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# Strong Convergence Theorems for Fixed Point Problems for Nonexpansive Mappings and Zero Point Problems for Accretive Operators Using Viscosity Implicit Midpoint Rules in Banach Spaces

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**Abstract:** This paper uses the viscosity implicit midpoint rule to find common points of the fixed point set of a nonexpansive mapping and the zero point set of an accretive operator in Banach space. Under certain conditions, this paper obtains the strong convergence results of the proposed algorithm and improves the relevant results of researchers in this field. In the end, this paper gives numerical examples to support the main results.

**Keywords:** viscosity implicit midpoint rule; nonexpansive mapping; accretive operator; zero point problem

## 1. Introduction

Let $E$ be a Banach Space and $E^*$ the dual space. $J$ denotes the normalized duality mapping from $E$ to $2^{E^*}$ and is defined by

$$J(x) = \left\{ f \in E^* :< x, f >= \|x\|^2 = \|f\|^2 \right\}, \ x \in E.$$

Let $C$ be a nonempty set of $E$. A mapping $f : C \to C$ is contractive, if $\|f(x) - f(y)\| \leq k\|x - y\|$, $\forall x, y \in C$, $k \in [0, 1)$. A mapping $S : C \to C$ is nonexpansive, if $\|S(x) - S(y)\| \leq \|x - y\|$, $\forall x, y \in C$. Let $F(S)$ denote the fixed point set of $S$.

$A : C \to E$ is called accretive operator, if there exists $j(x - y) \in J(x - y)$ such that $\langle Ax - Ay, j(x - y)\rangle \geq 0$, for any $x, y \in C$. If $R(I + rA) = E$, $\forall r > 0$, then $A$ is called $m$-accretive operator. $J_r : R(I + rA) \to D(A)$ is called the resolvent of $m$-accretive operator $A$ and defined by $J_r = (I + rA)^{-1}$, $\forall r > 0$. It is well known that $J_r$ is nonexpansive mapping and $N(A) = F(J_r)$, where $N(A) = \{x \in E : 0 \in Ax\}$ and $F(J_r)$ is the fixed point set of $J_r$. So fixed point theory of nonexpansive mappings has been applied to zero point problem of accretive operator, see [1–6] and the references therein.

The implicit midpoint rule is one of the powerful methods for solving ordinary differential equations; see [7–12] and the references therein. Moreover, viscosity iterative algorithms for finding common fixed points for nonlinear operators and solutions of variational inequality problems have been researched by many authors.

In 2009, Chang et al. [1] proposed a viscosity iterative algorithm for accretive operator and nonexpansive mapping:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) S J_r x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \forall n \geq 0. \end{cases}$$

In 2010, Jung [13] proposed a composite iterative algorithm by viscosity method for finding the zero point of an accretive operator:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \forall n \geq 0. \end{cases}$$

In 2016, Jung [14] extended the related results and proposed an iterative algorithm for finding common point of zero of accretive operator and fixed point of nonexpansive mapping:

$$x_{n+1} = J_{r_n}(\alpha_n f(x_n) + (1-\alpha_n) S x_n), \forall n \geq 0,$$
$$x_{n+1} = J_{r_n}(\alpha_n f(x_n) + (1-\alpha_n) S x_n + e_n), \forall n \geq 0.$$

In 2017, Li [15] introduced a new iterative algorithm in a real reflexive Banach space $E$ with the uniformly *Gâteaux* differentiable norm and $C$ is a nonempty closed convex subset of $E$ which has the normal structure:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n S J_{r_n}(e_n + x_n) + (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n T(x_n) + (1-\alpha_n) y_n, \forall n \geq 0. \end{cases}$$

In 2015, Xu et al. [16] used viscosity iterative algorithm to implicit midpoint rule for nonexpansive mapping in Hilbert space and proposed viscosity implicit midpoint rule: $\{x_n\}$ was generated by the following

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \geq 0.$$

Under some conditions on $\{\alpha_n\}$, they obtained that $\{x_n\}$ strongly converged to $q \in F(T)$, and $q$ was the solution of variational inequality $\langle (I-f)q, x-q \rangle \geq 0$, $x \in F(T)$.

In 2017, Luo et al. [17] extended the results of Xu et al. [16] from Hilbert space to uniformly smooth Banach space: $\{x_n\}$ was generated by the following

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \geq 0.$$

Under some conditions, they obtained that $\{x_n\}$ strongly converged to $q \in F(T)$, and $q$ was the solution of variational inequality $\langle (I-f)q, j(x-q) \rangle \geq 0$, $x \in F(T)$.

Motivated and inspired by the above papers, this paper uses the viscosity implicit midpoint rule to find common points of the fixed point set of a nonexpansive mapping and the zero point set of an accretive operator in Banach space and obtains the strong convergence results and improves the previous results. Finally, this paper gives numerical examples to support the main results.

## 2. Preliminaries

For all $\varepsilon \in [0,2]$, $\|x\| = \|y\| = 1$, $\|x-y\| \geq \varepsilon$, if there exists $\delta_\varepsilon > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta_\varepsilon$, then $E$ is called uniformly convex. A Banach space is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that

$$\|\lambda x+(1-\lambda)y\|^2 \leq \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)g(\|x-y\|). \tag{1}$$

For each $x, y \in U$, $U=\{x \in E : \|x\|=1\}$, if $\lim\limits_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists, then $E$ is said to be have a *Gâteaux* differentiable norm. If for each $y \in U$, the limit is attained uniformly for $x \in U$, then $E$ is said to be have a uniformly *Gâteaux* differentiable norm. It is well known that if $E$ has uniformly *Gâteaux* differentiable norm, then $J$ is single valued and norm-to-weak* uniformly continuous on each bounded subset of $E$, see [18].

Let $C$ be a closed convex subset of $E$. If for each bounded closed convex subset $D$ of $C$ which contains at least two points, there exists one element $x \in D$ which is not a diametral point of $D$ such that $diam(D) > \sup\{\|x-y\| : y \in D\}$, where $diam(D)$ is the diameter of $D$, then $C$ is said to have normal structure.

We need the following lemmas for the proof of our main results.

**Lemma 1** [19]. *For $\lambda, \mu > 0$, $x \in E$, so $J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1-\frac{\mu}{\lambda}\right)J_\lambda x\right)$.*

**Lemma 2** [20]. *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n,\ \forall n \geq 0,$$

*where $\{t_n\} \subset (0,1)$. If the following conditions are satisfied $\sum\limits_{n=0}^{\infty} t_n = \infty$; $b_n = o(t_n)$; $\sum\limits_{n=1}^{\infty} c_n < \infty$. So $\lim\limits_{n\to\infty} a_n = 0$.*

**Lemma 3** [4,21]. *Let $E$ be a real reflexive Banach space with the uniformly Gâteaux differentiable norm and $C$ be nonempty closed convex subset of $E$ which has normal structure. Let $S : C \to C$ be a nonexpansive mapping with a fixed point and $T : C \to C$ be a fixed contraction with the coefficient $\tau \in (0,1)$. Let $\{x_{S,T,t}\}$ be an sequence defined as follows*

$$x_{S,T,t} = tTx_t + (1-t)Sx_{S,T,t},$$

*where $t \in (0,1)$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $x^*$ of $S$, which is the unique solution in $F(S)$ to the following variational inequality*

$$\langle Tx^* - x^*, j(x^*-p)\rangle \geq 0,\ \forall p \in F(S).$$

**Lemma 4** [2]. *In a Banach space $E$, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle,\ \forall x, y \in E,$$

*where $j(x+y) \in J(x+y)$.*

## 3. Main Results

**Theorem 1.** *Let $E$ be a reflexive and uniformly convex Banach space which has uniformly Gâteaux differentiable norm and $C$ be a nonempty closed convex subset of $E$ which has normal structure. Let $f : C \to C$ be a contractive mapping with $k \in [0,1)$, $A$ be a m-accretive operator in $E$ and $S : C \to C$ be a nonexpansive mapping with $F(S) \cap N(A) \neq \varnothing$. For any $x_0 \in C$ and $\forall n \geq 0$, $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_n\left(\frac{x_n+x_{n+1}}{2}\right) + (1-\beta_n)J_{r_n}\left(\frac{x_n+x_{n+1}}{2}\right), \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n)Sy_n, \end{cases} \tag{2}$$

*where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,1)$ satisfy the following conditions:*

*(i)* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $|\alpha_n - \alpha_{n-1}| = o(\alpha_n)$; *(ii)* $\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$; *(iii)* $\lim_{n\to\infty} r_n = r$, $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$.

*Then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in F(S)\cap N(A)$, *where* $q$ *is the unique solution of the variational inequality* $\langle (I-f)q, J_\varphi(q-p)\rangle \le 0, \forall p \in F(S)\cap N(A)$.

**Proof.** The proof is split into eleven steps.

**Step 1:** Show that $\{x_n\}$ and $\{y_n\}$ are bounded.

Take $p \in F(S)\cap N(A)$, then we have

$$\begin{aligned}\|y_n - p\| &\le \beta_n \|\tfrac{x_n + x_{n+1}}{2} - p\| + (1-\beta_n)\|J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big) - p\| \\ &\le \beta_n \|\tfrac{x_n + x_{n+1}}{2} - p\| + (1-\beta_n)\|\tfrac{x_n+x_{n+1}}{2} - p\| \\ &\le \tfrac{1}{2}\|x_n - p\| + \tfrac{1}{2}\|x_{n+1} - p\|,\end{aligned}$$

and then we get

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\|fx_n - p\| + (1-\alpha_n)\|Sy_n - p\| \\ &\le k\alpha_n\|x_n - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|y_n - p\| \\ &\le k\alpha_n\|x_n - p\| + \alpha_n\|fp - p\| + \tfrac{1-\alpha_n}{2}\|x_n - p\| + \tfrac{1-\alpha_n}{2}\|x_{n+1} - p\| \\ &= \Big(k\alpha_n + \tfrac{1-\alpha_n}{2}\Big)\|x_n - p\| + \alpha_n\|fp - p\| + \tfrac{1-\alpha_n}{2}\|x_{n+1} - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &\le \frac{k\alpha_n + \frac{1-\alpha_n}{2}}{\frac{1+\alpha_n}{2}}\|x_n - p\| + \frac{2\alpha_n}{1+\alpha_n}\|fp - p\| \\ &= \Big[1 - \frac{2\alpha_n(1-k)}{1+\alpha_n}\Big]\|x_n - p\| + \frac{2\alpha_n(1-k)}{1+\alpha_n}\frac{\|fp-p\|}{1-k} \\ &\le \max\Big\{\|x_0 - p\|, \frac{\|fp-p\|}{1-k}\Big\}.\end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ are bounded. So $\{fx_n\}$, $\{fy_n\}$, $\{Sx_n\}$, $\{Sy_n\}$, $\{J_{r_n}x_n\}$ and $\{J_{r_n}y_n\}$ are also bounded.

**Step 2:** Show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

From (2), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n fx_n + (1-\alpha_n)Sy_n - \alpha_{n-1}fx_{n-1} - (1-\alpha_{n-1})Sy_{n-1}\| \\ &\le \alpha_n\|fx_n - fx_{n-1}\| + |\alpha_n - \alpha_{n-1}|\cdot\|fx_{n-1} - Sy_{n-1}\| + (1-\alpha_n)\|Sy_n - Sy_{n-1}\| \\ &\le k\alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\cdot\|fx_{n-1} - Sy_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\|.\end{aligned} \tag{3}$$

From (2), we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|\beta_n\Big(\tfrac{x_n+x_{n+1}}{2}\Big) + (1-\beta_n)J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big) - \beta_{n-1}\Big(\tfrac{x_{n-1}+x_n}{2}\Big) \\ &\quad -(1-\beta_{n-1})J_{r_{n-1}}\Big(\tfrac{x_{n-1}+x_n}{2}\Big)\| \\ &\le \beta_n\|\tfrac{x_{n+1}-x_{n-1}}{2}\| + |\beta_n - \beta_{n-1}|\cdot\|\tfrac{x_{n-1}+x_n}{2} - J_{r_{n-1}}\Big(\tfrac{x_{n-1}+x_n}{2}\Big)\| \\ &\quad +(1-\beta_n)\|J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big) - J_{r_{n-1}}\Big(\tfrac{x_{n-1}+x_n}{2}\Big)\|.\end{aligned} \tag{4}$$

From Lemma 1, we have

$$
\begin{aligned}
&\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - J_{r_{n-1}}\left(\tfrac{x_{n-1}+x_n}{2}\right)\| \\
&= \|J_{r_{n-1}}\left(\tfrac{r_{n-1}}{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) + \left(1-\tfrac{r_{n-1}}{r_n}\right)J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right)\right) - J_{r_{n-1}}\left(\tfrac{x_{n-1}+x_n}{2}\right)\| \\
&\leq \|\tfrac{r_{n-1}}{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) + \left(1-\tfrac{r_{n-1}}{r_n}\right)J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - \tfrac{x_{n-1}+x_n}{2}\| \\
&= \|\tfrac{r_{n-1}}{r_n}\left(\tfrac{x_{n+1}-x_{n-1}}{2}\right) + \left(1-\tfrac{r_{n-1}}{r_n}\right)\left(J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - \tfrac{x_{n-1}+x_n}{2}\right)\| \\
&\leq \tfrac{r_{n-1}}{r_n}\|\tfrac{x_{n+1}-x_{n-1}}{2}\| + \left|1-\tfrac{r_{n-1}}{r_n}\right|\cdot\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - \tfrac{x_n+x_{n+1}}{2}\| + \left|1-\tfrac{r_{n-1}}{r_n}\right|\cdot\|\tfrac{x_{n+1}-x_{n-1}}{2}\| \\
&= \|\tfrac{x_{n+1}-x_{n-1}}{2}\| + \left|1-\tfrac{r_{n-1}}{r_n}\right|\cdot\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - \tfrac{x_n+x_{n+1}}{2}\|.
\end{aligned}
\tag{5}
$$

Put (4) and (5) into (3), we get

$$
\begin{aligned}
\|x_{n+1}-x_n\| \quad &\leq k\alpha_n\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|\cdot\|fx_{n-1}-Sy_{n-1}\| + (1-\alpha_n)\|\tfrac{x_{n+1}-x_{n-1}}{2}\| \\
&+(1-\alpha_n)|\beta_n-\beta_{n-1}|\cdot\|\tfrac{x_{n-1}+x_n}{2} - J_{r_{n-1}}\left(\tfrac{x_{n-1}+x_n}{2}\right)\| \\
&+(1-\alpha_n)(1-\beta_n)\left|1-\tfrac{r_{n-1}}{r_n}\right|\cdot\|\tfrac{x_n+x_{n+1}}{2} - J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right)\| \\
&\leq k\alpha_n\|x_n-x_{n-1}\| + \tfrac{1-\alpha_n}{2}\|x_{n+1}-x_n\| + \tfrac{1-\alpha_n}{2}\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|M_1 \\
&+\left[(1-\alpha_n)|\beta_n-\beta_{n-1}| + (1-\alpha_n)(1-\beta_n)\left|1-\tfrac{r_{n-1}}{r_n}\right|\right]M_2 \\
&= \left(k\alpha_n + \tfrac{1-\alpha_n}{2}\right)\|x_n-x_{n-1}\| + \tfrac{1-\alpha_n}{2}\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|M_1 \\
&+\left[(1-\alpha_n)|\beta_n-\beta_{n-1}| + (1-\alpha_n)(1-\beta_n)\left|1-\tfrac{r_{n-1}}{r_n}\right|\right]M_2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-x_n\| \quad &\leq \tfrac{k\alpha_n+\frac{1-\alpha_n}{2}}{\frac{1+\alpha_n}{2}}\|x_n-x_{n-1}\| + \tfrac{2|\alpha_n-\alpha_{n-1}|}{1+\alpha_n}M_1 \\
&+\tfrac{2(1-\alpha_n)|\beta_n-\beta_{n-1}|+2(1-\alpha_n)(1-\beta_n)\left|1-\frac{r_{n-1}}{r_n}\right|}{1+\alpha_n}M_2 \\
&= \left[1-\tfrac{2\alpha_n(1-k)}{1+\alpha_n}\right]\|x_n-x_{n-1}\| + \tfrac{2|\alpha_n-\alpha_{n-1}|}{1+\alpha_n}M_1 \\
&+\tfrac{2(1-\alpha_n)|\beta_n-\beta_{n-1}|+2(1-\alpha_n)(1-\beta_n)\left|1-\frac{r_{n-1}}{r_n}\right|}{1+\alpha_n}M_2,
\end{aligned}
$$

where $M_1 = \max\|fx_{n-1}-Sy_{n-1}\|$ and $M_2 = \max\|\frac{x_n+x_{n+1}}{2} - J_{r_n}\left(\frac{x_n+x_{n+1}}{2}\right)\|$.

Take $t_n = \frac{2\alpha_n(1-k)}{1+\alpha_n}$, then $t_n > \alpha_n(1-k)$. From $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$, so $\sum\limits_{n=0}^{\infty}t_n = \infty$.

Take $b_n = \frac{2|\alpha_n-\alpha_{n-1}|}{1+\alpha_n}M_1$, then $\frac{b_n}{t_n} = \frac{|\alpha_n-\alpha_{n-1}|M_1}{\alpha_n(1-k)}$. From $|\alpha_n-\alpha_{n-1}| = o(\alpha_n)$, so $b_n = o(t_n)$.

Take $c_n = \frac{2(1-\alpha_n)|\beta_n-\beta_{n-1}|+2(1-\alpha_n)(1-\beta_n)\left|1-\frac{r_{n-1}}{r_n}\right|}{1+\alpha_n}M_2$. From $\lim\limits_{n\to\infty}r_n = r$, then $c_n < 2M_2\left(|\beta_n-\beta_{n-1}| + \frac{|r_n-r_{n-1}|}{r-\varepsilon}\right)(\forall\varepsilon>0)$. From $\sum\limits_{n=1}^{\infty}|\beta_n-\beta_{n-1}| < \infty$ and $\sum\limits_{n=1}^{\infty}|r_n-r_{n-1}| < \infty$, so $\sum\limits_{n=1}^{\infty}c_n < \infty$.

From Lemma 2, we get $\lim\limits_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

**Step 3:** Show that $\lim\limits_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0$.

Because $\|\cdot\|^2$ is convex function and (1), so we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|fx_n-p\|^2+(1-\alpha_n)\|Sy_n-p\|^2\\
&\le \alpha_n\|fx_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\
&\le \alpha_n\|fx_n-p\|^2+(1-\alpha_n)\Big[\beta_n\|\tfrac{x_n+x_{n+1}}{2}-p\|^2+(1-\beta_n)\|J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)-p\|^2\\
&\quad -\beta_n(1-\beta_n)g\Big(\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\|\Big)\Big]\\
&\le \alpha_n\|fx_n-p\|^2+(1-\alpha_n)\|\tfrac{x_n+x_{n+1}}{2}-p\|^2\\
&\quad -(1-\alpha_n)\beta_n(1-\beta_n)g\Big(\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\|\Big)\\
&\le \alpha_n\|fx_n-p\|^2+\tfrac{1-\alpha_n}{2}\|x_n-p\|^2+\tfrac{1-\alpha_n}{2}\|x_{n+1}-p\|^2\\
&\quad -(1-\alpha_n)\beta_n(1-\beta_n)g\Big(\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\|\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \tfrac{1-\alpha_n}{1+\alpha_n}\|x_n-p\|^2+\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2\\
&\quad -\tfrac{2(1-\alpha_n)\beta_n(1-\beta_n)}{1+\alpha_n}g\Big(\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\|\Big)\\
&\le \|x_n-p\|^2+\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2\\
&\quad -\tfrac{2(1-\alpha_n)\beta_n(1-\beta_n)}{1+\alpha_n}g\Big(\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\|\Big).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\tfrac{2(1-\alpha_n)\beta_n(1-\beta_n)}{1+\alpha_n}g\Big(\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\|\Big)-\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2\\
\le\ &\|x_n-p\|^2-\|x_{n+1}-p\|^2.
\end{aligned}$$

If $\frac{2(1-\alpha_n)\beta_n(1-\beta_n)}{1+\alpha_n}g\Big(\|\frac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\frac{x_n+x_{n+1}}{2}\Big)\|\Big)\le \frac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2$, so from $\lim_{n\to\infty}\alpha_n=0$ and the boundedness of $\{fx_n\}$, we get $\lim_{n\to\infty}g\Big(\|\frac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\frac{x_n+x_{n+1}}{2}\Big)\|\Big)=0$.

If $\frac{2(1-\alpha_n)\beta_n(1-\beta_n)}{1+\alpha_n}g\Big(\|\frac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\frac{x_n+x_{n+1}}{2}\Big)\|\Big)> \frac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2$, so

$$\begin{aligned}
&\sum_{n=0}^{N}\Big[\tfrac{2(1-\alpha_n)\beta_n(1-\beta_n)}{1+\alpha_n}g\Big(\Big\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\Big\|\Big)-\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2\Big]\\
&\le \|x_0-p\|^2-\|x_{N+1}-p\|^2\le\|x_0-p\|^2.
\end{aligned}$$

Then $\sum_{n=0}^{\infty}\Big[\frac{2(1-\alpha_n)\beta_n(1-\beta_n)}{1+\alpha_n}g\Big(\Big\|\frac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\frac{x_n+x_{n+1}}{2}\Big)\Big\|\Big)-\frac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2\Big]<\infty$.

So we get

$$\lim_{n\to\infty}\Big[\frac{2(1-\alpha_n)\beta_n(1-\beta_n)}{1+\alpha_n}g\Big(\|\frac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\frac{x_n+x_{n+1}}{2}\Big)\|\Big)-\frac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2\Big]=0,$$

and then $\lim_{n\to\infty}g\Big(\|\frac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\frac{x_n+x_{n+1}}{2}\Big)\|\Big)=0$.

From the property of $g$, so we get $\lim_{n\to\infty}\|\frac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\frac{x_n+x_{n+1}}{2}\Big)\|=0$.

We also have

$$\begin{aligned}
\|x_n-J_{r_n}x_n\| &\le \|x_n-\tfrac{x_n+x_{n+1}}{2}\|+\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\|+\|J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)-J_{r_n}x_n\|\\
&\le \|x_{n+1}-x_n\|+\|\tfrac{x_n+x_{n+1}}{2}-J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)\|.
\end{aligned}$$

Then from step 2, we get $\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0$.

**Step 4:** Show that $\lim_{n\to\infty}\|y_n - Sy_n\| = 0$.

$$\begin{aligned}
\|y_n - Sy_n\| &\leq \beta_n\|\tfrac{x_n+x_{n+1}}{2} - Sy_n\| + (1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - Sy_n\| \\
&\leq \|\tfrac{x_n+x_{n+1}}{2} - Sy_n\| + (1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - \tfrac{x_n+x_{n+1}}{2}\| \\
&\leq \|\tfrac{x_n+x_{n+1}}{2} - x_{n+1}\| + \|x_{n+1} - Sy_n\| + (1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - \tfrac{x_n+x_{n+1}}{2}\| \\
&= \tfrac{1}{2}\|x_n - x_{n+1}\| + \alpha_n\|fx_n - Sy_n\| + (1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}\right) - \tfrac{x_n+x_{n+1}}{2}\|.
\end{aligned}$$

From steps 2 and 3, $\lim_{n\to\infty}\alpha_n = 0$ and the boundedness of $\{fx_n\}$ and $\{Sy_n\}$, we get $\lim_{n\to\infty}\|y_n - Sy_n\| = 0$.

**Step 5:** Show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

$$\begin{aligned}
\|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| + \|Sy_n - y_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|fx_n - Sy_n\| + \|Sy_n - y_n\|.
\end{aligned}$$

From steps 2 and 4, we get $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

**Step 6:** Show that $\lim_{n\to\infty}\|y_n - J_{r_n}y_n\| = 0$.

$$\begin{aligned}
\|y_n - J_{r_n}y_n\| &\leq \|y_n - x_n\| + \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_{r_n}y_n\| \\
&\leq 2\|y_n - x_n\| + \|x_n - J_{r_n}x_n\|.
\end{aligned}$$

From steps 3 and 5, we get $\lim_{n\to\infty}\|y_n - J_{r_n}y_n\| = 0$.

**Step 7:** Show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - y_n\| + \|y_n - Sy_n\| + \|Sy_n - Sx_n\| \\
&\leq 2\|x_n - y_n\| + \|y_n - Sy_n\|.
\end{aligned}$$

From steps 4 and 5, we get $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

**Step 8:** Show that $\lim_{n\to\infty}\|y_n - J_r y_n\| = 0$.

$$\begin{aligned}
\|y_n - J_r y_n\| &\leq \|y_n - J_{r_n}y_n\| + \|J_{r_n}y_n - J_r y_n\| \\
&= \|y_n - J_{r_n}y_n\| + \|J_r\left(\tfrac{r}{r_n}y_n + \left(1 - \tfrac{r}{r_n}\right)J_{r_n}y_n\right) - J_r y_n\| \\
&\leq \|y_n - J_{r_n}y_n\| + \left|1 - \tfrac{r}{r_n}\right| \cdot \|J_{r_n}y_n - y_n\|.
\end{aligned}$$

From step 6 and $\lim_{n\to\infty} r_n = r$, we get $\lim_{n\to\infty}\|y_n - J_r y_n\| = 0$.

**Step 9:** Show that $\lim_{n\to\infty}\|x_n - J_r x_n\| = 0$.

$$\begin{aligned}
\|x_n - J_r x_n\| &\leq \|x_n - y_n\| + \|y_n - J_r y_n\| + \|J_r y_n - J_r x_n\| \\
&\leq 2\|x_n - y_n\| + \|y_n - J_r y_n\|.
\end{aligned}$$

From steps 5 and 8, we get $\lim_{n\to\infty}\|x_n - J_r x_n\| = 0$.

**Step 10:** Show that $\limsup_{n\to\infty}\langle q - fq, J(q - x_n)\rangle = 0$.

Let $\{x_t\}$ be defined by $x_t = tfx_t + (1-t)Sx_t$. From Lemma 3, we have that $\{x_t\}$ converges strongly to $q \in P_{F(S)\cap N(A)}fq$, which is also the unique solution of the variational inequality $\langle q - fq, J(q-p)\rangle \leq 0, \forall p \in F(S)\cap N(A)$.

We have

$$\begin{aligned}
\|x_t - x_n\|^2 &= (1-t)\langle Sx_t - Sx_n + Sx_n - x_n, J(x_t - x_n)\rangle + t\langle fx_t - x_t + x_t - x_n, J(x_t - x_n)\rangle \\
&\leq (1-t)\|x_t - x_n\|^2 + (1-t)\|Sx_n - x_n\| \cdot \|x_t - x_n\| + t\|x_t - x_n\|^2 + t\langle fx_t - x_t, J(x_t - x_n)\rangle \\
&= \|x_t - x_n\|^2 + (1-t)\|Sx_n - x_n\| \cdot \|x_t - x_n\| + t\|x_t - x_n\|^2 + t\langle fx_t - x_t, J(x_t - x_n)\rangle.
\end{aligned}$$

It follows that $\langle fx_t - x_t, J(x_t - x_n)\rangle \leq \frac{1-t}{t}\|Sx_n - x_n\| \cdot \|x_t - x_n\|$. From step 7, we get $\limsup\limits_{n\to\infty}\langle q - fq, J(q - x_n)\rangle = 0$.

**Step 11:** Show that $\lim\limits_{n\to\infty}\|x_n - q\| = 0$.

From Lemma 4, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq (1-\alpha_n)^2\|Sy_n - q\|^2 + 2\alpha_n\langle fx_n - q, J(x_{n+1} - q)\rangle \\
&\leq (1-\alpha_n)^2\|y_n - q\|^2 + 2k\alpha_n\|x_n - q\| \cdot \|x_{n+1} - q\| + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle \\
&\leq (1-\alpha_n)^2\left(\tfrac{1}{2}\|x_n - q\| + \tfrac{1}{2}\|x_{n+1} - q\|\right)^2 + 2k\alpha_n\|x_n - q\| \cdot \|x_{n+1} - q\| \\
&\quad + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle \\
&= \left(\tfrac{1-\alpha_n}{2}\right)^2\|x_n - q\|^2 + \left(\tfrac{1-\alpha_n}{2}\right)^2\|x_{n+1} - q\|^2 + \tfrac{(1-\alpha_n)^2}{2}\|x_n - q\| \cdot \|x_{n+1} - q\| \\
&\quad + 2k\alpha_n\|x_n - q\| \cdot \|x_{n+1} - q\| + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle \\
&= \left(\tfrac{1-\alpha_n}{2}\right)^2\|x_n - q\|^2 + \left(\tfrac{1-\alpha_n}{2}\right)^2\|x_{n+1} - q\|^2 \\
&\quad + \left[\tfrac{(1-\alpha_n)^2}{2} + 2k\alpha_n\right]\|x_n - q\| \cdot \|x_{n+1} - q\| + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle \\
&\leq \left(\tfrac{1-\alpha_n}{2}\right)^2\|x_n - q\|^2 + \left(\tfrac{1-\alpha_n}{2}\right)^2\|x_{n+1} - q\|^2 \\
&\quad + \left[\tfrac{(1-\alpha_n)^2}{4} + k\alpha_n\right]\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \frac{\left(\frac{1-\alpha_n}{2}\right)^2 + \frac{(1-\alpha_n)^2}{4} + k\alpha_n}{1 - \left[\left(\frac{1-\alpha_n}{2}\right)^2 + \frac{(1-\alpha_n)^2}{4} + k\alpha_n\right]}\|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \left[\left(\frac{1-\alpha_n}{2}\right)^2 + \frac{(1-\alpha_n)^2}{4} + k\alpha_n\right]}\langle fq - q, J(x_{n+1} - q)\rangle \\
&= \left[1 - \frac{1 - 2\left(\frac{(1-\alpha_n)^2}{2} + k\alpha_n\right)}{1 - \left(\frac{(1-\alpha_n)^2}{2} + k\alpha_n\right)}\right]\|x_n - q\|^2 + \frac{2\alpha_n}{1 - \left[\frac{(1-\alpha_n)^2}{2} + k\alpha_n\right]}\langle fq - q, J(x_{n+1} - q)\rangle.
\end{aligned}$$

Take $t_n = \frac{1 - 2\left(\frac{(1-\alpha_n)^2}{2} + k\alpha_n\right)}{1 - \left(\frac{(1-\alpha_n)^2}{2} + k\alpha_n\right)}$. From $\lim\limits_{n\to\infty}\alpha_n = 0$, we have

$$t_n \geq 1 - 2\left(\frac{(1-\alpha_n)^2}{2} + k\alpha_n\right) = \alpha_n(2 - 2k - \alpha_n) \geq \alpha_n(2 - 2k - \varepsilon)(\forall \varepsilon > 0).$$

From $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$, we get $\sum\limits_{n=0}^{\infty} t_n = \infty$.

Take $b_n = \frac{2\alpha_n}{1-\left[\frac{(1-\alpha_n)^2}{2}+k\alpha_n\right]}\langle fq-q, J(x_{n+1}-q)\rangle$, then we have

$$\frac{b_n}{t_n} = \frac{2\alpha_n\langle fq-q, J(x_{n+1}-q)\rangle}{1-2\left[\frac{(1-\alpha_n)^2}{2}+k\alpha_n\right]} = \frac{2\langle fq-q, J(x_{n+1}-q)\rangle}{2-2k-\alpha_n}.$$

From $\lim_{n\to\infty}\alpha_n = 0$ and step 10, we get $b_n = o(t_n)$.

Take $c_n = 0$, then we get $\sum_{n=0}^{\infty} c_n < \infty$.

From Lemma 2, we get $\lim_{n\to\infty}\|x_n - q\| = 0$. This completes the proof. □

The results of Theorem 1 improve the related results in [13,14,16,17]. For example, this paper uses the viscosity implicit midpoint rule to find common points of the fixed point set of a nonexpansive mapping and the zero point set of an accretive operator and the results improve the related results in [13,14]; If $\beta_n = 0$, the results of Theorem 1 can obtain the related results in [16,17].

**Corollary 1.** *Let $E$ be a reflexive and uniformly convex Banach space which has uniformly Gâteaux differentiable norm and $C$ be a nonempty closed convex subset of $E$ which has normal structure. Let $f : C \to C$ be a contractive mapping with $k \in [0,1)$, $A$ be a m-accretive operator in $E$ and $S : C \to C$ be a nonexpansive mapping with $F(S) \cap N(A) \neq \varnothing$. For any $x_0 \in C$ and $\forall n \geq 0$, $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_n\left(\frac{x_n+x_{n+1}}{2}\right) + (1-\beta_n)J_{r_n}\left(\frac{x_n+x_{n+1}}{2}\right) + e_n, \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n)Sy_n, \end{cases}$$

*where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{e_n\} \subset E$ and $\{r_n\} \subset (0,1)$ satisfy the following conditions:*

*(i)* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $|\alpha_n - \alpha_{n-1}| = o(\alpha_n)$;

*(ii)* $\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$;

*(iii)* $\lim_{n\to\infty} r_n = r$, $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$;

*(iv)* $\|e_n\| = o(\alpha_n)$.

*Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F(S) \cap N(A)$, where $q$ is the unique solution of the variational inequality $\langle (I-f)q, J_\varphi(q-p)\rangle \leq 0, \forall p \in F(S) \cap N(A)$.*

**Proof.** Assume

$$\begin{cases} w_n = \beta_n\left(\frac{z_n+z_{n+1}}{2}\right) + (1-\beta_n)J_{r_n}\left(\frac{z_n+z_{n+1}}{2}\right), \\ z_{n+1} = \alpha_n f z_n + (1-\alpha_n)Sw_n. \end{cases}$$

Then we have

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &\leq k\alpha_n\|x_n - z_n\| + (1-\alpha_n)\|y_n - w_n\| \\ &\leq k\alpha_n\|x_n - z_n\| + (1-\alpha_n)\left(\left\|\frac{x_n+x_{n+1}}{2} - \frac{z_n+z_{n+1}}{2}\right\| + \|e_n\|\right) \\ &\leq \left(k\alpha_n + \frac{1-\alpha_n}{2}\right)\|x_n - z_n\| + \frac{1-\alpha_n}{2}\|x_{n+1} - z_{n+1}\| + (1-\alpha_n)\|e_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &\leq \frac{k\alpha_n + \frac{1-\alpha_n}{2}}{\frac{1+\alpha_n}{2}}\|x_n - z_n\| + \frac{2(1-\alpha_n)}{1+\alpha_n}\|e_n\| \\ &= \left[1 - \frac{2\alpha_n(1-k)}{1+\alpha_n}\right]\|x_n - z_n\| + \frac{2(1-\alpha_n)}{1+\alpha_n}\|e_n\|. \end{aligned}$$

Take $t_n = \frac{2\alpha_n(1-k)}{1+\alpha_n}$, then $t_n \geq \alpha_n(1-k)$. From $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, we get $\sum\limits_{n=0}^{\infty} t_n = \infty$.

Take $b_n = \frac{2(1-\alpha_n)}{1+\alpha_n}\|e_n\|$, then $\frac{b_n}{t_n} = \frac{(1-\alpha_n)\|e_n\|}{\alpha_n(1-k)}$. From $\|e_n\| = o(\alpha_n)$, we get $b_n = o(t_n)$.

Take $c_n = 0$, then we get $\sum\limits_{n=0}^{\infty} c_n < \infty$.

From Lemma 2, we get $\lim\limits_{n\to\infty}\|x_n - z_n\| = 0$. From Theorem 1, we have $\{z_n\}$ and $\{w_n\}$ converge strongly to $q \in F(S) \cap N(A)$, where $q$ is the unique solution of the variational inequality $\langle (I-f)q, J_\varphi(q-p)\rangle \leq 0$, $\forall p \in F(S) \cap N(A)$. So $\{x_n\}$ and $\{y_n\}$ also converge strongly to $q \in F(S) \cap N(A)$. This completes the proof. □

The results of Corollary 1 improve the related results in [14,16,17].

**Theorem 2.** *Let $E$ be a reflexive and uniformly convex Banach space which has uniformly Gâteaux differentiable norm and $C$ be a nonempty closed convex subset of $E$ which has normal structure. Let $f : C \to C$ be a contractive mapping with $k \in [0,1)$, $A$ be a m-accretive operator in $E$ and $S : C \to C$ be a nonexpansive mapping with $F(S) \cap N(A) \neq \varnothing$. For any $x_0 \in C$ and $\forall n \geq 0$, $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_n\left(\frac{x_n+x_{n+1}}{2}\right) + (1-\beta_n)J_{r_n}\left(\frac{x_n+x_{n+1}}{2} + e_n\right), \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n)Sy_n, \end{cases} \tag{6}$$

*where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{e_n\} \subset E$ and $\{r_n\} \subset (0,1)$ satisfy the following conditions:*

*(i)* $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $|\alpha_n - \alpha_{n-1}| = o(\alpha_n)$;

*(ii)* $\sum\limits_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

*(iii)* $\lim\limits_{n\to\infty} r_n = r$, $\sum\limits_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$;

*(iv)* $\sum\limits_{n=1}^{\infty} \|e_n\| < \infty$.

*Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F(S) \cap N(A)$, where $q$ is the unique solution of the variational inequality $\langle (I-f)q, J_\varphi(q-p)\rangle \leq 0$, $\forall p \in F(S) \cap N(A)$.*

**Proof.** The proof is split into eleven steps.

**Step 1:** Show that $\{x_n\}$ and $\{y_n\}$ are bounded.

Take $p \in F(S) \cap N(A)$, then we have

$$\begin{aligned} \|y_n - p\| &\leq \beta_n\|\tfrac{x_n+x_{n+1}}{2} - p\| + (1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2} + e_n\right) - p\| \\ &\leq \beta_n\|\tfrac{x_n+x_{n+1}}{2} - p\| + (1-\beta_n)\|\tfrac{x_n+x_{n+1}}{2} - p\| + (1-\beta_n)\|e_n\| \\ &\leq \tfrac{1}{2}\|x_n - p\| + \tfrac{1}{2}\|x_{n+1} - p\| + \|e_n\|, \end{aligned}$$

and then we get

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|fx_n - p\| + (1-\alpha_n)\|Sy_n - p\| \\ &\leq k\alpha_n\|x_n - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|y_n - p\| \\ &\leq \left(k\alpha_n + \tfrac{1-\alpha_n}{2}\right)\|x_n - p\| + \tfrac{1-\alpha_n}{2}\|x_{n+1} - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|e_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1}-p\| &\leq \left[1-\tfrac{2\alpha_n(1-k)}{1+\alpha_n}\right]\|x_n-p\|+\tfrac{2\alpha_n}{1+\alpha_n}\|fp-p\|+\tfrac{2(1-\alpha_n)}{1+\alpha_n}\|e_n\| \\ &\leq \left[1-\tfrac{2\alpha_n(1-k)}{1+\alpha_n}\right]\|x_n-p\|+\tfrac{2\alpha_n(1-k)}{1+\alpha_n}\tfrac{\|fp-p\|}{1-k}+2\|e_n\| \\ &\leq \max\left\{\|x_0-p\|, \tfrac{\|fp-p\|}{1-k}+2\|e_n\|\right\}. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ are bounded. So $\{fx_n\}$, $\{fy_n\}$, $\{Sx_n\}$, $\{Sy_n\}$, $\{J_{r_n}x_n\}$ and $\{J_{r_n}y_n\}$ are also bounded.

**Step 2:** Show that $\lim\limits_{n\to\infty}\|x_{n+1}-x_n\|=0$.

From (6), we have

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|\alpha_n fx_n+(1-\alpha_n)Sy_n-\alpha_{n-1}fx_{n-1}-(1-\alpha_{n-1})Sy_{n-1}\| \\ &\leq \alpha_n\|fx_n-fx_{n-1}\|+|\alpha_n-\alpha_{n-1}|\cdot\|fx_{n-1}-Sy_{n-1}\|+(1-\alpha_n)\|Sy_n-Sy_{n-1}\| \\ &\leq k\alpha_n\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\cdot\|fx_{n-1}-Sy_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\|. \end{aligned} \tag{7}$$

From (6), we have

$$\begin{aligned} \|y_n-y_{n-1}\| &\leq \beta_n\|\tfrac{x_{n+1}-x_{n-1}}{2}\|+|\beta_n-\beta_{n-1}|\cdot\|\tfrac{x_{n-1}+x_n}{2}-J_{r_{n-1}}\Big(\tfrac{x_{n-1}+x_n}{2}+e_{n-1}\Big)\| \\ &\quad+(1-\beta_n)\|J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)-J_{r_{n-1}}\Big(\tfrac{x_{n-1}+x_n}{2}+e_{n-1}\Big)\|. \end{aligned} \tag{8}$$

From Lemma 1, we have

$$\begin{aligned} &\|J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)-J_{r_{n-1}}\Big(\tfrac{x_{n-1}+x_n}{2}+e_{n-1}\Big)\| \\ &= \|J_{r_{n-1}}\Big(\tfrac{r_{n-1}}{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)+\Big(1-\tfrac{r_{n-1}}{r_n}\Big)J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)\Big)-J_{r_{n-1}}\Big(\tfrac{x_{n-1}+x_n}{2}+e_{n-1}\Big)\| \\ &\leq \|\tfrac{r_{n-1}}{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)+\Big(1-\tfrac{r_{n-1}}{r_n}\Big)J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)-\Big(\tfrac{x_{n-1}+x_n}{2}+e_{n-1}\Big)\| \\ &\leq \tfrac{r_{n-1}}{r_n}\|\tfrac{x_{n+1}-x_{n-1}}{2}+e_n-e_{n-1}\|+\Big|1-\tfrac{r_{n-1}}{r_n}\Big|\cdot\|J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)-\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)\| \\ &\quad+\Big|1-\tfrac{r_{n-1}}{r_n}\Big|\cdot\|\tfrac{x_{n+1}-x_{n-1}}{2}+e_n-e_{n-1}\| \\ &= \|\tfrac{x_{n+1}-x_{n-1}}{2}+e_n-e_{n-1}\|+\Big|1-\tfrac{r_{n-1}}{r_n}\Big|\cdot\|J_{r_n}\Big(\tfrac{x_n+x_{n+1}}{2}\Big)-\Big(\tfrac{x_n+x_{n+1}}{2}+e_n\Big)\|. \end{aligned} \tag{9}$$

Put (8) and (9) into (7), we get

$$\begin{aligned} \|x_{n+1}-x_n\| &\leq k\alpha_n\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1+(1-\alpha_n)\|\tfrac{x_{n+1}-x_{n-1}}{2}\| \\ &\quad+(1-\alpha_n)|\beta_n-\beta_{n-1}|M_3+(1-\alpha_n)(1-\beta_n)\|e_n-e_{n-1}\| \\ &\quad+(1-\alpha_n)(1-\beta_n)\Big|1-\tfrac{r_{n-1}}{r_n}\Big|M_4 \\ &\leq \Big(k\alpha_n+\tfrac{1-\alpha_n}{2}\Big)\|x_n-x_{n-1}\|+\tfrac{1-\alpha_n}{2}\|x_{n+1}-x_n\|+|\alpha_n-\alpha_{n-1}|M_1 \\ &\quad+(1-\alpha_n)|\beta_n-\beta_{n-1}|M_3+(1-\alpha_n)(1-\beta_n)\|e_n-e_{n-1}\| \\ &\quad+(1-\alpha_n)(1-\beta_n)\Big|1-\tfrac{r_{n-1}}{r_n}\Big|M_4. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1}-x_n\| = &\left[1-\tfrac{2\alpha_n(1-k)}{1+\alpha_n}\right]\|x_n-x_{n-1}\|+\tfrac{2|\alpha_n-\alpha_{n-1}|}{1+\alpha_n}M_1+\tfrac{2(1-\alpha_n)}{1+\alpha_n}|\beta_n-\beta_{n-1}|M_3 \\ &+\tfrac{2(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}(\|e_n\|+\|e_{n-1}\|)+\tfrac{2(1-\alpha_n)(1-\beta_n)|r_n-r_{n-1}|}{(1+\alpha_n)r_n}M_4, \end{aligned}$$

where

$$M_3 = \max\|\frac{x_n + x_{n+1}}{2} - J_{r_n}\left(\frac{x_n + x_{n+1}}{2} + e_n\right)\|,\ M_4 = \max\|J_{r_n}\left(\frac{x_n + x_{n+1}}{2} + e_n\right) - \left(\frac{x_n + x_{n+1}}{2} + e_n\right)\|.$$

Take $t_n = \frac{2\alpha_n(1-k)}{1+\alpha_n}$, then $t_n > \alpha_n(1-k)$. From $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, so $\sum\limits_{n=0}^{\infty} t_n = \infty$.

Take $b_n = \frac{2|\alpha_n - \alpha_{n-1}|}{1+\alpha_n} M_1$, then $\frac{b_n}{t_n} = \frac{|\alpha_n - \alpha_{n-1}|M_1}{\alpha_n(1-k)}$. From $|\alpha_n - \alpha_{n-1}| = o(\alpha_n)$, so $b_n = o(t_n)$.

Take

$$c_n = \frac{2(1-\alpha_n)}{1+\alpha_n}|\beta_n - \beta_{n-1}|M_3 + \frac{2(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}(\|e_n\| + \|e_{n-1}\|) + \frac{2(1-\alpha_n)(1-\beta_n)|r_n - r_{n-1}|}{(1+\alpha_n)r_n} M_4,$$

then $c_n < 2|\beta_n - \beta_{n-1}|M_3 + 2(\|e_n\| + \|e_{n-1}\|) + \frac{2|r_n - r_{n-1}|M_4}{r+\varepsilon}$ $(\forall \varepsilon > 0)$.

From $\sum\limits_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, $\sum\limits_{n=1}^{\infty} \|e_n\| < \infty$, $\lim\limits_{n\to\infty} r_n = r$ and $\sum\limits_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, so $\sum\limits_{n=1}^{\infty} c_n < \infty$.

From Lemma 2, we get $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

**Step 3:** Show that $\lim\limits_{n\to\infty} \|x_n - J_{r_n} x_n\| = 0$.

Because $\|\cdot\|^2$ is convex function and (1), so we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)\|S y_n - p\|^2 \\
&\le \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)\beta_n \|\tfrac{x_n + x_{n+1}}{2} - p\|^2 \\
&+ (1-\alpha_n)(1-\beta_n)\|J_{r_n}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right) - p\|^2 \\
&= \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)[\beta_n \|\tfrac{x_n + x_{n+1}}{2} - p\|^2 \\
&+ (1-\beta_n)\|J_{\frac{r_n}{2}}\left(\tfrac{1}{2}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right) + \tfrac{1}{2} J_{r_n}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right)\right) - p\|^2 ] \\
&\le \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)\beta_n \|\tfrac{x_n + x_{n+1}}{2} - p\|^2 \\
&+ (1-\alpha_n)(1-\beta_n)\|\tfrac{1}{2}\left(\tfrac{x_n + x_{n+1}}{2} + e_n - p\right) + \tfrac{1}{2}\left(J_{r_n}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right) - p\right)\|^2 \\
&\le \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)\beta_n \|\tfrac{x_n + x_{n+1}}{2} - p\|^2 \\
&+ (1-\alpha_n)(1-\beta_n)\|\tfrac{x_n + x_{n+1}}{2} + e_n - p\|^2 \\
&- \tfrac{(1-\alpha_n)(1-\beta_n)}{4} g\left(\|\tfrac{x_n + x_{n+1}}{2} + e_n - J_{r_n}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right)\|\right) \\
&\le \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)\beta_n \|\tfrac{x_n + x_{n+1}}{2} - p\|^2 \\
&+ (1-\alpha_n)(1-\beta_n)\|\tfrac{x_n + x_{n+1}}{2} + e_n - p\|^2 \\
&- \tfrac{(1-\alpha_n)(1-\beta_n)}{4} g\left(\|\tfrac{x_n + x_{n+1}}{2} + e_n - J_{r_n}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right)\|\right) \\
&\le \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)\beta_n \|\tfrac{x_n + x_{n+1}}{2} - p\|^{2}2 \\
&+ (1-\alpha_n)(1-\beta_n)\|\tfrac{x_n + x_{n+1}}{2} + e_n - p\| \\
&+ (1-\alpha_n)(1-\beta_n)\left(\|\tfrac{x_n + x_{n+1}}{2} - p\|^2 + 2\langle e_n, J\left(\tfrac{x_n + x_{n+1}}{2} + e_n - p\right)\rangle\right) \\
&- \tfrac{(1-\alpha_n)(1-\beta_n)}{4} g\left(\|\tfrac{x_n + x_{n+1}}{2} + e_n - J_{r_n}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right)\|\right) \\
&= \alpha_n \|f x_n - p\|^2 + (1-\alpha_n)\|\tfrac{x_n + x_{n+1}}{2} - p\|^2 \\
&+ 2(1-\alpha_n)(1-\beta_n)\langle e_n, J\left(\tfrac{x_n + x_{n+1}}{2} + e_n - p\right)\rangle \\
&- \tfrac{(1-\alpha_n)(1-\beta_n)}{4} g\left(\|\tfrac{x_n + x_{n+1}}{2} + e_n - J_{r_n}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right)\|\right) \\
&\le \alpha_n \|f x_n - p\|^2 + \tfrac{1-\alpha_n}{2}\|x_n - p\|^2 + \tfrac{1-\alpha_n}{2}\|x_{n+1} - p\|^2 \\
&+ 2(1-\alpha_n)(1-\beta_n)\langle e_n, J\left(\tfrac{x_n + x_{n+1}}{2} + e_n - p\right)\rangle \\
&- \tfrac{(1-\alpha_n)(1-\beta_n)}{4} g\left(\|\tfrac{x_n + x_{n+1}}{2} + e_n - J_{r_n}\left(\tfrac{x_n + x_{n+1}}{2} + e_n\right)\|\right).
\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \tfrac{1-\alpha_n}{1+\alpha_n}\|x_n-p\|^2+\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2+\tfrac{(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}\|e_n\|\cdot\|\tfrac{x_n+x_{n+1}}{2}+e_n-p\|\\ &-\tfrac{(1-\alpha_n)(1-\beta_n)}{4}g\left(\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|\right)\\ &\le \|x_n-p\|^2+\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2+\tfrac{(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}\|e_n\|\cdot\|\tfrac{x_n+x_{n+1}}{2}+e_n-p\|\\ &-\tfrac{(1-\alpha_n)(1-\beta_n)}{4}g\left(\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|\right).\end{aligned}$$

Then we have

$$\begin{aligned}&\tfrac{(1-\alpha_n)(1-\beta_n)}{4}g\left(\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|\right)-\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2-\\ &\tfrac{(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}\|e_n\|\cdot\|\tfrac{x_n+x_{n+1}}{2}+e_n-p\|\le\|x_n-p\|^2-\|x_{n+1}-p\|^2.\end{aligned}$$

If $\frac{(1-\alpha_n)(1-\beta_n)}{4}g\left(\|\frac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\frac{x_n+x_{n+1}}{2}+e_n\right)\|\right)\le\frac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2+\frac{(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}\|e_n\|\cdot\|\frac{x_n+x_{n+1}}{2}+e_n-p\|$, so from $\lim\limits_{n\to\infty}\alpha_n=0$, step 1 and $\sum\limits_{n=0}^{\infty}\|e_n\|<\infty$, we get $\lim\limits_{n\to\infty}g\left(\|\frac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\frac{x_n+x_{n+1}}{2}+e_n\right)\|\right)=0$.

If $\frac{(1-\alpha_n)(1-\beta_n)}{4}g\left(\|\frac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\frac{x_n+x_{n+1}}{2}+e_n\right)\|\right)\ge\frac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2+\frac{(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}\|e_n\|\cdot\|\frac{x_n+x_{n+1}}{2}+e_n-p\|$, so

$$\begin{aligned}&\sum_{n=0}^{N}\Big[\tfrac{(1-\alpha_n)(1-\beta_n)}{4}g\left(\left\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\right\|\right)-\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2-\\ &\tfrac{(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}\|e_n\|\cdot\|\tfrac{x_n+x_{n+1}}{2}+e_n-p\|\Big]\le\|x_0-p\|^2-\|x_{N+1}-p\|^2\le\|x_0-p\|^2.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{n=0}^{\infty}\Big[\tfrac{(1-\alpha_n)(1-\beta_n)}{4}g\left(\left\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\right\|\right)-\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2-\\ &\tfrac{(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}\|e_n\|\cdot\|\tfrac{x_n+x_{n+1}}{2}+e_n-p\|\Big]<\infty.\end{aligned}$$

So we get

$$\begin{aligned}&\lim_{n\to\infty}\Big[\tfrac{(1-\alpha_n)(1-\beta_n)}{4}g\left(\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|\right)-\tfrac{2\alpha_n}{1+\alpha_n}\|fx_n-p\|^2-\\ &\tfrac{(1-\alpha_n)(1-\beta_n)}{1+\alpha_n}\|e_n\|\cdot\|\tfrac{x_n+x_{n+1}}{2}+e_n-p\|\Big]=0\end{aligned}$$

and then $\lim\limits_{n\to\infty}g\left(\|\frac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\frac{x_n+x_{n+1}}{2}+e_n\right)\|\right)=0$.

From the property of $g$, so we get $\lim\limits_{n\to\infty}\|\frac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\frac{x_n+x_{n+1}}{2}+e_n\right)\|=0$.

We also have

$$\begin{aligned}\|x_n-J_{r_n}x_n\| &\le\|x_n-\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|+\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|\\ &+\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)-J_{r_n}x_n\|\\ &\le 2\|x_n-\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|+\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|\\ &\le\|x_n-x_{n+1}\|+2\|e_n\|+\|\tfrac{x_n+x_{n+1}}{2}+e_n-J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2}+e_n\right)\|.\end{aligned}$$

Then from step 2 and $\sum\limits_{n=0}^{\infty}\|e_n\|<\infty$, we get $\lim\limits_{n\to\infty}\|x_n-J_{r_n}x_n\|=0$.

**Step 4:** Show that $\lim_{n\to\infty}\|y_n - Sy_n\| = 0$.

$$\begin{aligned}\|y_n - Sy_n\| &\le \beta_n\|\tfrac{x_n+x_{n+1}}{2} - Sy_n\| + (1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2} + e_n\right) - Sy_n\| \\ &\le \|\tfrac{x_n+x_{n+1}}{2} - Sy_n\| + (1-\beta_n)\|e_n\| \\ &+(1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2} + e_n\right) - \left(\tfrac{x_n+x_{n+1}}{2} + e_n\right)\| \\ &\le \|\tfrac{x_n+x_{n+1}}{2} - x_{n+1}\| + \|x_{n+1} - Sy_n\| + (1-\beta_n)\|e_n\| \\ &+(1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2} + e_n\right) - \left(\tfrac{x_n+x_{n+1}}{2} + e_n\right)\| \\ &\le \tfrac{1}{2}\|x_n - x_{n+1}\| + \alpha_n\|fx_n - Sy_n\| + (1-\beta_n)\|e_n\| \\ &+(1-\beta_n)\|J_{r_n}\left(\tfrac{x_n+x_{n+1}}{2} + e_n\right) - \left(\tfrac{x_n+x_{n+1}}{2} + e_n\right)\|.\end{aligned}$$

From step 1, step 2, step 3, $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\|e_n\| < \infty$, we get $\lim_{n\to\infty}\|y_n - Sy_n\| = 0$.

**Step 5:** Show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| + \|Sy_n - y_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\|fx_n - Sy_n\| + \|Sy_n - y_n\|.\end{aligned}$$

From step1, step 2, step 4 and $\lim_{n\to\infty}\alpha_n = 0$, we get $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

**Step 6:** Show that $\lim_{n\to\infty}\|y_n - J_{r_n}y_n\| = 0$.

$$\begin{aligned}\|y_n - J_{r_n}y_n\| &\le \|y_n - x_n\| + \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_{r_n}y_n\| \\ &\le 2\|y_n - x_n\| + \|x_n - J_{r_n}x_n\|.\end{aligned}$$

From steps 3 and 5, we get $\lim_{n\to\infty}\|y_n - J_{r_n}y_n\| = 0$.

**Step 7:** Show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - y_n\| + \|y_n - Sy_n\| + \|Sy_n - Sx_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - Sy_n\|.\end{aligned}$$

From steps 4 and 5, we get $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

**Step 8:** Show that $\lim_{n\to\infty}\|y_n - J_ry_n\| = 0$.

$$\begin{aligned}\|y_n - J_ry_n\| &\le \|y_n - J_{r_n}y_n\| + \|J_{r_n}y_n - J_ry_n\| \\ &= \|y_n - J_{r_n}y_n\| + \|J_r\left(\tfrac{r}{r_n}y_n + \left(1 - \tfrac{r}{r_n}\right)J_{r_n}y_n\right) - J_ry_n\| \\ &\le \|y_n - J_{r_n}y_n\| + \left|1 - \tfrac{r}{r_n}\right| \cdot \|J_{r_n}y_n - y_n\|.\end{aligned}$$

From step 6 and $\lim_{n\to\infty} r_n = r$, we get $\lim_{n\to\infty}\|y_n - J_ry_n\| = 0$.

**Step 9:** Show that $\lim_{n\to\infty}\|x_n - J_rx_n\| = 0$.

$$\begin{aligned}\|x_n - J_rx_n\| &\le \|x_n - y_n\| + \|y_n - J_ry_n\| + \|J_ry_n - J_rx_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - J_ry_n\|.\end{aligned}$$

From steps 5 and 8, we get $\lim_{n\to\infty}\|x_n - J_rx_n\| = 0$.

**Step 10:** Show that $\limsup_{n\to\infty}\langle q - fq, J(q - x_n)\rangle = 0$.

From Theorem 1, we have that $\{x_t\}$ converges strongly to $q \in P_{F(S)\cap N(A)}fq$, which is also the unique solution of the variational inequality $\langle q - fq, J(q - p)\rangle \leq 0, \forall p \in F(S)\cap N(A)$.

We have

$$\begin{aligned}\|x_t - x_n\|^2 &= (1-t)\langle Sx_t - Sx_n + Sx_n - x_n, J(x_t - x_n)\rangle + t\langle fx_t - x_t + x_t - x_n, J(x_t - x_n)\rangle \\ &\leq (1-t)\|x_t - x_n\|^2 + (1-t)\|Sx_n - x_n\|\cdot\|x_t - x_n\| + t\|x_t - x_n\|^2 + t\langle fx_t - x_t, J(x_t - x_n)\rangle \\ &= \|x_t - x_n\|^2 + (1-t)\|Sx_n - x_n\|\cdot\|x_t - x_n\| + t\|x_t - x_n\|^2 + t\langle fx_t - x_t, J(x_t - x_n)\rangle.\end{aligned}$$

It follows that $\langle fx_t - x_t, J(x_t - x_n)\rangle \leq \frac{1-t}{t}\|Sx_n - x_n\|\cdot\|x_t - x_n\|$. From step 1 and step 7, we get $\limsup_{n\to\infty}\langle q - fq, J(q - x_n)\rangle = 0$.

**Step 11:** Show that $\lim_{n\to\infty}\|x_n - q\| = 0$.

From Lemma 4, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq (1-\alpha_n)^2\|Sy_n - q\|^2 + 2\alpha_n\langle fx_n - q, J(x_{n+1} - q)\rangle \\ &\leq (1-\alpha_n)^2\|y_n - q\|^2 + 2k\alpha_n\|x_n - q\|\cdot\|x_{n+1} - q\| + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle \\ &\leq (1-\alpha_n)^2\Big(\tfrac{1}{2}\|x_n - q\| + \tfrac{1}{2}\|x_{n+1} - q\| + \|e_n\|\Big)^2 + 2k\alpha_n\|x_n - q\|\cdot\|x_{n+1} - q\| \\ &\quad + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle \\ &= \Big(\tfrac{1-\alpha_n}{2}\Big)^2\|x_n - q\|^2 + \Big(\tfrac{1-\alpha_n}{2}\Big)^2\|x_{n+1} - q\|^2 + (1-\alpha_n)^2\|e_n\|^2 \\ &\quad + \tfrac{(1-\alpha_n)^2}{2}\|x_n - q\|\cdot\|x_{n+1} - q\| + (1-\alpha_n)^2\|x_n - q\|\cdot\|e_n\| \\ &\quad + (1-\alpha_n)^2\|x_{n+1} - q\|\cdot\|e_n\| + 2k\alpha_n\|x_n - q\|\cdot\|x_{n+1} - q\| \\ &\quad + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle \\ &\leq \Big(\tfrac{1-\alpha_n}{2}\Big)^2\|x_n - q\|^2 + \Big(\tfrac{1-\alpha_n}{2}\Big)^2\|x_{n+1} - q\|^2 \\ &\quad + \Big[\tfrac{(1-\alpha_n)^2}{4} + k\alpha_n\Big]\Big(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\Big) + (1-\alpha_n)^2\|e_n\|^2 \\ &\quad + (1-\alpha_n)^2\|e_n\|(\|x_n - q\| + \|x_{n+1} - q\|) + 2\alpha_n\langle fq - q, J(x_{n+1} - q)\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \frac{\big(\frac{1-\alpha_n}{2}\big)^2 + \frac{(1-\alpha_n)^2}{4} + k\alpha_n}{1 - \Big[\big(\frac{1-\alpha_n}{2}\big)^2 + \frac{(1-\alpha_n)^2}{4} + k\alpha_n\Big]}\|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \Big[\big(\frac{1-\alpha_n}{2}\big)^2 + \frac{(1-\alpha_n)^2}{4} + k\alpha_n\Big]}\langle fq - q, J(x_{n+1} - q)\rangle \\ &\quad + \frac{(1-\alpha_n)^2\|e_n\|}{1 - \Big[\big(\frac{1-\alpha_n}{2}\big)^2 + \frac{(1-\alpha_n)^2}{4} + k\alpha_n\Big]}(\|x_n - q\| + \|x_{n+1} - q\| + \|e_n\|) \\ &= \left[1 - \frac{1 - 2\Big(\frac{(1-\alpha_n)^2}{2} + k\alpha_n\Big)}{1 - \Big(\frac{(1-\alpha_n)^2}{2} + k\alpha_n\Big)}\right]\|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \Big[\frac{(1-\alpha_n)^2}{2} + k\alpha_n\Big]}\langle fq - q, J(x_{n+1} - q)\rangle \\ &\quad + \frac{(1-\alpha_n)^2\|e_n\|}{1 - \Big[\frac{(1-\alpha_n)^2}{2} + k\alpha_n\Big]}(\|x_n - q\| + \|x_{n+1} - q\| + \|e_n\|).\end{aligned}$$

Take $t_n = \frac{1-2\left(\frac{(1-\alpha_n)^2}{2}+k\alpha_n\right)}{1-\left(\frac{(1-\alpha_n)^2}{2}+k\alpha_n\right)}$. From $\lim_{n\to\infty}\alpha_n = 0$, we have

$$t_n \geq 1-2\left(\frac{(1-\alpha_n)^2}{2}+k\alpha_n\right) = \alpha_n(2-2k-\alpha_n) \geq \alpha_n(2-2k-\varepsilon)(\forall\varepsilon > 0).$$

From $\sum_{n=0}^{\infty}\alpha_n = \infty$, we get $\sum_{n=0}^{\infty} t_n = \infty$.

Take $b_n = \frac{2\alpha_n}{1-\left[\frac{(1-\alpha_n)^2}{2}+k\alpha_n\right]}\langle fq-q, J(x_{n+1}-q)\rangle$, then we have

$$\frac{b_n}{t_n} = \frac{2\alpha_n\langle fq-q, J(x_{n+1}-q)\rangle}{1-2\left[\frac{(1-\alpha_n)^2}{2}+k\alpha_n\right]} = \frac{2\langle fq-q, J(x_{n+1}-q)\rangle}{2-2k-\alpha_n}.$$

From $\lim_{n\to\infty}\alpha_n = 0$ and step 10, we get $b_n = o(t_n)$.

Take $c_n = \frac{(1-\alpha_n)^2\|e_n\|}{1-\left[\frac{(1-\alpha_n)^2}{2}+k\alpha_n\right]}(\|x_n-q\|+\|x_{n+1}-q\|+\|e_n\|)$, then

$$c_n = \frac{2(1-\alpha_n)^2\|e_n\|}{1+\alpha_n(2-2k-\alpha_n)}(\|x_n-q\|+\|x_{n+1}-q\|+\|e_n\|).$$

From $k \in (0,1)$ and $\alpha_n \in (0,1)$, we get $\alpha_n(2-2k-\alpha_n) > 0$, and then $c_n < 2(\|x_n-q\|+\|x_{n+1}-q\|+\|e_n\|)\|e_n\|$. From $\sum_{n=0}^{\infty}\|e_n\| < \infty$, we get $\sum_{n=0}^{\infty} c_n < \infty$.

From Lemma 2, we get $\lim_{n\to\infty}\|x_n-q\| = 0$. This completes the proof. □

The results of Theorem 2 improve the related results in [15–17]. For example, the results of Theorem 2 is can obtain the related results in [15–17]; the rate of convergence and computational accuracy is better than their in [15–17].

## 4. Numerical Examples

We give four numerical examples to support the main results.

**Example 1.** *Let $R$ be the real line with Euclidean norm, $f: R \to R$ be defined by $f(x) = \frac{x}{6}$, $S: R \to R$ be defined by $S(x) = \frac{x}{4}$ and $J_{r_n}x = \frac{r_n x}{2}$. So $F(T) = \{0\}$. Let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n}$ and $r_n = 1-\frac{1}{n}$, then they satisfy the conditions of Theorem 1. $\{x_n\}$ is generated by (2). From Theorem 1, we can obtain $\{x_n\}$ converges strongly to 0.*

Next, we simplify the form of (2) and get

$$x_{n+1} = \frac{3-9n+9n^2+3n^3-14n^4}{-3+9n-9n^2-51n^3+6n^4}x_n. \tag{10}$$

Next, we take $x_1 = 1$ into (10). Finally, we get the following numerical results in Figure 1.

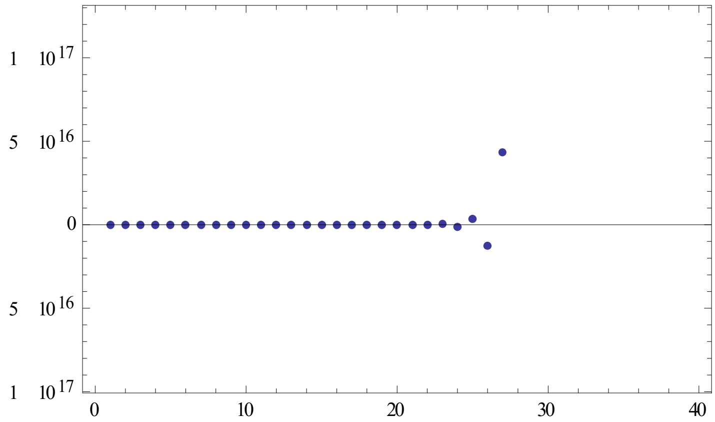


**Figure 1.** Numerical results.

**Example 2.** *Let $R$ be the real line with Euclidean norm, $f : R \to R$ be defined by $f(x) = \frac{x}{6}$, $S : R \to R$ be defined by $S(x) = \frac{x}{4}$ and $J_{r_n}x = \frac{r_n x}{2}$. So $F(T) = \{0\}$. Let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n}$, $r_n = 1 - \frac{1}{n}$ and $e_n = \frac{1}{n^2}$, then they satisfy the conditions of Theorem 2. $\{x_n\}$ is generated by (6). From Theorem 2, we can obtain $\{x_n\}$ converges strongly to 0.*

Next, we simplify the form of (6) and get

$$x_{n+1} = -\frac{2(n-1)^2 n^4}{-1 + 3n - 3n^2 - 17n^3 + 2n^4} x_n. \tag{11}$$

Next, we take $x_1 = 1$ into (11). Finally, we get the following numerical results in Figure 2.

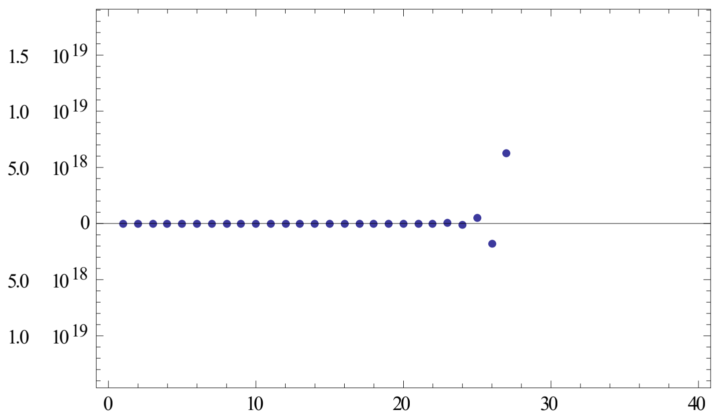


**Figure 2.** Numerical results.

**Example 3.** *Let $\langle \cdot, \cdot \rangle : R^3 \times R^3 \to R$ be the inner product and defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_2 y_3.$$

*Let $\| \cdot \| : R^3 \to R$ be the usual norm and defined by $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for any $x = (x_1, x_2, x_3)$.*

*For any $x \in R^3$, let $f : R^3 \to R^3$ be defined by $f(x) = \frac{x}{6}$, $S : R^3 \to R^3$ be defined by $S(x) = \frac{x}{4}$ and $J_{r_n}x = \frac{r_n x}{2}$. So $F(T) = \{0\}$. Let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n}$ and $r_n = 1 - \frac{1}{n}$, then they satisfy the conditions of Theorem 1. $\{x_n\}$ is generated by (2). From Theorem 1, we can obtain $\{x_n\}$ converges strongly to 0.*

Next, we simplify the form of (2) and get

$$x_{n+1} = \frac{3 - 9n + 9n^2 + 3n^3 - 14n^4}{-3 + 9n - 9n^2 - 51n^3 + 6n^4} x_n. \tag{12}$$

Next, we take $x_1 = (1, 2, 3)$ into (12). Finally, we get the following numerical results in Figure 3.

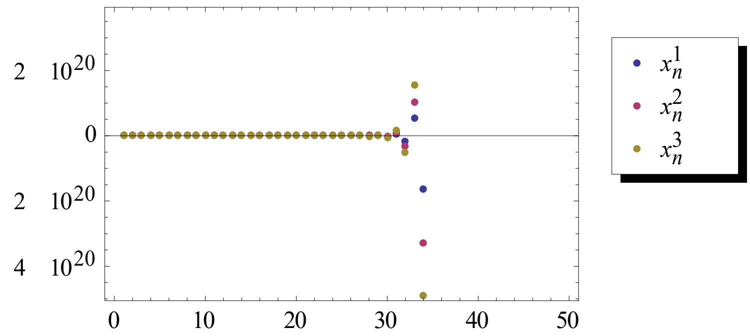


**Figure 3.** Numerical results.

**Example 4.** *Let* $\langle \cdot, \cdot \rangle : R^3 \times R^3 \to R$ *be the inner product and defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_2 y_3.$$

*Let* $\| \cdot \| : R^3 \to R$ *be the usual norm and defined by* $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ *for any* $x = (x_1, x_2, x_3)$.

*For any* $x \in R^3$, *let* $f : R^3 \to R^3$ *be defined by* $f(x) = \frac{x}{6}$, $S : R^3 \to R^3$ *be defined by* $S(x) = \frac{x}{4}$ *and* $J_{r_n} x = \frac{r_n x}{2}$. *So* $F(T) = \{0\}$. *Let* $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n}$, $r_n = 1 - \frac{1}{n}$ *and* $e_n = \frac{1}{n^2}$, *then they satisfy the conditions of Theorem 2.* $\{x_n\}$ *is generated by (6). From Theorem 2, we can obtain* $\{x_n\}$ *converges strongly to 0.*

Next, we simplify the form of (6) and get

$$x_{n+1} = -\frac{2(n-1)^2 n^4}{-1 + 3n - 3n^2 - 17n^3 + 2n^4} x_n. \tag{13}$$

Next, we take $x_1 = (1, 10, 100)$ into (13). Finally, we get the following numerical results in Figure 4.

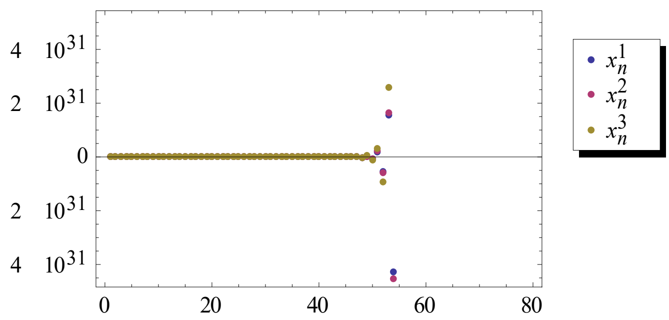


**Figure 4.** Numerical results.

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