (MADM). For this set, the condition of the sum of the degrees that is replaced with their sum of squares is less than one; hence, the PFS is more general than the IFS. Further, it is clear that $0.8^2 + 0.5^2 \leq 1$; hence, PFS stands for such cases. After this discovery, Zhang and Xu [20] presented the PFS TOPSIS for MADM. Zhang [21] presented a PFS similarity measure for solving MADM. Peng and Yang [22] proposed some fundamental properties for PFNs. Reformat and Yager [23] used the PFSs in solving the recommender system. Zeng et al. [24] proposed a hybrid method for Pythagorean fuzzy multiple-criteria decision making. Garg [25,26] proposed some generalized PFS aggregation operators based on Einstein Operations. Zhang [27] extended the PFS to the interval-valued PFSs (IVPFSs). Garg [28] presented some aggregation operators with interval-valued Pythagorean fuzzy numbers (IVPFNs). Also, a new accuracy function has been presented to rank the IVPFNs. However, a novel accuracy function [28], correlation coefficient [29], improved accuracy function [30], and improved score function [31] have been defined under the PFS and IVPFS, and have been used to solve the

MADM. Recently, some confidence operators with PFNs have been developed by Garg [32]. Other scholars also studied the MADM under the PFS or IVPFS [24,27,30,31,33–41].

Both Bonferroni mean (BM) operators [42–47] and the Heronian mean (HM) [48–53] operators consider the interrelationships of aggregated arguments. The Hamy mean (HM) [54] operator can consider interrelationships among any number of arguments; both BM operator and Maclaurin symmetric mean [55] are the special cases of HM operator. Qin and Liu [56] proposed some HM operators with 2-tuple linguistic information. Because IVPFNs can easily describe the fuzzy information, and the HM operator and dual HM (DHM) operator can capture interrelationships among any number of arguments, it is necessary to expand the HM and DHM operator to deal with the IVPFNs. The purpose of this paper is to propose some HM and DHM operators with IVPFNs, then to study some properties of these operators, and apply them to solve MADM problems with IVPFNs.

To achieve this goal, the rest of this paper is set out as follows. Section 2 introduces some basic definitions of IVPFSs. In Section 3, we propose some HM and DHM operators with IVPFNs. In Section 4, we propose two MADM methods for IVPFNs with the interval-valued Pythagorean fuzzy weighted Hamy mean (IVPFWHM) operator and interval-valued Pythagorean fuzzy weighted dual Hamy mean (IVPFWDHM) operator. In Section 5, an illustrative example is given. In Section 6, we give some conclusions of this study.

2. Basic Concepts

In this section, we introduce some fundamental concepts of IVPFSs and HM, which will be used in the next section. These concepts are based on a fixed set X will be used on the next sections.

2.1. Pythagorean Fuzzy Set (PFS)

Definition 1 ([18,19]). A PFS P is defined as

$$P = \{ \langle x, (\mu_P(x), \nu_P(x)) \rangle | x \in X \} \quad (1)$$

where the functions $\mu_P : X \rightarrow [0, 1]$ and $\nu_P : X \rightarrow [0, 1]$ defines the degrees of membership and non-membership of the element $x \in X$ to P , such that for each x , the condition $(\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$, holds.

Definition 2 ([33]). The $T = (\mu, \nu)$ be called as Pythagorean fuzzy number (PFN) and defined the score and accuracy functions as $S(T) = \mu^2 - \nu^2$ and $H(T) = \mu^2 + \nu^2$. In order to compare two or more PFNs T_1 and T_2 , a comparison law is defined as:

- (1) If $S(T_1) < S(T_2)$, then $T_1 < T_2$;
- (2) If $S(T_1) = S(T_2)$, then,
 - (i) If $H(T_1) = H(T_2)$, then $T_1 = T_2$;
 - (ii) If $H(T_1) < H(T_2)$, then $T_1 < T_2$.

Example 1. Let $t_1 = (0.7, 0.5)$, $t_2 = (0.5, 0.1)$, $t_3 = (0.4, 0.1)$ be three PFNs, according to Definitions 1 and 2, we get $S(t_1) = 0.7^2 - 0.5^2 = 0.2400$, $S(t_2) = 0.5^2 - 0.1^2 = 0.2400$, $S(t_3) = 0.4^2 - 0.1^2 = 0.1500$, $S(t_1) = S(t_2) > S(t_3)$, $H(t_1) = 0.7^2 + 0.5^2 = 0.7400$, $H(t_2) = 0.5^2 + 0.1^2 = 0.2600$, $S(t_1) > S(t_2) > S(t_3)$; therefore, $t_1 > t_2 > t_3$.

2.2. Interval-Valued Pythagorean Fuzzy Set (IVPFS)

Zhang [27] extended the PFS to the IVPFSs which is defined, as follows, over the fixed set X .

Definition 3 ([27]). An IVPFS \tilde{b} is defined as:

$$\tilde{b} = \{ \langle x, (\mu_{\tilde{b}}(x), \nu_{\tilde{b}}(x)) \rangle | x \in X \} \quad (2)$$

where $\mu_{\tilde{b}}(x) = [\mu_{\tilde{b}}^Z(x), \mu_{\tilde{b}}^Y(x)]$, $\nu_{\tilde{b}}(x) = [\nu_{\tilde{b}}^Z(x), \nu_{\tilde{b}}^Y(x)]$ are the interval numbers of $[0, 1]$ with the condition $0 \leq (\mu_{\tilde{b}}^Y(x))^2 + (\nu_{\tilde{b}}^Y(x))^2 \leq 1$, $\forall x \in X$. The pair $\tilde{a} = ([\mu_{\tilde{b}}^Z, \mu_{\tilde{b}}^Y], [\nu_{\tilde{b}}^Z, \nu_{\tilde{b}}^Y])$ is called an IVPF number (IVPFN), where $\mu_{\tilde{b}}, \nu_{\tilde{b}} \subseteq [0, 1]$ and $(\mu_{\tilde{b}}^Y)^2 + (\nu_{\tilde{b}}^Y)^2 \leq 1$.

Definition 4 ([28]). For three IVPFNs $\tilde{b}_1 = ([\mu_{\tilde{b}_1}^Z, \mu_{\tilde{b}_1}^Y], [\nu_{\tilde{b}_1}^Z, \nu_{\tilde{b}_1}^Y])$, $\tilde{b}_2 = ([\mu_{\tilde{b}_2}^Z, \mu_{\tilde{b}_2}^Y], [\nu_{\tilde{b}_2}^Z, \nu_{\tilde{b}_2}^Y])$, and $\tilde{b} = ([\mu_{\tilde{b}}^Z, \mu_{\tilde{b}}^Y], [\nu_{\tilde{b}}^Z, \nu_{\tilde{b}}^Y])$, the basic operational laws are defined as follows:

$$\begin{aligned} \tilde{b}_1 \oplus \tilde{b}_2 &= \left(\left[\frac{\sqrt{(\mu_{\tilde{b}_1}^Z)^2 + (\mu_{\tilde{b}_2}^Z)^2 - (\mu_{\tilde{b}_1}^Z)^2 (\mu_{\tilde{b}_2}^Z)^2}}{\sqrt{(\mu_{\tilde{b}_1}^Y)^2 + (\mu_{\tilde{b}_2}^Y)^2 - (\mu_{\tilde{b}_1}^Y)^2 (\mu_{\tilde{b}_2}^Y)^2}}, \frac{[\nu_{\tilde{b}_1}^Z \nu_{\tilde{b}_2}^Z, \nu_{\tilde{b}_1}^Y \nu_{\tilde{b}_2}^Y]}{\sqrt{(\mu_{\tilde{b}_1}^Y)^2 + (\mu_{\tilde{b}_2}^Y)^2 - (\mu_{\tilde{b}_1}^Y)^2 (\mu_{\tilde{b}_2}^Y)^2}} \right]; \right. \\ \tilde{b}_1 \otimes \tilde{b}_2 &= \left([\mu_{\tilde{b}_1}^Z \mu_{\tilde{b}_2}^Z, \mu_{\tilde{b}_1}^Y \mu_{\tilde{b}_2}^Y], \left[\frac{\sqrt{(\nu_{\tilde{b}_1}^Z)^2 + (\nu_{\tilde{b}_2}^Z)^2 - (\nu_{\tilde{b}_1}^Z)^2 (\nu_{\tilde{b}_2}^Z)^2}}{\sqrt{(\nu_{\tilde{b}_1}^Y)^2 + (\nu_{\tilde{b}_2}^Y)^2 - (\nu_{\tilde{b}_1}^Y)^2 (\nu_{\tilde{b}_2}^Y)^2}}, \right] \right); \\ k\tilde{b} &= \left(\left[\sqrt{1 - (1 - (\mu_{\tilde{b}}^Z)^2)^k}, \sqrt{1 - (1 - (\mu_{\tilde{b}}^Y)^2)^k} \right], [\nu_{\tilde{b}}^Z]^k, [\nu_{\tilde{b}}^Y]^k \right), k > 0; \\ (\tilde{b})^k &= \left([(\mu_{\tilde{b}}^Z)^k, (\mu_{\tilde{b}}^Y)^k], \left[\sqrt{1 - (1 - (\nu_{\tilde{b}}^Z)^2)^k}, \sqrt{1 - (1 - (\nu_{\tilde{b}}^Y)^2)^k} \right] \right), k > 0; \\ \tilde{b}^c &= ([\nu_{\tilde{b}}^Z, \nu_{\tilde{b}}^Y], [\mu_{\tilde{b}}^Z, \mu_{\tilde{b}}^Y]). \end{aligned}$$

Example 2. Suppose that $t_1 = ([0.3, 0.6], [0.1, 0.4])$, $t_2 = ([0.3, 0.7], [0.2, 0.4])$, and $t_3 = ([0.5, 0.7], [0.3, 0.5])$ be three IVPFNs, and $k = 0.5$, then we have:

$$\begin{aligned} t_1 \oplus t_2 &= \left(\left[\frac{\sqrt{0.3^2 + 0.3^2 - 0.3^2 \times 0.3^2}}{\sqrt{0.6^2 + 0.7^2 - 0.6^2 \times 0.7^2}}, \frac{[0.1 \times 0.2, 0.4 \times 0.4]}{\sqrt{0.6^2 + 0.7^2 - 0.6^2 \times 0.7^2}} \right]; \right. \\ &= ([0.4146, 0.8207], [0.0200, 0.1600]) \end{aligned}$$

$$\begin{aligned} t_1 \otimes t_2 &= ([0.3 \times 0.3, 0.6 \times 0.7], \left[\frac{\sqrt{0.1^2 + 0.2^2 - 0.1^2 \times 0.2^2}}{\sqrt{0.4^2 + 0.4^2 - 0.4^2 \times 0.4^2}}, \right]) \\ &= ([0.0900, 0.4200], [0.2227, 0.5426]) \end{aligned}$$

$$\begin{aligned} kt_3 &= \left(\left[\sqrt{1 - (1 - 0.5)^{0.5}}, \sqrt{1 - (1 - 0.7)^{0.5}} \right], [0.3^{0.5}, 0.5^{0.5}] \right) \\ &= ([0.5412, 0.6725], [0.5477, 0.7071]) \end{aligned}$$

$$\begin{aligned} (t_3)^k &= ([0.5^{0.5}, 0.7^{0.5}], \left[\sqrt{1 - (1 - 0.3)^{0.5}}, \sqrt{1 - (1 - 0.5)^{0.5}} \right]) \\ &= ([0.7071, 0.8367], [0.4042, 0.5412]) \end{aligned}$$

$$t_1^c = ([0.3, 0.5], [0.5, 0.7])$$

Based on the Definition 4, Garg [28] derived the following properties easily.

Theorem 1. Let $\tilde{b}_1 = ([\mu_{\tilde{b}_1}^Z, \mu_{\tilde{b}_1}^Y], [v_{\tilde{b}_1}^Z, v_{\tilde{b}_1}^Y])$ and $\tilde{b}_2 = ([\mu_{\tilde{b}_2}^Z, \mu_{\tilde{b}_2}^Y], [v_{\tilde{b}_2}^Z, v_{\tilde{b}_2}^Y])$ be two IVPFNs, and $k, k_1, k_2 > 0$, be three real numbers, then:

$$\begin{aligned}\tilde{b}_1 \oplus \tilde{b}_2 &= \tilde{b}_2 \oplus \tilde{b}_1; \\ \tilde{b}_1 \otimes \tilde{b}_2 &= \tilde{b}_2 \otimes \tilde{b}_1; \\ k(\tilde{b}_1 \oplus \tilde{b}_2) &= k\tilde{b}_1 \oplus k\tilde{b}_2; \\ (\tilde{b}_1 \otimes \tilde{b}_2)^k &= (\tilde{b}_1)^k \otimes (\tilde{b}_2)^k; \\ k_1\tilde{b}_1 \oplus k_2\tilde{b}_1 &= (k_1 + k_2)\tilde{b}_1; \\ ((\tilde{b}_1)^{k_1})^{k_2} &= (\tilde{b}_1)^{k_1 k_2}.\end{aligned}$$

Definition 5. For an IVPFN $\tilde{b} = ([\mu_{\tilde{b}}^Z, \mu_{\tilde{b}}^Y], [v_{\tilde{b}}^Z, v_{\tilde{b}}^Y])$, the score and accuracy functions of it are defined as $S(\tilde{b}) = \frac{1}{4} \left[\left(1 + (\mu_{\tilde{b}}^Z)^2 - (v_{\tilde{b}}^Z)^2 \right) + \left(1 + (\mu_{\tilde{b}}^Y)^2 - (v_{\tilde{b}}^Y)^2 \right) \right]$ and $H(\tilde{b}) = \frac{(\mu_{\tilde{b}}^Z)^2 + (\mu_{\tilde{b}}^Y)^2 + (v_{\tilde{b}}^Z)^2 + (v_{\tilde{b}}^Y)^2}{2}$, respectively. Further, in order to compare two different IVPFNs \tilde{b}_1 and \tilde{b}_2 , an order relation is defined as:

- (1) if $S(\tilde{b}_1) < S(\tilde{b}_2)$, then $\tilde{b}_1 < \tilde{b}_2$
- (2) if $S(\tilde{b}_1) = S(\tilde{b}_2)$, then,
 - (i) if $H(\tilde{b}_1) = H(\tilde{b}_2)$, then $\tilde{b}_1 = \tilde{b}_2$.
 - (ii) if $H(\tilde{b}_1) < H(\tilde{b}_2)$, then $\tilde{b}_1 < \tilde{b}_2$.

Example 3. Let $t_1 = ([0.4, 0.7], [0, 0.5])$, $t_2 = ([0.5, 0.5], [0.1, 0.3])$, and $t_3 = ([0.3, 0.4], [0.2, 0.5])$ be three IVPFNs, according to Definition 5, we get:

$$\begin{aligned}S(t_1) &= \frac{(1 + 0.4^2 - 0^2) + (1 + 0.7^2 - 0.5^2)}{4} = 0.6000, \\ S(t_2) &= \frac{(1 + 0.5^2 - 0.1^2) + (1 + 0.5^2 - 0.5^2)}{4} = 0.6000, \\ S(t_3) &= \frac{(1 + 0.3^2 - 0.2^2) + (1 + 0.4^2 - 0.5^2)}{4} = 0.4900, S(t_1) = S(t_2) > S(t_3). \\ H(t_1) &= \frac{0.4^2 + 0.7^2 + 0^2 + 0.5^2}{2} = 0.4500, \\ H(t_2) &= \frac{0.5^2 + 0.5^2 + 0.1^2 + 0.3^2}{2} = 0.3000\end{aligned}$$

$S(t_1) > S(t_2) > S(t_3)$, therefore $t_1 > t_2 > t_3$.

2.3. HM Operator

Definition 6 ([54]). The HM operator is defined as the following:

$$HM^{(x)}(b_1, b_2, \dots, b_k) = \frac{\sum_{1 \leq i_1 < \dots < i_x \leq k} \left(\prod_{j=1}^x b_{i_j} \right)^{\frac{1}{x}}}{C_k^x} \quad (3)$$

where x is a parameter and x are x integer values taken from the set $\{1, 2, \dots, k\}$ of k integer values, and C_k^x denotes the binomial coefficient and $C_k^x = \frac{k!}{x!(k-x)!}$.

The properties of the operator are shown as follows:

- (i) when $b_i = b (i = 1, 2, \dots, k)$, $HM^{(x)}(b_1, b_2, \dots, b_k) = b$;
- (ii) when $b_i \leq \pi_i (i = 1, 2, \dots, k)$, $HM^{(x)}(b_1, b_2, \dots, b_k) \leq HM^{(x)}(\pi_1, \pi_2, \dots, \pi_k)$;
- (iii) when $\min\{b_i\} \leq HM^{(x)}(b_1, b_2, \dots, b_k) \leq \max\{b_i\}$.

Two particular cases of the HM operator are given as follows:

- (1) when $x = 1$, $HM^{(k)}(b_1, b_2, \dots, b_k) = \frac{1}{k} \sum_{i=1}^k b_i$, it becomes the arithmetic mean operator.
- (2) when $x = k$, $HM^{(k)}(b_1, b_2, \dots, b_k) = \left(\prod_{i=1}^k b_i \right)^{\frac{1}{k}}$, it becomes the arithmetic mean operator. Which is the arithmetic averaging operator.

3. Certain HM and DHM Operators with IVPFNs

In this section, we shall develop certain Hamy mean operators with IVPFNs.

3.1. IVPFHM Operator

The HM operator has usually been utilized in the situation of an interaction relationship. Next, we extend the HM operator to IVPFS. From Definitions 4 and 6, we can obtain:

Definition 7. Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y]) (i = 1, 2, \dots, k)$ be a set of IVPFNs, then the interval-valued Pythagorean Fuzzy Hamy mean (IVPFHM) operator is defined as:

$$IVPFHM^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \frac{\bigoplus_{1 \leq i_1 < \dots < i_x \leq k} \left(\bigotimes_{j=1}^x \tilde{b}_{i_j} \right)^{\frac{1}{x}}}{C_k^x} \quad (4)$$

where x is a parameter and $x = 1, 2, \dots, k$, i_1, i_2, \dots, i_x are x integer values taken from the set $\{1, 2, \dots, k\}$ of k integer values, and C_k^x denotes the binomial coefficient and $C_k^x = \frac{k!}{x!(k-x)!}$.

Based on the operations of the IVPFN described, we can obtain Theorem 2.

Theorem 2. Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y]) (i = 1, 2, \dots, k)$ be a set of IVPFNs, then their aggregated result by utilizing IVPFHM operator is also an IVPFN, and

$$IVPFHM^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \frac{\bigoplus_{1 \leq i_1 < \dots < i_x \leq k} \left(\bigotimes_{j=1}^x \tilde{b}_{i_j} \right)^{\frac{1}{x}}}{C_k^x} = \left(\left[\sqrt{\frac{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}}}{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}}} \right]^{\frac{1}{C_k^x}}, \left[\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}}, \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \right] \right) \quad (5)$$

Proof.

$$\bigotimes_{j=1}^x \tilde{b}_{i_j} = \left(\left[\prod_{j=1}^x (\mu_{i_j}^Z), \prod_{j=1}^x (\mu_{i_j}^Y) \right], \left[\sqrt{1 - \prod_{j=1}^x (1 - (v_{i_j}^Z)^2)}, \sqrt{1 - \prod_{j=1}^x (1 - (v_{i_j}^Y)^2)} \right] \right) \quad (6)$$

$$\left(\bigotimes_{j=1}^x \tilde{b}_{ij}\right)^{\frac{1}{x}} = \left(\left[\left(\prod_{j=1}^x \mu_{ij}^Z \right)^{\frac{1}{x}}, \left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right], \left[\sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Z)^2) \right)^{\frac{1}{x}}}, \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Y)^2) \right)^{\frac{1}{x}}} \right] \right) \quad (7)$$

Thereafter,

$$\begin{aligned} & \left(\bigoplus_{1 \leq i_1 < \dots < i_x \leq k} \left(\bigotimes_{j=1}^x \tilde{a}_{ij} \right)^{\frac{1}{x}} \right)^{\frac{1}{C_k^x}} \\ &= \left(\left[\sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Z \right)^{\frac{1}{x}} \right)^2 \right)} \right]^{\frac{1}{C_k^x}}, \left[\sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right)} \right]^{\frac{1}{C_k^x}} \right] \right)^{\frac{1}{C_k^x}} \quad (8) \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{\left(\bigoplus_{1 \leq i_1 < \dots < i_x \leq k} \left(\bigotimes_{j=1}^x \tilde{b}_{ij} \right)^{\frac{1}{x}} \right)^{\frac{1}{C_k^x}}}{C_k^x} = \\ & \left(\left[\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Z \right)^{\frac{1}{x}} \right)^2 \right) \right)} \right]^{\frac{1}{C_k^x}}, \left[\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)} \right]^{\frac{1}{C_k^x}} \right] \right)^{\frac{1}{C_k^x}} \quad (9) \end{aligned}$$

And then, we can know:

$$0 \leq \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)}^{\frac{1}{C_k^x}} \leq 1 \quad (10)$$

$$0 \leq \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Y)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \leq 1 \quad (11)$$

We can obtain $(\mu_p(x))^2 + (v_p(x))^2 \leq 1$ from the definition of IVPFS, so

$$\begin{aligned} & \left(\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)} \right)^{\frac{1}{C_k^x}} + \left(\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Y)^2) \right)^{\frac{1}{x}}} \right) \right)^{\frac{1}{C_k^x}} \\ & \leq 1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Y)^2) \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{C_k^x}} + \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Y)^2) \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{C_k^x}} = 1 \quad (12) \end{aligned}$$

We complete the proof. \square

In the following, we give some properties of the IVPFMM operator.

Property 1. (Idempotency.) Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y]) = \tilde{b} = ([\mu^Z, \mu^Y], [v^Z, v^Y])$ ($i = 1, 2, 3, \dots, k$), then

$$\text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \tilde{b} \quad (13)$$

Proof.

$$\begin{aligned}
 & \text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) \\
 &= \left(\left[\sqrt{\frac{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}}}{\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}}} \right]^{\frac{1}{C_k^x}}, \left[\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Z)^2) \right)^{\frac{1}{x}}}} \right)^{\frac{1}{C_k^x}}, \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Y)^2) \right)^{\frac{1}{x}}}} \right)^{\frac{1}{C_k^x}} \right]^{\frac{1}{C_k^x}} \right) \\
 &= \left(\left[\sqrt{\frac{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} (1 - (\mu_i^Z)^2) \right)^{\frac{1}{C_k^x}}}}{\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} (1 - (\mu_i^Y)^2) \right)^{\frac{1}{C_k^x}}}} \right]^{\frac{1}{C_k^x}}, \left[\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - (1 - (v_i^Z)^2)} \right)^{\frac{1}{C_k^x}}, \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - (1 - (v_i^Y)^2)} \right)^{\frac{1}{C_k^x}} \right]^{\frac{1}{C_k^x}} \right) \\
 &= \left(\left[\sqrt{1 - (1 - (\mu_i^Z)^2)}, \sqrt{1 - (1 - (1 - (\mu_i^Y)^2))} \right], \left[\sqrt{1 - (1 - (v_i^Z)^2)}, \sqrt{1 - (1 - (v_i^Y)^2)} \right] \right) \\
 &= \left(\left[\sqrt{1 - (1 - (\mu^Z)^2)}, \sqrt{1 - (1 - (\mu^Y)^2)} \right], \left[\sqrt{1 - (1 - (v^Z)^2)}, \sqrt{1 - (1 - (v^Y)^2)} \right] \right) \\
 &= ([\mu^Z, \mu^Y], [v^Z, v^Y]) = \tilde{b}
 \end{aligned} \tag{14}$$

□

Property 2. (Monotonicity.) Let $\tilde{b}_i = ([\mu_{i_j}^Z, \mu_{i_j}^Y], [v_{i_j}^Z, v_{i_j}^Y])$ and $\tilde{c}_i = ([\mu_{\theta_j}^Z, \mu_{\theta_j}^Y], [v_{\theta_j}^Z, v_{\theta_j}^Y])$ ($i = 1, 2, \dots, k$) be two sets of IVPFNs, If $(\mu_{i_j}^Z)^2 + (\mu_{i_j}^Y)^2 \leq (\mu_{\theta_j}^Z)^2 + (\mu_{\theta_j}^Y)^2$ and $(v_{i_j}^Z)^2 + (v_{i_j}^Y)^2 \geq (v_{\theta_j}^Z)^2 + (v_{\theta_j}^Y)^2$, then,

$$\text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) \leq \text{IVPFHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k) \tag{15}$$

Proof.

$$\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \leq \left(\prod_{j=1}^x \mu_{\theta_j}^Y \right)^{\frac{1}{x}} \tag{16}$$

$$\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right) \geq \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{\theta_j}^Y \right)^{\frac{1}{x}} \right)^2 \right) \tag{17}$$

Therefore,

$$\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}} \leq \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{\theta_j}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}} \tag{18}$$

Similarly, we also can obtain

$$\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \geq \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{\theta_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \tag{19}$$

And

$$\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}} \leq \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{\theta_j}^Z \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}} \tag{20}$$

$$\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \geq \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{\theta_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \quad (21)$$

then, the proof is completed. Then,

If $(\mu_{i_j}^Z)^2 + (\mu_{i_j}^Y)^2 < (\mu_{\theta_j}^Z)^2 + (\mu_{\theta_j}^Y)^2$ and $(v_{i_j}^Z)^2 + (v_{i_j}^Y)^2 > (v_{\theta_j}^Z)^2 + (v_{\theta_j}^Y)^2$ then

$$\text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) < \text{IVPFHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k);$$

If $(\mu_{i_j}^Z)^2 + (\mu_{i_j}^Y)^2 < (\mu_{\theta_j}^Z)^2 + (\mu_{\theta_j}^Y)^2$ and $(v_{i_j}^Z)^2 + (v_{i_j}^Y)^2 = (v_{\theta_j}^Z)^2 + (v_{\theta_j}^Y)^2$ then

$$\text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) < \text{IVPFHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k);$$

If $(\mu_{i_j}^Z)^2 + (\mu_{i_j}^Y)^2 = (\mu_{\theta_j}^Z)^2 + (\mu_{\theta_j}^Y)^2$ and $(v_{i_j}^Z)^2 + (v_{i_j}^Y)^2 > (v_{\theta_j}^Z)^2 + (v_{\theta_j}^Y)^2$ then

$$\text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) < \text{IVPFHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k);$$

If $(\mu_{i_j}^Z)^2 + (\mu_{i_j}^Y)^2 = (\mu_{\theta_j}^Z)^2 + (\mu_{\theta_j}^Y)^2$ and $(v_{i_j}^Z)^2 + (v_{i_j}^Y)^2 = (v_{\theta_j}^Z)^2 + (v_{\theta_j}^Y)^2$ then

$$\text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \text{IVPFHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k).$$

□

Property 3. (Boundedness.) Let $\tilde{b}_i = ([\mu_{i_j}^Z, \mu_{i_j}^Y], [v_{i_j}^Z, v_{i_j}^Y])$, $\tilde{b}^+ = ([\mu_{\max i_j}^Z, \mu_{\max i_j}^Y], [v_{\min i_j}^Z, v_{\min i_j}^Y])$ and $\tilde{b}^- = ([\mu_{\min i_j}^Z, \mu_{\min i_j}^Y], [v_{\max i_j}^Z, v_{\max i_j}^Y])$ ($i = 1, 2, \dots, k$) be a set of IVPFNs. According to the process of property of monotonicity and idempotency, it is easy to obtain that,

$$\tilde{b}_i^- < \text{IVPFHM}_\omega^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) < \tilde{b}_i^+ \quad (22)$$

Example 4. Let $\tilde{b}_1 = ([0.1, 0.4], [0.2, 0.5])$, $\tilde{b}_2 = ([0.4, 0.6], [0.3, 0.4])$, $\tilde{b}_3 = ([0.5, 0.7], [0.2, 0.3])$, $\tilde{b}_4 = ([0.3, 0.5], [0.2, 0.4])$ be four IVPFNs. Then we use the proposed IVPFHM operator to aggregate four IVPFNs. (suppose $x = 2$):

$$\begin{aligned} & \text{IVPFHM}^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) \\ &= \left(\left[\sqrt{\frac{1 - \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(1 - \left(\left(\prod_{j=1}^2 \mu_{i_j}^Z \right)^{\frac{1}{2}} \right)^2 \right) \right)^{\frac{1}{C_4^2}}}}{\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(1 - \left(\left(\prod_{j=1}^2 \mu_{i_j}^Y \right)^{\frac{1}{2}} \right)^2 \right) \right)^{\frac{1}{C_4^2}}}} \right], \left[\left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \sqrt{1 - \left(\prod_{j=1}^2 (1 - (v_{i_j}^Z)^2) \right)^{\frac{1}{2}}} \right)^{\frac{1}{C_4^2}}, \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \sqrt{1 - \left(\prod_{j=1}^2 (1 - (v_{i_j}^Y)^2) \right)^{\frac{1}{2}}} \right)^{\frac{1}{C_4^2}} \right] \right) \end{aligned}$$

$$= \left(\left[\begin{array}{c} \sqrt{1 - \left(\frac{(1 - 0.1 \times 0.4) \times (1 - 0.1 \times 0.5) \times (1 - 0.1 \times 0.3)}{(1 - 0.4 \times 0.5) \times (1 - 0.4 \times 0.3) \times (1 - 0.5 \times 0.3)} \right)^{\frac{1}{6}}}, \\ \sqrt{1 - \left(\frac{(1 - 0.4 \times 0.6) \times (1 - 0.4 \times 0.7) \times (1 - 0.4 \times 0.5)}{(1 - 0.6 \times 0.7) \times (1 - 0.6 \times 0.5) \times (1 - 0.7 \times 0.5)} \right)^{\frac{1}{6}}} \end{array} \right], \right. \\ \left. \left[\begin{array}{c} \left(\frac{\sqrt{1 - ((1 - 0.2^2) \times (1 - 0.3^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.2^2) \times (1 - 0.2^2))^{\frac{1}{2}}}}{\sqrt{1 - ((1 - 0.2^2) \times (1 - 0.2^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.3^2) \times (1 - 0.2^2))^{\frac{1}{2}}}} \right)^{\frac{1}{6}}, \\ \left(\frac{\sqrt{1 - ((1 - 0.5^2) \times (1 - 0.4^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.5^2) \times (1 - 0.3^2))^{\frac{1}{2}}}}{\sqrt{1 - ((1 - 0.5^2) \times (1 - 0.4^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.4^2) \times (1 - 0.3^2))^{\frac{1}{2}}}} \right)^{\frac{1}{6}} \end{array} \right] \right) \\ = ([0.3171, 0.5497], [0.2097, 0.4036])$$

Finally, we get $IVPFHM^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = ([0.3171, 0.5497], [0.1317, 0.2401])$.

3.2. IVPFWMH Operator

It can be seen that the IVPFHM operator doesn't consider the importance of the aggregated arguments. To overcome the limitation of IVPFHM, we develop the interval-valued Pythagorean fuzzy weighted HM (IVPFWMH) operator as follows.

Definition 8. Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y]) (i = 1, 2, \dots, k)$ be a set of IVPFNs with weight vector being $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$, $\omega_i \in [0, 1]$ and $\sum_{k=1}^k \omega_i = 1$, then the IVPFWMH operator is given as:

$$IVPFWMH_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \begin{cases} \frac{1_{\leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{i_j} \right) \left(\bigotimes_{j=1}^x \tilde{b}_{i_j} \right)^{\frac{1}{x}}}{C_{k-1}^x} & (1 \leq x < k) \\ \bigotimes_{i=1}^x \tilde{b}_i^{1-\frac{\omega_i}{k-1}} & (x = k) \end{cases} \quad (23)$$

Based on the operations of the IVPFN described, we can obtain Theorem 3.

Theorem 3. Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y]) (i = 1, 2, \dots, k)$ be a collection of IVPFNs, then their aggregated result of IVPFWMH operator is also an IVPFN, and

$$IVPFWMH_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \frac{1_{\leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{i_j} \right) \left(\bigotimes_{j=1}^x \tilde{b}_{i_j} \right)^{\frac{1}{x}}}{C_{k-1}^x} \\ = \left(\left[\begin{array}{c} \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{1 - \sum_{j=1}^x \omega_{i_j}} \right)^{\frac{1}{C_{k-1}^x}}}, \\ \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{1 - \sum_{j=1}^x \omega_{i_j}} \right)^{\frac{1}{C_{k-1}^x}}}, \\ \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{1 - \sum_{j=1}^x \omega_{i_j}} \right)^{\frac{1}{C_{k-1}^x}}, \\ \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{1 - \sum_{j=1}^x \omega_{i_j}} \right)^{\frac{1}{C_{k-1}^x}} \end{array} \right] \right) \quad (1 \leq x < k) \quad (24)$$

or

$$\begin{aligned} \text{IVPFWHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) &= \bigotimes_{i=1}^x \tilde{b}_i^{\frac{1-\omega_i}{k-1}} \\ &= \left(\left[\prod_{j=1}^k (\mu_{ij}^Z)^{\frac{1-\omega_i}{k-1}}, \prod_{j=1}^k (\mu_{ij}^Y)^{\frac{1-\omega_i}{k-1}} \right], \right. \\ &\quad \left. \left[\sqrt{1 - \prod_{i=1}^k (1 - (v_i^Z)^2)^{\frac{1-\omega_i}{k-1}}}, \sqrt{1 - \prod_{i=1}^k (1 - (v_i^Y)^2)^{\frac{1-\omega_i}{k-1}}} \right] \right) (x = k) \end{aligned} \quad (25)$$

Proof.

$$\left(\bigotimes_{j=1}^x \tilde{b}_{ij} \right)^{\frac{1}{x}} = \left(\left[\left(\prod_{j=1}^x \mu_{ij}^Z \right)^{\frac{1}{x}}, \left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right], \left[\sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Z)^2) \right)^{\frac{1}{x}}}, \sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Y)^2) \right)^{\frac{1}{x}}} \right] \right) \quad (26)$$

Thereafter,

$$\begin{aligned} \left(1 - \sum_{j=1}^x \omega_{ij} \right) \left(\bigotimes_{j=1}^x \tilde{b}_{ij} \right)^{\frac{1}{x}} &= \\ \left(\left[\sqrt{1 - \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{ij})}} \right], \left[\sqrt{1 - \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{ij})}} \right] \right) & \quad (27) \end{aligned}$$

Moreover,

$$\begin{aligned} \bigoplus_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{ij} \right) \left(\bigotimes_{j=1}^x \tilde{b}_{ij} \right)^{\frac{1}{x}} &= \\ \left(\left[\sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{ij})}} \right], \left[\sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{ij})}} \right] \right) & \quad (28) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\bigoplus_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{ij} \right) \left(\bigotimes_{j=1}^x \tilde{b}_{ij} \right)^{\frac{1}{x}}}{C_{k-1}^x} &= \\ \left(\left[\sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{ij})} \right]^{\frac{1}{C_{k-1}^x}}, \left[\sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{ij})} \right]^{\frac{1}{C_{k-1}^x}} \right) & \quad (29) \end{aligned}$$

For the second case, when $(x = k)$, we get

$$\tilde{b}_i^{\frac{1-\omega_i}{k-1}} = \left(\left[(\mu_i^Z)^{\frac{1-\omega_i}{k-1}}, (\mu_i^Y)^{\frac{1-\omega_i}{k-1}} \right], \left[\sqrt{1 - (1 - (v_i^Z)^2)^{\frac{1-\omega_i}{k-1}}}, \sqrt{1 - (1 - (v_i^Y)^2)^{\frac{1-\omega_i}{k-1}}} \right] \right) \quad (30)$$

Then,

$$\bigotimes_{i=1}^k \tilde{b}_i^{\frac{1-\omega_i}{k-1}} = \left(\left[\prod_{i=1}^k \left((\mu_i^Z)^{\frac{1-\omega_i}{k-1}} \right), \prod_{i=1}^k \left((\mu_i^Y)^{\frac{1-\omega_i}{k-1}} \right) \right], \left[\sqrt{1 - \prod_{i=1}^k \left(1 - (v_i^Z)^2 \right)^{\frac{1-\omega_i}{k-1}}}, \sqrt{1 - \prod_{i=1}^k \left(1 - (v_i^Y)^2 \right)^{\frac{1-\omega_i}{k-1}}} \right] \right) \quad (31)$$

and we can easily obtain,

$$0 \leq \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}}} \leq 1 \quad (32)$$

$$0 \leq \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x \left(1 - (v_{i_j}^Y)^2 \right) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \leq 1 \quad (33)$$

and,

$$0 \leq \prod_{i=1}^k \left((\mu_i^Y)^{\frac{1-\omega_i}{k-1}} \right) \leq 1 \quad (34)$$

$$0 \leq \sqrt{1 - \prod_{i=1}^k \left(1 - (v_i^Y)^2 \right)^{\frac{1-\omega_i}{k-1}}} \leq 1 \quad (35)$$

Therefore,

$$\begin{aligned} & \left(\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}}} \right)^2 + \\ & \left(\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x \left(1 - (v_{i_j}^Y)^2 \right) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \right)^2 \leq \\ & \left(1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x \left(1 - (v_{i_j}^Y)^2 \right) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \right)^2 \\ & + \left(\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x \left(1 - (v_{i_j}^Y)^2 \right) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \right)^2 = 1 \end{aligned} \quad (36)$$

For the second case, $x = k$, we can easily prove that it is kept. Therefore, the aggregation result produced by Definition 8 is still an IVPFN. Next, we shall deduce some desirable properties of the IVPFWMH operator. \square

In the following, we give some properties of the IVPFWMH operator.

Property 4. (Idempotency.) Let $\tilde{b}_i = \left([\mu_{i_j}^Z, \mu_{i_j}^Y], [v_{i_j}^Z, v_{i_j}^Y] \right) = ([\mu^Z, \mu^Y], [v^Z, v^Y]) = \tilde{b}$, and weight vector meets $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$, $\omega_i \in [0, 1]$ and $\sum_{i=1}^k \omega_i = 1$, then,

$$\text{IVPFWHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \tilde{b} \quad (37)$$

Property 5. (Monotonicity.) Let $\tilde{b}_i = ([\mu_{i_j}^Z, \mu_{i_j}^Y], [\nu_{i_j}^Z, \nu_{i_j}^Y])$ and $\tilde{c}_i = ([\mu_{\theta_j}^Z, \mu_{\theta_j}^Y], [\nu_{\theta_j}^Z, \nu_{\theta_j}^Y])$ ($i = 1, 2, \dots, k$) be two sets of IVPFNs, and weight vector meets $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$, $\omega_i \in [0, 1]$ and $\sum_{i=1}^k \omega_i = 1$. If $(\mu_{i_j}^Z)^2 + (\mu_{i_j}^Y)^2 \leq (\mu_{\theta_j}^Z)^2 + (\mu_{\theta_j}^Y)^2$ and $(\nu_{i_j}^Z)^2 + (\nu_{i_j}^Y)^2 \geq (\nu_{\theta_j}^Z)^2 + (\nu_{\theta_j}^Y)^2$ then,

$$\text{IVPFWHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) \leq \text{IVPFWHM}_{\omega}^{(x)}(\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k) \quad (38)$$

Property 6. (Boundedness.) Let $\tilde{b}_i = ([\mu_{i_j}^Z, \mu_{i_j}^Y], [\nu_{i_j}^Z, \nu_{i_j}^Y])$, $\tilde{b}^+ = ([\mu_{\max i_j}^Z, \mu_{\max i_j}^Y], [\nu_{\min i_j}^Z, \nu_{\min i_j}^Y])$ and $\tilde{b}^- = ([\mu_{\min i_j}^Z, \mu_{\min i_j}^Y], [\nu_{\max i_j}^Z, \nu_{\max i_j}^Y])$ ($i = 1, 2, \dots, k$) be a set of IVPFNs with weight vector being $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$, $\omega_i \in [0, 1]$, $\sum_{i=1}^k \omega_i = 1$, because of Property 5, then,

$$\tilde{b}_i^- < \text{IVPFWHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) < \tilde{b}_i^+ \quad (39)$$

Example 5. Let $\tilde{b}_1 = ([0.1, 0.4], [0.2, 0.3])$, $\tilde{b}_2 = ([0.2, 0.4], [0.3, 0.5])$, $\tilde{b}_3 = ([0.3, 0.4], [0.2, 0.3])$, $\tilde{b}_4 = ([0.4, 0.7], [0.1, 0.3])$ be four IVPFNs, the weighting vector of attributes be $\omega = \{0.2, 0.1, 0.3, 0.4\}$. Then we use the proposed IVPFWHM operator to aggregate four IVPFNs, (suppose $x = 2$).

$$\begin{aligned} \text{IVPFWHM}_{\omega}^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}) &= \left(\left[\sqrt{\frac{1 - \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(1 - \left(\prod_{j=1}^2 \mu_{i_j}^Z \right)^{\frac{1}{2}} \right)^2 \right)^{\left(1 - \sum_{j=1}^2 \omega_{i_j} \right)^{\frac{1}{2}}}}}{\left(1 - \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(1 - \left(\prod_{j=1}^2 \mu_{i_j}^Y \right)^{\frac{1}{2}} \right)^2 \right)^{\left(1 - \sum_{j=1}^2 \omega_{i_j} \right)^{\frac{1}{2}}}} \right)^{\frac{1}{2}}}, \left[\left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(\sqrt{1 - \left(\prod_{j=1}^2 (1 - (\nu_{i_j}^Z)^2) \right)^{\frac{1}{2}}} \right)^{\left(1 - \sum_{j=1}^2 \omega_{i_j} \right)^{\frac{1}{2}}}} \right)^{\frac{1}{2}}}, \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(\sqrt{1 - \left(\prod_{j=1}^2 (1 - (\nu_{i_j}^Y)^2) \right)^{\frac{1}{2}}} \right)^{\left(1 - \sum_{j=1}^2 \omega_{i_j} \right)^{\frac{1}{2}}}} \right)^{\frac{1}{2}} \right] \right) \\ &= \left(\left[\sqrt{\frac{1 - \left(\frac{(1 - 0.1 \times 0.2)^{1-0.2-0.1} \times (1 - 0.1 \times 0.3)^{1-0.2-0.3} \times (1 - 0.1 \times 0.4)^{1-0.2-0.4}}{(1 - 0.2 \times 0.3)^{1-0.1-0.3} \times (1 - 0.2 \times 0.4)^{1-0.1-0.4} \times (1 - 0.3 \times 0.4)^{1-0.3-0.4}} \right)^{\frac{1}{3}}}}{\sqrt{\frac{(1 - 0.4 \times 0.4)^{1-0.2-0.1} \times (1 - 0.4 \times 0.4)^{1-0.2-0.3} \times (1 - 0.4 \times 0.7)^{1-0.2-0.4}}{(1 - 0.4 \times 0.4)^{1-0.1-0.3} \times (1 - 0.4 \times 0.7)^{1-0.1-0.4} \times (1 - 0.4 \times 0.7)^{1-0.3-0.4}} \right)^{\frac{1}{3}}}} \right]^{\frac{1}{3}}, \left[\left(\frac{\left(\sqrt{1 - (1 - 0.2^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.2^2) \times (1 - 0.1^2)} \right)^{\frac{1}{2}}} \right)^{1-0.2-0.4} \times \left(\frac{\left(\sqrt{1 - (1 - 0.2^2) \times (1 - 0.2^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.2^2)} \right)^{\frac{1}{2}}} \right)^{1-0.1-0.3} \right. \\ &\quad \times \left(\frac{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.1^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.1^2)} \right)^{\frac{1}{2}}} \right)^{1-0.1-0.4} \times \left(\frac{\left(\sqrt{1 - (1 - 0.2^2) \times (1 - 0.1^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.2^2) \times (1 - 0.1^2)} \right)^{\frac{1}{2}}} \right)^{1-0.3-0.4} \right]^{\frac{1}{3}}, \\ &\quad \left[\left(\frac{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.5^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}} \right)^{1-0.2-0.1} \times \left(\frac{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.5^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}} \right)^{1-0.2-0.3} \right. \\ &\quad \times \left(\frac{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.5^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}} \right)^{1-0.2-0.4} \times \left(\frac{\left(\sqrt{1 - (1 - 0.5^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.5^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}} \right)^{1-0.1-0.3} \\ &\quad \times \left(\frac{\left(\sqrt{1 - (1 - 0.5^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}} \right)^{1-0.1-0.4} \times \left(\frac{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}}{\left(\sqrt{1 - (1 - 0.3^2) \times (1 - 0.3^2)} \right)^{\frac{1}{2}}} \right)^{1-0.3-0.4} \right]^{\frac{1}{3}} \end{aligned}$$

$$= ([0.2299, 0.4585], [0.5479, 0.6893])$$

Finally, we get $\text{IVPFWHM}_{\omega}^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}) = ([0.2299, 0.4585], [0.5479, 0.6893])$.

3.3. IVPFDHM Operator

Wu et al. [32] proposed the dual Hamy mean (DHM) operator.

Definition 9. The DHM operator is defined as follows:

$$\text{DHM}^{(x)}(b_1, b_2, \dots, b_k) = \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\frac{\sum_{j=1}^x b_{i_j}}{x} \right) \right)^{\frac{1}{C_k^x}} \quad (40)$$

where x is a parameter and $x = 1, 2, \dots, k$, i_1, i_2, \dots, i_x are x integer values taken from the set $\{1, 2, \dots, k\}$ of k integer values, and C_n^x denotes the binomial coefficient and $C_k^x = \frac{k!}{x!(k-x)!}$.

In the following, we proposed the interval-valued Pythagorean fuzzy DHM (IVPFDHM) operator for IVPFNs.

Definition 10. Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y])$ ($i = 1, 2, \dots, k$) be a set of IVPFNs, then the IVPFDHM operator is:

$$\text{IVPFDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \left(\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(\frac{\bigoplus_{j=1}^x \tilde{b}_{i_j}}{x} \right) \right)^{\frac{1}{C_k^x}} \quad (41)$$

where x is a parameter and $x = 1, 2, \dots, k$, i_1, i_2, \dots, i_x are x integer values taken from the set $\{1, 2, \dots, k\}$ of k integer values, and C_n^x denotes the binomial coefficient and $C_k^x = \frac{k!}{x!(k-x)!}$.

Based on the operations of the IVPFN described, we can obtain Theorem 4.

Theorem 4. Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y])$ ($i = 1, 2, \dots, k$) be a collection of IVPFNs, then the aggregated value of IVPFDHM operator is also an IVPFN, and

$$\begin{aligned} \text{IVPFDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) &= \left(\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(\frac{\bigoplus_{j=1}^x \tilde{b}_{i_j}}{x} \right) \right)^{\frac{1}{C_k^x}} \\ &= \left(\left[\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}}, \left[\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}}, \right. \right. \right. \\ &\quad \left. \left. \left[\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}}, \left[\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}} \right] \right] \right) \right] \end{aligned} \quad (42)$$

Proof.

$$\bigoplus_{j=1}^x \tilde{b}_{i_j} = \left(\left[\sqrt{1 - \prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2)}, \sqrt{1 - \prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2)} \right], \left[\prod_{j=1}^x (v_{i_j}^Z), \prod_{j=1}^x (v_{i_j}^Y) \right] \right) \quad (43)$$

$$\frac{\bigoplus_{j=1}^x \tilde{b}_{i_j}}{x} = \left(\left[\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}}, \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right], \left[\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}}, \left(\prod_{j=1}^x v_{i_j}^Y \right)^{\frac{1}{x}} \right] \right) \quad (44)$$

Therefore,

$$\left(\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(\frac{\bigoplus_{j=1}^x \tilde{b}_{ij}}{x} \right) \right)^{\frac{1}{C_k^x}} = \left(\left[\begin{array}{l} \prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Z)^2) \right)^{\frac{1}{x}}} , \prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Y)^2) \right)^{\frac{1}{x}}} \\ \sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{ij}^Z \right)^{\frac{1}{x}} \right)^2 \right)} , \sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right)} \end{array} \right] \right)^{\frac{1}{C_k^x}} \quad (45)$$

Then, we can get

$$\left(\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(\frac{\bigoplus_{j=1}^x \tilde{b}_{ij}}{x} \right) \right)^{\frac{1}{C_k^x}} = \left(\left[\begin{array}{l} \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Z)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} , \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Y)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \\ \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Z)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} , \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Y)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \end{array} \right] \right)^{\frac{1}{C_k^x}} \quad (46)$$

From the aggregation result above, we prove the result of IVPFDHM aggregation is also an IVPFN in the following, then,

$$0 \leq \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Z)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \leq 1 \quad (47)$$

$$0 \leq \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}} \leq 1 \quad (48)$$

And, we can prove

$$\begin{aligned} & \left(\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Z)^2) \right)^{\frac{1}{x}}} \right)^{\frac{1}{C_k^x}} \right)^2 + \left(\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{ij}^Y \right)^{\frac{1}{x}} \right)^2 \right) \right)^{\frac{1}{C_k^x}}} \right)^2 \\ & \leq \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\prod_{j=1}^x (1 - (\mu_{ij}^Z)^2) \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{C_k^x}} + 1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\prod_{j=1}^x (1 - (v_{ij}^Y)^2) \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{C_k^x}} = 1 \end{aligned} \quad (49)$$

Therefore, we proved that the aggregation result of IVPFDHM is also an IVPFN. Next, we will talk about some properties of the IVPFDHM operator. \square

Property 7. (Idempotency.) If all $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y]) (i = 1, 2, \dots, k)$ are equal, i.e., $\tilde{b}_i = \tilde{b} = ([\mu^Z, \mu^Y], [v^Z, v^Y])$, then,

$$\text{IVPFDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) = \tilde{b} \quad (50)$$

Property 8. (Monotonicity.) Let $\tilde{b}_i = ([\mu_{ij}^Z, \mu_{ij}^Y], [v_{ij}^Z, v_{ij}^Y])$ and $\tilde{c}_i = ([\mu_{\theta j}^Z, \mu_{\theta j}^Y], [v_{\theta j}^Z, v_{\theta j}^Y])$ ($i = 1, 2, \dots, k$) be two sets of IVPFNs. If $(\mu_{ij}^Z)^2 + (\mu_{ij}^Y)^2 \leq (\mu_{\theta j}^Z)^2 + (\mu_{\theta j}^Y)^2$ and $(v_{ij}^Z)^2 + (v_{ij}^Y)^2 \geq (v_{\theta j}^Z)^2 + (v_{\theta j}^Y)^2$, then,

$$\text{IVPFDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) \leq \text{IVPFDHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k) \quad (51)$$

Property 9. (Boundedness.) Let $\tilde{b}_i = ([\mu_{ij}^Z, \mu_{ij}^Y], [v_{ij}^Z, v_{ij}^Y])$ be a set of IVPFNs. If $\tilde{b}^+ = ([\mu_{maxij}^Z, \mu_{maxij}^Y], [v_{maxij}^Z, v_{maxij}^Y])$ and $\tilde{b}^- = ([\mu_{minij}^Z, \mu_{minij}^Y], [v_{minij}^Z, v_{minij}^Y])$ ($i = 1, 2, \dots, k$), because of Properties 7 and 8, then,

$$\tilde{b}_i^- < \text{IVPFDHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) < \tilde{b}_i^+ \quad (52)$$

Example 6. Let $\tilde{b}_1 = ([0.2, 0.5], [0.1, 0.2])$, $\tilde{b}_2 = ([0.1, 0.3], [0.3, 0.4])$, $\tilde{b}_3 = ([0.4, 0.6], [0.2, 0.5])$, $\tilde{b}_4 = ([0.5, 0.7], [0.4, 0.6])$ be four IVPFNs. Then we use the proposed IVPFDHM operator to aggregate four IVPFNs, (suppose $x = 2$).

$$\begin{aligned} & \text{IVPFDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) \\ &= \left(\left[\left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \sqrt{1 - \left(\prod_{j=1}^2 (1 - (\mu_{ij}^Z)^2) \right)^{\frac{1}{2}}} \right)^{\frac{1}{C_4^2}}, \left[\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(1 - \left(\left(\prod_{j=1}^2 v_{ij}^Z \right)^{\frac{1}{2}} \right)^2 \right) \right)^{\frac{1}{C_4^2}}}, \right. \right. \\ &= \left(\left[\left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \sqrt{1 - \left(\prod_{j=1}^2 (1 - (\mu_{ij}^Y)^2) \right)^{\frac{1}{2}}} \right)^{\frac{1}{C_4^2}}, \left[\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(1 - \left(\left(\prod_{j=1}^2 v_{ij}^Y \right)^{\frac{1}{2}} \right)^2 \right) \right)^{\frac{1}{C_4^2}}}, \right. \right. \\ &= \left(\left[\left(\frac{\sqrt{1 - ((1 - 0.2^2) \times (1 - 0.1^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.2^2) \times (1 - 0.4^2))^{\frac{1}{2}}}}{\sqrt{1 - ((1 - 0.1^2) \times (1 - 0.5^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.1^2) \times (1 - 0.4^2))^{\frac{1}{2}}}} \right)^{\frac{1}{6}}, \right. \right. \\ &= \left(\left[\left(\frac{\sqrt{1 - ((1 - 0.5^2) \times (1 - 0.3^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.5^2) \times (1 - 0.6^2))^{\frac{1}{2}}}}{\sqrt{1 - ((1 - 0.5^2) \times (1 - 0.7^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.3^2) \times (1 - 0.6^2))^{\frac{1}{2}}}} \right)^{\frac{1}{6}}, \right. \right. \\ &= \left(\left[\left(\frac{\sqrt{1 - ((1 - 0.3^2) \times (1 - 0.7^2))^{\frac{1}{2}}} \times \sqrt{1 - ((1 - 0.6^2) \times (1 - 0.7^2))^{\frac{1}{2}}}}{\sqrt{1 - ((1 - 0.1 \times 0.3) \times (1 - 0.1 \times 0.2) \times (1 - 0.1 \times 0.4) \times (1 - 0.3 \times 0.2) \times (1 - 0.3 \times 0.4) \times (1 - 0.2 \times 0.4) \right)^{\frac{1}{6}}}} \right. \right. \\ &= \left(\left[\left(\frac{\sqrt{1 - ((1 - 0.2 \times 0.4) \times (1 - 0.2 \times 0.5) \times (1 - 0.2 \times 0.6) \times (1 - 0.4 \times 0.7) \times (1 - 0.4 \times 0.6) \times (1 - 0.5 \times 0.6) \right)^{\frac{1}{6}}}}{\sqrt{1 - ((1 - 0.1 \times 0.3) \times (1 - 0.1 \times 0.2) \times (1 - 0.1 \times 0.4) \times (1 - 0.3 \times 0.2) \times (1 - 0.3 \times 0.4) \times (1 - 0.2 \times 0.4) \right)^{\frac{1}{6}}}} \right. \right. \\ &= ([0.3156, 0.5432], [0.2428, 0.4377]) \end{aligned}$$

At last, we get $\text{IVPFDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = ([0.3156, 0.5432], [0.2428, 0.4377])$.

3.4. IVPFWDHM Operator

It can be seen that the IVPFDHM operator doesn't consider the importance of the aggregated arguments and that the weights of attributes play an important role in the process of aggregation. To overcome the limitation of IVPFDHM, we develop the interval-valued Pythagorean fuzzy weighted DHM (IVPFWDHM) operator.

Definition 11. Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y])$ ($i = 1, 2, \dots, k$) be a set of IVPFNs with weight vector being $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$, $\omega_i \in [0, 1]$, $\sum_{i=1}^k \omega_i = 1$, then the IVPFWDHM operator is given as:

$$\text{IVPFWDHM}_{\omega}^{(x)}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k) = \begin{cases} \left(\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{i_j} \left(\frac{\bigoplus_{j=1}^x \tilde{a}_{i_j}}{x} \right) \right) \right)^{\frac{1}{C_{k-1}^x}} & (1 \leq x < k) \\ \bigoplus_{i=1}^x \tilde{a}_i^{\frac{1-\omega_i}{k-1}} & (x = k) \end{cases} \quad (53)$$

Based on the operations of the IVPFN described, we can obtain Theorem 5.

Theorem 5. Let $\tilde{b}_i = ([\mu_i^Z, \mu_i^Y], [v_i^Z, v_i^Y]) (i = 1, 2, \dots, k)$ be a collection of IVPFNs, then the aggregated result of IVPFWDHM operator is also an IVPFN, and

$$\text{IVPFWDHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \left(\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{i_j} \right) \left(\frac{\bigoplus_{j=1}^x \tilde{a}_{i_j}}{x} \right) \right)^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}} =$$

$$\left(\left[\begin{array}{c} \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} \right)^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}} \\ \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} \right)^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}} \end{array} \right]^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}}, \left[\begin{array}{c} \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} \right)^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}}} \\ \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} \right)^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}}} \end{array} \right]^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}} \right) \quad (1 \leq x < k) \quad (54)$$

or

$$\text{IVPFWDHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \bigoplus_{i=1}^x \tilde{a}_i^{\frac{1-\omega_i}{k-1}} = \left(\left[\begin{array}{c} \sqrt{1 - \prod_{i=1}^k (1 - (\mu_i^Z)^2)^{\frac{1-\omega_i}{k-1}}}, \sqrt{1 - \prod_{i=1}^k (1 - (\mu_i^Y)^2)^{\frac{1-\omega_i}{k-1}}} \\ \left[\prod_{i=1}^k (v_i^Z)^{\frac{1-\omega_i}{k-1}}, \prod_{i=1}^k (v_i^Y)^{\frac{1-\omega_i}{k-1}} \right] \end{array} \right], \right) \quad (x = k) \quad (55)$$

Proof.

$$\frac{\bigoplus_{j=1}^x \tilde{a}_{i_j}}{x} = \left(\left[\begin{array}{c} \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}}, \sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \\ \left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}}, \left(\prod_{j=1}^x v_{i_j}^Y \right)^{\frac{1}{x}} \end{array} \right] \right) \quad (56)$$

Thereafter,

$$\left(1 - \sum_{j=1}^x \omega_{i_j} \right) \left(\frac{\bigoplus_{j=1}^x \tilde{a}_{i_j}}{x} \right) = \left(\left[\begin{array}{c} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} \\ \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} \end{array} \right]^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}}, \left[\begin{array}{c} \sqrt{1 - \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} } \\ \sqrt{1 - \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} } \end{array} \right]^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}} \right) \quad (57)$$

Moreover,

$$\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{i_j} \right) \left(\frac{\bigoplus_{j=1}^x \tilde{a}_{i_j}}{x} \right) =$$

$$\left(\left[\begin{array}{c} \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} \\ \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} \end{array} \right]^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}}, \left[\begin{array}{c} \sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} } \\ \sqrt{1 - \prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})} } \end{array} \right]^{\frac{1}{\sum_{j=1}^x \omega_{i_j}}} \right) \quad (58)$$

Therefore,

$$\left(\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{i_j} \right) \left(\frac{\bigoplus_{j=1}^x \tilde{a}_{i_j}}{x} \right) \right)^{\frac{1}{C_{k-1}^x}} =$$

$$\left[\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}}, \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \right] \left[\left(\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}}, \right. \right. \quad (59)$$

$$\left. \left. \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \right] \right]$$

For the second case, when $(x = k)$, we get:

$$\tilde{b}_i^{\frac{1-\omega_i}{k-1}} = \left(\left[(\mu_i^Z)^{\frac{1-\omega_i}{k-1}}, (\mu_i^Y)^{\frac{1-\omega_i}{k-1}} \right], \left[\sqrt{1 - (1 - (v_i^Z)^2)^{\frac{1-\omega_i}{k-1}}}, \sqrt{1 - (1 - (v_i^Y)^2)^{\frac{1-\omega_i}{k-1}}} \right] \right) \quad (60)$$

Then,

$$\bigoplus_{i=1}^k \tilde{b}_i^{\frac{1-\omega_i}{k-1}} = \left(\left[\sqrt{1 - \prod_{i=1}^k (1 - (\mu_i^Z)^2)^{\frac{1-\omega_i}{k-1}}}, \sqrt{1 - \prod_{i=1}^k (1 - (\mu_i^Y)^2)^{\frac{1-\omega_i}{k-1}}} \right], \left[\prod_{i=1}^k ((v_i^Z)^{\frac{1-\omega_i}{k-1}}), \prod_{i=1}^k ((v_i^Y)^{\frac{1-\omega_i}{k-1}}) \right] \right) \quad (61)$$

Then, we can obtain,

$$0 \leq \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \leq 1 \quad (62)$$

$$0 \leq \sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}}} \leq 1 \quad (63)$$

And,

$$0 \leq \sqrt{1 - \prod_{i=1}^k (1 - (\mu_i^Z)^2)^{\frac{1-\omega_i}{k-1}}} \leq 1 \quad (64)$$

$$0 \leq \prod_{i=1}^k ((v_i^Y)^{\frac{1-\omega_i}{k-1}}) \leq 1 \quad (65)$$

Therefore,

$$\left(\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \right)^2$$

$$+ \left(\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}}} \right)^2 \leq \quad (66)$$

$$\left(\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \right)^2$$

$$+ 1 - \left(\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (v_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{C_{k-1}^x}}} \right)^{\frac{1}{C_{k-1}^x}} \right)^2 = 1$$

For the second case, $x = k$, we can easily prove that it is kept. Therefore, the aggregation result produced by Definition 11 is still an IVPFN. Next, we shall deduce some desirable properties of the IVPFWDHM operator. \square

Property 10. (Idempotency.) Let $\tilde{b}_i = ([\mu_{i_j}^Z, \mu_{i_j}^Y], [v_{i_j}^Z, v_{i_j}^Y]) = ([\mu^Z, \mu^Y], [v^Z, v^Y]) = \tilde{b}$, and weight vector meets $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$, $\omega_i \in [0, 1]$ and $\sum_{i=1}^k \omega_i = 1$, then,

$$\text{IVPFWDHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \tilde{b} \quad (67)$$

Property 11. (Monotonicity.) Let $\tilde{b}_i = ([\mu_{i_j}^Z, \mu_{i_j}^Y], [v_{i_j}^Z, v_{i_j}^Y])$ and $\tilde{c}_i = ([\mu_{\theta_j}^Z, \mu_{\theta_j}^Y], [v_{\theta_j}^Z, v_{\theta_j}^Y])$ ($i = 1, 2, \dots, k$) be two sets of IVPFNs with weight vector being $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$, $\omega_i \in [0, 1]$, $\sum_{i=1}^k \omega_i = 1$. If $(\mu_{i_j}^Z)^2 + (\mu_{i_j}^Y)^2 \leq (\mu_{\theta_j}^Z)^2 + (\mu_{\theta_j}^Y)^2$ and $(v_{i_j}^Z)^2 + (v_{i_j}^Y)^2 \geq (v_{\theta_j}^Z)^2 + (v_{\theta_j}^Y)^2$, then,

$$\text{IVPFWDHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) \leq \text{IVPFWDHM}_{\omega}^{(x)}(\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k) \quad (68)$$

Property 12. (Boundedness.) Let $\tilde{b}_i = ([\mu_{i_j}^Z, \mu_{i_j}^Y], [v_{i_j}^Z, v_{i_j}^Y])$, $\tilde{b}^+ = ([\mu_{\max i_j}^Z, \mu_{\max i_j}^Y], [v_{\min i_j}^Z, v_{\min i_j}^Y])$ and $\tilde{b}^- = ([\mu_{\min i_j}^Z, \mu_{\min i_j}^Y], [v_{\max i_j}^Z, v_{\max i_j}^Y])$ ($i = 1, 2, \dots, k$) be a set of IVPFNs with weight vector being $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$, $\omega_i \in [0, 1]$, $\sum_{i=1}^k \omega_i = 1$, because of Property 11, then,

$$\tilde{b}_i^- < \text{IVPFWDHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) < \tilde{b}_i^+ \quad (69)$$

Example 7. Let $\tilde{b}_1 = ([0.3, 0.5], [0.2, 0.4])$, $\tilde{b}_2 = ([0.4, 0.6], [0.6, 0.7])$, $\tilde{b}_3 = ([0.5, 0.7], [0.2, 0.3])$, $\tilde{b}_4 = ([0.4, 0.8], [0.1, 0.2])$ be four IVPFNs. the weighting vector of attributes be $\omega = \{0.3, 0.2, 0.3, 0.2\}$. Then, we use the proposed IVPFWDHM operator to aggregate four IVPFNs, (suppose $x = 2$).

$$\text{IVPFWDHM}_{\omega}^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = \left(\left[\left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(\sqrt{1 - \left(\prod_{j=1}^2 (1 - (\mu_{i_j}^Z)^2) \right)^{\frac{1}{2}}} \right)^{(1 - \sum_{j=1}^2 \omega_{i_j})} \right)^{\frac{1}{C_{4-1}^2}}, \right. \right. \\ \left. \left[\left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(\sqrt{1 - \left(\prod_{j=1}^2 (1 - (\mu_{i_j}^Y)^2) \right)^{\frac{1}{2}}} \right)^{(1 - \sum_{j=1}^2 \omega_{i_j})} \right)^{\frac{1}{C_{4-1}^2}}, \right. \right. \\ \left. \left[\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(1 - \left(\left(\prod_{j=1}^2 v_{i_j}^Z \right)^{\frac{1}{2}} \right)^2 \right)^{(1 - \sum_{j=1}^2 \omega_{i_j})} \right)^{\frac{1}{C_{4-1}^2}}}, \right. \right. \\ \left. \left[\sqrt{1 - \left(\prod_{1 \leq i_1 < \dots < i_2 \leq 4} \left(1 - \left(\left(\prod_{j=1}^2 v_{i_j}^Y \right)^{\frac{1}{2}} \right)^2 \right)^{(1 - \sum_{j=1}^2 \omega_{i_j})} \right)^{\frac{1}{C_{4-1}^2}}} \right] \right)$$

$$= \left(\left[\left(\left(\sqrt{(1 - (1 - 0.3^2) \times (1 - 0.4^2))}^{\frac{1}{2}} \right)^{1-0.3-0.2} \times \left(\sqrt{(1 - (1 - 0.3^2) \times (1 - 0.5^2))}^{\frac{1}{2}} \right)^{1-0.3-0.3} \right)^{\frac{1}{3}} \right. \right. \\ \left. \left. \times \left(\sqrt{(1 - (1 - 0.3^2) \times (1 - 0.4^2))}^{\frac{1}{2}} \right)^{1-0.3-0.2} \times \left(\sqrt{(1 - (1 - 0.4^2) \times (1 - 0.5^2))}^{\frac{1}{2}} \right)^{1-0.2-0.3} \right)^{\frac{1}{3}} \right. \right. \\ \left. \left. \times \left(\sqrt{(1 - (1 - 0.4^2) \times (1 - 0.3^2))}^{\frac{1}{2}} \right)^{1-0.2-0.2} \times \left(\sqrt{(1 - (1 - 0.5^2) \times (1 - 0.4^2))}^{\frac{1}{2}} \right)^{1-0.3-0.2} \right)^{\frac{1}{3}} \right. \right. \\ \left. \left. \times \left(\sqrt{(1 - (1 - 0.5^2) \times (1 - 0.6^2))}^{\frac{1}{2}} \right)^{1-0.3-0.2} \times \left(\sqrt{(1 - (1 - 0.5^2) \times (1 - 0.7^2))}^{\frac{1}{2}} \right)^{1-0.3-0.3} \right)^{\frac{1}{3}} \right. \right. \\ \left. \left. \times \left(\sqrt{(1 - (1 - 0.5^2) \times (1 - 0.8^2))}^{\frac{1}{2}} \right)^{1-0.3-0.2} \times \left(\sqrt{(1 - (1 - 0.6^2) \times (1 - 0.7^2))}^{\frac{1}{2}} \right)^{1-0.2-0.3} \right)^{\frac{1}{3}} \right. \right. \\ \left. \left. \times \left(\sqrt{(1 - (1 - 0.6^2) \times (1 - 0.8^2))}^{\frac{1}{2}} \right)^{1-0.2-0.2} \times \left(\sqrt{(1 - (1 - 0.7^2) \times (1 - 0.8^2))}^{\frac{1}{2}} \right)^{1-0.3-0.2} \right)^{\frac{1}{3}} \right. \right. \\ \left. \left. \sqrt{1 - \left(\frac{(1 - 0.2 \times 0.6)^{1-0.3-0.2} \times (1 - 0.2 \times 0.2)^{1-0.3-0.3} \times (1 - 0.2 \times 0.1)^{1-0.3-0.2}}{(1 - 0.6 \times 0.2)^{1-0.2-0.3} \times (1 - 0.6 \times 0.1)^{1-0.2-0.2} \times (1 - 0.2 \times 0.1)^{1-0.3-0.2}} \right)^{\frac{1}{3}}} \right. \right. \\ \left. \left. \sqrt{1 - \left(\frac{(1 - 0.4 \times 0.7)^{1-0.3-0.2} \times (1 - 0.4 \times 0.3)^{1-0.3-0.3} \times (1 - 0.4 \times 0.2)^{1-0.3-0.2}}{(1 - 0.7 \times 0.3)^{1-0.2-0.3} \times (1 - 0.7 \times 0.2)^{1-0.2-0.2} \times (1 - 0.3 \times 0.2)^{1-0.3-0.2}} \right)^{\frac{1}{3}}} \right. \right. \\ \left. \left. = ([0.7306, 0.9099], [0.2547, 0.3905]) \right) \right)$$

Finally, we get $IVPFWDHM_{\omega}^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = ([0.7306, 0.9099], [0.2547, 0.3905])$.

4. Models for MADM with IVPFNs

Based on the IVPFWMH and IVPFWDHM operators, in this section, we shall propose the model for MADM with IVPFNs. Let $E = \{E_1, E_2, \dots, E_m\}$ be a discrete set of alternatives, and $D = \{D_1, D_2, \dots, D_n\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weight of the attribute $D_j (j = 1, 2, \dots, n)$, where $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$. Suppose that $\tilde{B}_i = (\tilde{b}_{ij})_{m \times n} = ([\mu_{ij}^Z, \mu_{ij}^Y], [\nu_{ij}^Z, \nu_{ij}^Y])_{m \times n}$ is the interval-valued Pythagorean fuzzy decision matrix, where $[\mu_{ij}^Z, \mu_{ij}^Y]$ indicates the degree that the alternative E_i satisfies the attribute D_j given by the decision maker, $[\nu_{ij}^Z, \nu_{ij}^Y]$ depicts the degree that the alternative E_i doesn't satisfy the attribute D_j given by the decision maker, $[\mu_{ij}^Z, \mu_{ij}^Y] \subset [0, 1]$, $[\nu_{ij}^Z, \nu_{ij}^Y] \subset [0, 1]$, $(\mu_{ij}^Y)^2 + (\nu_{ij}^Y)^2 \leq 1, i = 1, 2, \dots, m, j = 1, 2, \dots, n$.

In the following, we use the IVPFWMH (IVPFWDHM) operator to solve the MADM problems with IVPFNs.

Step 1. We use the IVPFNs given in matrix \tilde{B} , and the IVPFWMH operator,

$$IVPFWMH_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) \\ = \frac{\bigoplus_{1 \leq i_1 < \dots < i_x \leq k} \left(\left(\prod_{j=1}^x \omega_{i_j} \right) \left(\bigotimes_{j=1}^x \tilde{b}_{i_j} \right) \right)^{\frac{1}{x}}}{C_{k-1}^x} = \\ \left(\left[\sqrt{\frac{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{x-1}}} \right)^{\frac{1}{x-1}}}{1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x \mu_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{x-1}}} \right)^{\frac{1}{x-1}}} \right)^{\frac{1}{x-1}} \right. \right. \\ \left. \left. \left[\left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\nu_{i_j}^Z)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{x-1}}} \right)^{\frac{1}{x-1}} \right. \right. \right. \right. \\ \left. \left. \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\nu_{i_j}^Y)^2) \right)^{\frac{1}{x}}} \right)^{(1 - \sum_{j=1}^x \omega_{i_j})^{\frac{1}{x-1}}} \right)^{\frac{1}{x-1}} \right)^{\frac{1}{x-1}} \right] \right] \right) \quad (1 \leq x < k)$$

$$IVPFWMH_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \bigotimes_{i=1}^x \tilde{b}_i^{\frac{1-\omega_i}{x-1}} = \left(\left[\frac{\left[\prod_{i=1}^k (\mu_i^Z)^{\frac{1-\omega_i}{k-1}}, \prod_{i=1}^k (\mu_i^Y)^{\frac{1-\omega_i}{k-1}} \right]}{\sqrt{1 - \prod_{i=1}^k (1 - (\nu_i^Z)^2)^{\frac{1-\omega_i}{k-1}}}, \sqrt{1 - \prod_{i=1}^k (1 - (\nu_i^Y)^2)^{\frac{1-\omega_i}{k-1}}} \right]} \right) \quad (x = k)$$

Or

$$\text{IVPFWDHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \left(\bigotimes_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \sum_{j=1}^x \omega_{i_j} \right) \left(\frac{\bigoplus_{j=1}^x \tilde{a}_{i_j}}{x} \right) \right)^{\frac{1}{C_{k-1}^x}} =$$

$$\left(\left[\begin{array}{c} \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Z)^2) \right)} \right)^{\frac{1}{x}} \right)^{\frac{(1 - \sum_{j=1}^x \omega_{i_j})}{C_{k-1}^x}} \\ \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^x (1 - (\mu_{i_j}^Y)^2) \right)} \right)^{\frac{1}{x}} \right)^{\frac{(1 - \sum_{j=1}^x \omega_{i_j})}{C_{k-1}^x}} \end{array} \right], \left[\begin{array}{c} 1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Z \right)^{\frac{1}{x}} \right)^2 \right)^{\frac{(1 - \sum_{j=1}^x \omega_{i_j})}{C_{k-1}^x}} \right)^{\frac{1}{C_{k-1}^x}} \\ 1 - \left(\prod_{1 \leq i_1 < \dots < i_x \leq k} \left(1 - \left(\left(\prod_{j=1}^x v_{i_j}^Y \right)^{\frac{1}{x}} \right)^2 \right)^{\frac{(1 - \sum_{j=1}^x \omega_{i_j})}{C_{k-1}^x}} \right)^{\frac{1}{C_{k-1}^x}} \end{array} \right] \right) \quad (72)$$

(1 ≤ x < k)

$$\text{IVPFWDHM}_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_k) = \bigoplus_{i=1}^x \tilde{a}_i^{\frac{1-\omega_i}{k-1}} = \left(\left[\begin{array}{c} \sqrt{1 - \prod_{i=1}^k (1 - (\mu_i^Z)^2)}^{\frac{1-\omega_i}{k-1}}, \sqrt{1 - \prod_{i=1}^k (1 - (\mu_i^Y)^2)}^{\frac{1-\omega_i}{k-1}} \\ \left[\prod_{i=1}^k (v_i^Z)^{\frac{1-\omega_i}{k-1}}, \prod_{i=1}^k (v_i^Y)^{\frac{1-\omega_i}{k-1}} \right] \end{array} \right], \right)$$

(x = k)

To derive the $\tilde{b}_i (i = 1, 2, \dots, m)$ of the alternative E_i .

Step 2. Calculate the $S(\tilde{b}_i)$ and $H(\tilde{b}_i)$ ($i = 1, 2, \dots, m$) of the overall IVPFNs \tilde{b}_i to rank all the alternatives $E_i (i = 1, 2, \dots, m)$.

Step 3. Rank all the alternatives $E_i (i = 1, 2, \dots, m)$ and select the best one(s) in accordance with $S(\tilde{b}_i)$ and $H(\tilde{b}_i)$ ($i = 1, 2, \dots, m$).

Step 4. End.

5. Numerical Example and Comparative Analysis

5.1. Numerical Example

With the rapid development of economic globalization and the growing competitive enterprise environment, the competition between modern enterprises has become the competition between supply chain and supply chain. Owing to the diversity of the people's consumption concept, the new product life cycles are getting shorter, volatility of the demand market and those from external factors drives enterprises to achieve effective supply chain integration and management, and strategic alliance with other enterprises in order to enhance core competitiveness and resist external risk. The key measure to achieving this goal is the supplier selection. Therefore, supplier selection has gained great attention whether in supply chain management theory or in actual production management problems. A company wants to select an air-conditioning and know there are four possible suppliers as candidates $E_i (i = 1, 2, 3, 4)$ to select. The experts select four attributes to assess the four possible suppliers: (1) D_1 the quality of product; (2) D_2 the price of product; (3) D_3 product life cycle; (4) D_4 after-sale service. The weight vector of attributes is $\omega = (0.3, 0.2, 0, 1, 0.4)^T$, (suppose $x = 2$) as shown in Table 1.

In the following, in order to select the best suppliers in supply chain management (SCM), we utilize the IVPFHM, IVPFWM, IVPFDHM, and IVPFWDHM operators to solve the MADM problem with IVPFNs, which includes the following calculating steps:

Step 1. According to Table 1, aggregate all IVPFNs $b_{ij} (i = 1, 2, 3, 4, j = 1, 2, \dots, n)$ by using the IVPFHM, IVPFWM, IVPFDHM, and IVPFWDHM operators to derive the overall IVPFNs $b_{ij} (i = 1, 2, 3, 4, j = 1, 2, \dots, n)$ of the alternative E_i . The aggregation results are shown in Table 3, (suppose $x = 2$).

Step 2. According to the results listed in Table 3, the score functions of the suppliers are shown in Table 2.

Table 1. Decision matrix.

	D ₁	D ₂	D ₃	D ₄
E ₁	([0.10,0.40] [0.20,0.50])	([0.50,0.60] [0.20,0.30])	([0.30,0.40] [0.30,0.50])	([0.40,0.50] [0.20,0.30])
E ₂	([0.50,0.60] [0.10,0.30])	([0.50,0.60] [0.10,0.30])	([0.50,0.70] [0.20,0.30])	([0.60,0.70] [0.40,0.60])
E ₃	([0.60,0.70] [0.20,0.40])	([0.20,0.30] [0.70,0.80])	([0.50,0.60] [0.10,0.20])	([0.20,0.50] [0.20,0.30])
E ₄	([0.30,0.70] [0.20,0.30])	([0.30,0.70] [0.10,0.20])	([0.20,0.50] [0.40,0.60])	([0.50,0.80] [0.30,0.70])

Table 2. The rank and score of the suppliers by using IVPFHM, IVPFWM, IVPFDHM, and IVPFWDHM operators.

	E ₁	E ₂	E ₃	E ₄	Order
IVPFHM	0.5603	0.6654	0.5623	0.6033	E ₂ > E ₄ > E ₃ > E ₁
IVPFWM	0.6436	0.7352	0.6514	0.6862	E ₂ > E ₄ > E ₃ > E ₁
IVPFDHM	0.4891	0.5980	0.5042	0.5428	E ₂ > E ₄ > E ₃ > E ₁
IVPFWDHM	0.3972	0.5155	0.4156	0.4470	E ₂ > E ₄ > E ₃ > E ₁

Table 3. The aggregation result of IVPFHM, IVPFWM, IVPFDHM, and IVPFWDHM operators.

	E ₁	E ₂	E ₃	E ₄
IVPFHM	([0.2428,0.6315] [0.2261,0.4068])	([0.4490,0.7434] [0.1259,0.2769])	([0.2655,0.6007] [0.2310,0.3588])	([0.3201,0.6608] [0.1788,0.3062])
IVPFWM	([0.2421,0.8608] [0.2317,0.4138])	([0.4599,0.9083] [0.1318,0.2813])	([0.2765,0.8403] [0.2317,0.3578])	([0.3056,0.8760] [0.1726,0.2937])
IVPFDHM	([0.2598,0.4798] [0.2237,0.5395])	([0.4566,0.5856] [0.1225,0.3802])	([0.2790,0.4590] [0.2168,0.4742])	([0.3315,0.5039] [0.1687,0.4051])
IVPFWDHM	([0.2542,0.4787] [0.2281,0.8080])	([0.4650,0.5997] [0.1265,0.7055])	([0.2982,0.4644] [0.2121,0.7729])	([0.3156,0.4967] [0.1657,0.7286])

According to the result of the company order, we know that the best choice is supplier 2. We obtain the same result by different aggregation, proving the effectiveness of result.

5.2. Influence of the Parameter on the Final Result

The aggregation method to extend IVPFS with HM has two advantages, one is that it can reduce the bad effects of the unduly high and low assessments on the final result, the other is that it can capture the interrelationship between IVPFNs. These aggregation operators have a parameter vector, which makes the extended operator more flexible, causing a different vector to lead to different aggregation results, scores, and ranking results. In order to illustrate the influence of the parameter vector x on the ranking result, we discuss the influence with several parameter vectors. The results of which are given in Tables 4 and 5.

We can see that the different parameters lead to different results and different ranking orders. The more attributes we consider, the bigger the scores and the bigger the attribute value, the lower the scores. Therefore, the parameter vector can be considered as the decision maker's risk preference.

Table 4. Ranking results by utilizing a different parameter vector x in the IVPFWHM operator.

$(x = 1, 2, 3, 4)$	Scores				Order
	E ₁	E ₂	E ₃	E ₄	
$x = 1$	0.5259	0.6225	0.5475	0.5654	$E_2 > E_4 > E_3 > E_1$
$x = 2$	0.6436	0.7352	0.6514	0.6962	$E_2 > E_4 > E_3 > E_1$
$x = 3$	0.5115	0.6006	0.5170	0.5521	$E_2 > E_4 > E_3 > E_1$
$x = 4$	0.5079	0.6101	0.5144	0.5473	$E_2 > E_4 > E_3 > E_1$

Table 5. Ranking results by utilizing a different parameter vector x in the IVPFWDHM operator.

$(x = 1, 2, 3, 4)$	Scores				Order
	E ₁	E ₂	E ₃	E ₄	
$x = 1$	0.5079	0.6101	0.5158	0.5473	$E_2 > E_4 > E_3 > E_1$
$x = 2$	0.3972	0.5155	0.4156	0.4470	$E_2 > E_4 > E_3 > E_1$
$x = 3$	0.5236	0.6226	0.5464	0.5620	$E_2 > E_4 > E_3 > E_1$
$x = 4$	0.5259	0.6244	0.5578	0.5661	$E_2 > E_4 > E_3 > E_1$

5.3. Comparative Analysis

Then, we compare the proposed method with the IVPFWA operator and IVPFWG operator proposed by Garg [28].

Definition 12. ([28]). Suppose that $\tilde{B} = (\tilde{b}_{ij})_{m \times n} = \left([\mu_{ij}^Z, \mu_{ij}^Y], [v_{ij}^Z, v_{ij}^Y] \right)_{m \times n}$ be an IVPFN matrix, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ be the weight of w_j , $0 \leq \omega_i \leq 1$, $\sum_{i=1}^n \omega_i = 1$. Then,

$$\begin{aligned} \tilde{b}_i &= \text{IVPFWA}_\omega(\tilde{b}_{i1}, \tilde{b}_{i2}, \dots, \tilde{b}_{in}) = \bigoplus_{j=1}^n (\omega_j \tilde{b}_{ij}) \\ &= \left(\left[\sqrt{1 - \prod_{j=1}^n \left(1 - (\mu_{ij}^Z)^2 \right)^{\omega_j}}, \sqrt{1 - \prod_{j=1}^n \left(1 - (\mu_{ij}^Y)^2 \right)^{\omega_j}} \right], \left[\prod_{j=1}^n (v_{ij}^Z)^{\omega_j}, \prod_{j=1}^n (v_{ij}^Y)^{\omega_j} \right] \right) \end{aligned} \quad (74)$$

$i = 1, 2, \dots, m, j = 1, 2, \dots, n$

$$\begin{aligned} \tilde{b}_i &= \text{IVPFWG}_\omega(\tilde{b}_{i1}, \tilde{b}_{i2}, \dots, \tilde{b}_{in}) = \bigotimes_{j=1}^n (\tilde{b}_{ij})^{\omega_j} \\ &= \left(\left[\prod_{j=1}^n (\mu_{ij}^Z)^{\omega_j}, \prod_{j=1}^n (\mu_{ij}^Y)^{\omega_j} \right], \left[\sqrt{1 - \prod_{j=1}^n \left(1 - (v_{ij}^Z)^2 \right)^{\omega_j}}, \sqrt{1 - \prod_{j=1}^n \left(1 - (v_{ij}^Y)^2 \right)^{\omega_j}} \right] \right) \end{aligned} \quad (75)$$

$i = 1, 2, \dots, m, j = 1, 2, \dots, n$

By utilizing \tilde{B} and the IVPFWA and IVPFWG operators, the aggregation results were derived and are shown in Table 6.

Table 6. The aggregation results of green suppliers by the IVPFWA (IVPFWG) operators.

	IVPFWA	IVPFWG
E ₁	([0.2939, 0.4896], [0.2083, 0.3680])	([0.2232, 0.4732], [0.2124, 0.3971])
E ₂	([0.4355, 0.5500], [0.1072, 0.2551])	([0.4076, 0.5181], [0.1142, 0.2652])
E ₃	([0.2518, 0.4540], [0.2144, 0.3478])	([0.3192, 0.4300], [0.2498, 0.3751])
E ₄	([0.3914, 0.5291], [0.1663, 0.3005])	([0.3534, 0.5144], [0.2139, 0.3673])

According to Table 6, the score values were derived and are shown in Table 7.

Table 7. The score functions of the suppliers.

	IVPFWA	IVPFWG
E ₁	0.5368	0.5180
E ₂	0.6039	0.5878
E ₃	0.5257	0.5074
E ₄	0.5788	0.5522

Following from Table 7, the order of the suppliers is given in Table 8.

Table 8. Order of the green suppliers.

Order	
IVPFWA	$E_2 > E_4 > E_1 > E_3$
IVPFWG	$E_2 > E_4 > E_1 > E_3$

From above, we can observe that we consistently identify the same best suppliers, showing the effectiveness of our approaches. However, the existing aggregation operators, such as IVPFWA operator and IVPFWG operator, don't consider interrelationship among arguments being aggregated, and thus can't eliminate the influence of unfair arguments. Our proposed IVPFWM and IVPFWDHM operators consider the information concerning the interrelationship among arguments being aggregated.

In a word, we have verified the effectiveness of the proposed method and shown the advantages of solving the MADM problem with incomplete and indeterminate information.

6. Conclusions

IVPFNs can easily describe incomplete and indeterminate information by degrees of membership and non-membership, and the HM operator and dual HM (DMM) operators are good tools to deal with multiple attribute decision making (MADM) problems because they can capture the interrelationship among the multi-input arguments. Motivated by the studies regarding the HM operator and dual HM operator, in this paper, we proposed some new HM and DHM operators to cope with MADM with IVPFNs, including the IVPFHM operator, IVPFWM operator, IVPFDHM operator, and IVPFWDHM operator. Moreover, these proposed operators are utilized to solve the MADM problems with IVPFNs. Finally, we used an illustrative example for supplier selections in SCM to show the feasibility and validity of the proposed operators by comparison with the other existing methods. In subsequent studies, we shall extend the proposed operators to the different fields [56–72] as well as propose some new aggregation operators under the uncertain environment [73–80]. On the other hand, we can develop the potential applications of the proposed method to different domains [81–87].

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