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The Coefficients of Powers of Bazilević Functions

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Abstract: In the present work, a sharp bound on the modulus of the initial coefficients for powers of strongly Bazilević functions is obtained. As an application of these results, certain conditions are investigated under which the Littlewood-Paley conjecture holds for strongly Bazilević functions for large values of the parameters involved therein. Further, sharp estimate on the generalized Fekete-Szegö functional is also derived. Relevant connections of our results with the existing ones are also made.

Keywords: univalent function; starlike function; strongly Bazilević function; Littlewood-Paley conjecture; Fekete-Szegö functional

MSC: 30C45; 30C50

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ having Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. The famous Bieberbach conjecture (now de Branges's theorem [1]) states that the coefficient of functions in the class $\mathcal{S}$ satisfy $|a_n| \leq n$ with equality in case of the Koebe function $k(z) = z/(1-z)^2$. Denote, by $\mathcal{S}^*$, the subclass of $\mathcal{S}$ consisting of starlike functions, so that $\mathrm{Re}(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$).

The class of strongly starlike functions of order β $(0 < \beta \leq 1)$ is defined by

$$\mathcal{SS}^*(\beta) := \left\{ f \in \mathcal{S} : \left| \arg \frac{zf'(z)}{f(z)} \right| \leq \frac{\beta\pi}{2},\ z \in \mathbb{D} \right\}.$$

This class was introduced by Brannan and Kirwan [2].

Kaplan [3] introduce the class $\mathcal{C}$ of close-to-convex functions consisting of the functions $f \in \mathcal{S}$ satisfying $\mathrm{Re}\,(zf'(z)/g(z)) > 0\,(z \in \mathbb{D})$, where $g \in \mathcal{S}^*$. The class $\mathcal{B}_\alpha$ is a generalization of the class of starlike functions and was considered by Thomas [4] in 1967. For a starlike function g, Thomas [4] defined a class $\mathcal{B}_\alpha$ consisting of functions $f \in \mathcal{S}$ and satisfying the condition $\mathrm{Re}(zf'(z)f(z)^{\alpha-1}/g(z)^{\alpha}) > 0$ $(z \in \mathbb{D})$ and called it as the class of Bazilevič functions of type α and it has been proved that the functions in this class are univalent. Clearly $\mathcal{B}_1 =: \mathcal{C}$.

Singh [5], in 1973, considered a special case of the class $\mathcal{B}_\alpha$, by setting $g(z) = z$. For $\alpha \geq 0$, let $\mathcal{B}_1(\alpha)$ be the subclass of $\mathcal{B}_\alpha$ of Bazilevič functions defined by

$$\mathcal{B}_1(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} > 0,\ z \in \mathbb{D} \right\}.$$

For $f \in \mathcal{B}_1(\alpha)$, Singh gave sharp estimates for the first four coefficients, together with other results, and in 2017 Marjono et al. [6] obtained sharp estimates for the fifth and sixth coefficients for some values of α and conjectured that when $\alpha \geq 1$

$$|a_n| \leq \frac{2}{n-1+\alpha}$$

for $n \geq 2$.

The above conjecture has recently been verified by Cho and Kumar [7] when $n = 5$ and $n = 6$ for a certain range of α, and other coefficient results for $f \in \mathcal{B}_1(\alpha)$ been obtained by Thomas [8]. For other results concerning concerning Bazilevič functions, see [4,9–18].

Let $\gamma > 0$, f be given by (1), and $k(z) = z/(1-z)^2$. Write

$$\left(\frac{f(z)}{z} \right)^{\frac{1}{\gamma}} = 1 + \sum_{n=1}^{\infty} a_n(\gamma) z^n \tag{2}$$

and

$$\left(\frac{k(z)}{z} \right)^{\frac{1}{\gamma}} = 1 + \sum_{n=1}^{\infty} b_n(\gamma) z^n. \tag{3}$$

Equating coefficients in (2) and (3), we obtain

$$a_1(\gamma) = \frac{1}{\gamma} a_2, \quad a_2(\gamma) = \frac{1}{\gamma} \left(a_3 - \frac{\gamma - 1}{2\gamma} a_2^2 \right) \tag{4}$$

and

$$b_n(\gamma) = \frac{2(2+\gamma)(2+2\gamma)\cdots(2+(n-1)\gamma)}{(n!)\gamma^n}. \tag{5}$$

We now consider the validity of the inequality

$$|a_n(\gamma)| \leq b_n(\gamma), \tag{6}$$

whenever $f \in \mathcal{S}$.

First note that when $\gamma = 1$, (6) becomes de Branges theorem, and when $\gamma = 2$, (6) reduces to the Littlewood-Paley conjecture [19], which was shown to be false by Fekete and Szegö [20].

When $f \in \mathcal{S}$ and $\gamma \leq 1$ Hayman and Hummel [21] showed that (6) is true, but false when $\gamma > 1$. On the other hand (6) is true for $\gamma > 0$ when $f \in \mathcal{S}^*$ [22,23].

In the case of close-to-convex functions, Jahangiri [22] showed that (6) is valid when $n = 2$ provided $0 < \gamma \leq 3$, but is false when $\gamma > 1$. Similar problems were considered by Darus and Thomas in [24].

We now introduce the class of strongly Bazilević functions as follows.

Definition 1. *A function f defined by (1) belongs to the class $\mathcal{B}(\alpha, \beta)$ if there exists a normalized analytic function $g \in \mathcal{S}^*$ such that*

$$\left| \arg \frac{z f'(z) f^{\alpha-1}(z)}{g^\alpha(z)} \right| \leq \frac{\pi \beta}{2} \quad (z \in \mathbb{D},\ 0 < \alpha,\ 0 < \beta \leq 1).$$

All powers in the above definition are understood to be the principal ones. Clearly $\mathcal{B}(0,1) = \mathcal{S}^*$ and $\mathcal{B}(1,1) = \mathcal{C}$. Also $\mathcal{B}(\alpha,\beta)$ is a subclass of $\mathcal{B}_\alpha$ and hence contains only univalent functions.

In this paper we will obtain some sharp upper bounds for $|a_1(\gamma)|$, $|a_2(\gamma)|$ and $M(\gamma)$ for $f \in \mathcal{B}(\alpha,\beta)$. Our results will infer that when $f \in \mathcal{B}(\alpha,\beta)$, (6) holds for some large values of γ, thus extending the work of several authors, e.g., Jahangiri [22], London [25], Eenigenberg, Silvia [26], Keogh, Merkes [27] and Abdel-Gawad, Thomas [28].

2. Powers of Bazilević Functions

First we prove the following theorem, which gives sharp estimates of $|a_1(\gamma)|$ and $|a_2(\gamma)|$. These estimates will be later used to discuss the Littlewood-Paley conjecture for functions in the class $\mathcal{B}(\alpha,\beta)$.

Theorem 1. *Let $f \in \mathcal{B}(\alpha,\beta)$ $(0 < \alpha,\ 0 < \beta \le 1)$ and $a_n(\gamma)$ $(n = 1, 2)$ be given by (4). Then the following sharp bounds hold.*

$$|a_1(\gamma)| \le \frac{2(\alpha+\beta)}{\gamma(\alpha+1)} \quad (0 < \gamma)$$

and

$$|a_2(\gamma)| \le \begin{cases} \frac{\tau}{(\alpha+1)^2(\alpha+2)\gamma^2}, & 0 < \gamma \le \frac{(2+\alpha)(\alpha+\beta)}{\alpha^2+\alpha-\beta+1}; \\ \frac{\upsilon}{\gamma(\alpha+2)((\alpha+1)^2\gamma-\beta(\alpha+\gamma+2))}, & \gamma \ge \frac{(2+\alpha)(\alpha+\beta)}{\alpha^2+\alpha-\beta+1}, \end{cases}$$

where

$$\tau := \alpha^3(\gamma+2) + 4\alpha^2(\beta+\gamma+1) + \alpha\left(2\beta^2 + 4\beta(\gamma+2) + \gamma\right) + 2\beta^2(\gamma+2)$$

and

$$\upsilon := \alpha^3(\gamma+2) + \alpha^2(2\beta\gamma - \beta + 4\gamma + 4) + \alpha\left(-2\beta^2 + \beta(3\gamma-2) + \gamma\right) + 2\beta(\gamma - \beta(\gamma+2)).$$

We note that the above theorem generalizes many existing results in literature. For example, for $\alpha = 1 = \beta$, the above theorem gives the following result due to Jahangiri [22].

Corollary 1. [22] (Theorem 1, p. 1141) *Let $f \in \mathcal{C}$ and $a_n(\gamma)$ $(n = 1, 2)$ be as given in (4). Then the following sharp bounds hold.*

$$|a_1(\gamma)| \le \frac{2}{\gamma} \quad (0 < \gamma)$$

and

$$|a_2(\gamma)| \le \begin{cases} \frac{2+\gamma}{\gamma^2}, & 0 < \gamma \le 3; \\ \frac{11\gamma-3}{9\gamma(\gamma-1)}, & \gamma \ge 3. \end{cases}$$

Remark 1. *First note from (5) that $b_1(\gamma) = 2/\gamma$ and $b_2(\gamma) = (2+\gamma)/\gamma^2$. Obviously $|a_1(\gamma)| \le b_1(\gamma)$ holds for all $0 < \alpha \le 1, 0 < \beta \le 1$ and $\gamma > 0$. This verifies the Littlewood-Paley conjecture for $n = 1$ and all $\gamma > 0$. We now consider the case when $n = 2$ and $\gamma = 1$. For this case we have*

$$|a_2(\gamma)| \le \begin{cases} S_1, & \left(0 < \alpha < 1,\ \frac{1-\alpha}{\alpha+3} \le \beta \le 1\right) \text{ or } (\alpha \ge 1,\ 0 < \beta \le 1); \\ T_1, & 0 < \alpha < 1,\ 0 < \beta < \frac{1-\alpha}{\alpha+3}, \end{cases}$$

where

$$S_1 := \frac{3\alpha^3 + 4\alpha^2(\beta+2) + \alpha\left(2\beta^2 + 12\beta + 1\right) + 6\beta^2}{(\alpha+1)^2(\alpha+2)}$$

and

$$T_1 := \frac{3\alpha^3 + \alpha^2(\beta+8) + \alpha\left(-2\beta^2 + \beta + 1\right) + 2\beta(1-3\beta)}{(\alpha+2)\left(\alpha^2 - \alpha(\beta-2) - 3\beta + 1\right)}.$$

It is easy to verify that $S_1 \le 3 = b_2(1)$ *holds for* $(0 < \alpha < 1$ *and* $(1-\alpha)/(\alpha+3) \le \beta \le 1)$ *or* $(\alpha \ge 1$ *and* $0 < \beta \le 1)$ *and* $T_1 \le 3 = b_2(1)$ *holds for* $0 < \alpha < 1$*and* $0 < \beta < (1-\alpha)/(\alpha+3)$. *Thus* $|a_2(1)| \le b_2(1)$ *is true for all* $0 < \alpha$ *and* $0 < \beta \le 1$.

Consider the case when $n = 2$ *and* $\gamma = 2$. *For this case we have*

$$|a_2(\gamma)| \le \begin{cases} S_2, & \left(0 < \alpha < 2,\ \frac{\alpha^2+2}{\alpha+4} \le \beta \le 1\right) \text{ or } (\alpha = 2,\ \beta = 1); \\ T_2, & (\alpha, \beta) \in \Omega, \end{cases}$$

where Ω *is the set of* (α, β) *such that either of*

$$0 < \alpha \le 2 \ \text{ and } \ 0 < \beta < \frac{\alpha^2+2}{\alpha+4}$$

or

$$\alpha > 2 \ \text{ and } \ 0 < \beta \le \frac{5\alpha^2+16\alpha+20}{4(\alpha+4)} - \frac{1}{4}\sqrt{\frac{25\alpha^4+128\alpha^3+184\alpha^2+144}{(\alpha+4)^2}}$$

holds, where S_2 *and* T_2 *are given by*

$$S_2 := \frac{4\alpha^3+4\alpha^2(\beta+3)+\alpha\left(2\beta^2+16\beta+2\right)+8\beta^2}{4(\alpha+1)^2(\alpha+2)}$$

and

$$T_2 := \frac{4\alpha^3+\alpha^2(3\beta+12)+\alpha\left(-2\beta^2+4\beta+2\right)+2\beta(2-4\beta)}{2(\alpha+2)\left(2(\alpha+1)^2-(\alpha+4)\beta\right)}.$$

It is a simple matter to check that if $\left(0 < \alpha < 2 \text{ and } (\alpha^2+2)/(\alpha+4) \le \beta \le 1\right)$ *or* $(\alpha = 2$ *and* $\beta = 1)$, *then* $S_2 \le 1 = b_2(2)$ *and if* $(\alpha, \beta) \in \Omega$, *then* $T_2 \le 1 = b_2(2)$. *Thus under certain conditions the Littlewood-Paley conjecture is also true for* $n = 2 = \gamma$.

Next assume that $n = 2$ *and* $\gamma = 3$. *In this case we see that* $|a_2(3)| \le 5/9 = b_2(3)$ *holds if either of the conditions*

$$\left(0 < \alpha < 1 \ \text{ and } \ \frac{2\alpha^2+\alpha+3}{\alpha+5} \le \beta \le 1\right) \text{ or } (\alpha = 1 \ \text{ and } \ \beta = 1)$$

or

$$0 < \alpha \le 1 \ \text{ and } \ 0 < \beta < \frac{2\alpha^2+\alpha+3}{\alpha+5}$$

or

$$\alpha > 1 \ \text{ and } \ 0 < \beta \le \frac{5\alpha^2+14\alpha+17}{3(\alpha+5)} - \frac{1}{3}\sqrt{\frac{25\alpha^4+122\alpha^3+177\alpha^2-64\alpha+64}{(\alpha+5)^2}}$$

is true. In a similar way we can check that for $n = 2$ *and* $\gamma = 4$, *the inequality* $|a_2(4)| \le 3/8 = b_2(4)$ *holds if either of the following conditions is true.*

$$\left(0 < \alpha < \frac{2}{3} \ \text{ and } \ \frac{3\alpha^2+2\alpha+4}{\alpha+6} \le \beta \le 1\right) \text{ or } \left(\alpha = \frac{2}{3} \ \text{ and } \ \beta = 1\right)$$

or

$$0 < \alpha \le \frac{2}{3} \ \text{ and } \ 0 < \beta < \frac{3\alpha^2+2\alpha+4}{\alpha+6}$$

or

$$\alpha > \frac{2}{3} \ \text{ and } \ 0 < \beta \le \frac{17\alpha^2+44\alpha+52}{8(\alpha+6)} - \frac{1}{8}\sqrt{\frac{289\alpha^4+1368\alpha^3+2104\alpha^2-800\alpha+400}{(\alpha+6)^2}}.$$

It should be noted that inequality (6) in many cases is not true for large values of γ, for example the case of close-to-convex functions it does not hold for value of $\gamma > 3$. Another example can be found in [29] (Equation (23), p. 93) due to Farahmand and Jahangiri. They proved that for a subclass of close-to-convex function the inequality $|a_2(\gamma)| \leq b_2(\gamma)$ not even true for $\gamma = 4$. It is therefore interesting to investigate the cases for which this inequality holds for large values of γ. The following corollary describes certain conditions under which inequality (6) holds for large values of γ for functions $f \in \mathcal{B}(\alpha, \beta)$.

3. Proof of Theorems

Corollary 2. *Let $f \in \mathcal{B}(\alpha, \beta)$ and $a_2(\gamma)$ and $b_2(\gamma)$ be given by (4) and (5), respectively. Then $|a_2(\gamma)| \leq b_2(\gamma)$ is true if either of the following sets of conditions holds:*

$$\alpha > 0,\ 0 < \beta \leq 1 \ \text{ and }\ 0 < \gamma \leq \frac{(\alpha+2)(\alpha+\beta)}{\alpha^2+\alpha-\beta+1}$$

or

$$\alpha > 0,\ 0 < \beta \leq \frac{1}{2}\left(\alpha^2+2\alpha+2\right) - \frac{1}{2}\sqrt{\alpha^4+4\alpha^3+8\alpha^2} \ \text{ and }\ \gamma > \frac{(\alpha+2)(\alpha+\beta)}{\alpha^2+\alpha-\beta+1}$$

or

$$\alpha > 0,\ \frac{1}{2}\left(\alpha^2+2\alpha+2\right) - \frac{1}{2}\sqrt{\alpha^4+4\alpha^3+8\alpha^2} < \beta < 1 \ \text{ and }\ \frac{(\alpha+2)(\alpha+\beta)}{\alpha^2+\alpha-\beta+1} < \gamma \leq \gamma_0,$$

where

$$\gamma_0 := \frac{2\alpha^2+\alpha\left(\beta^2-2\beta+5\right)+2(\beta-1)^2}{2\alpha^2\beta+4\alpha(\beta-1)-2(\beta-1)^2} + \frac{1}{2}\sqrt{\frac{(\alpha+2)^2\left(1-\beta^2\right)\left(4\alpha^2+4\alpha-\beta^2+1\right)}{\left(\alpha^2\beta+2\alpha(\beta-1)-(\beta-1)^2\right)^2}}.$$

To establish Theorem 1, we need the following lemma. The inequality (7) was proved by Carathéodory [30] (see also Duren [31], p. 41) and the inequality (8) can be found in [22] (Lemma 1, p. 142).

Lemma 1. *Let $p \in \mathcal{P}$, the class of functions satisfying* $\operatorname{Re} p(z) > 0$ *for $z \in \mathbb{D}$, with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then*

$$|p_n| \leq 2 \tag{7}$$

and

$$\left|p_2 + \mu p_1^2\right| \leq 2 + \mu |p_1|^2 \quad (\mu \geq -1/2). \tag{8}$$

Proof of Theorem 1. Since $f \in \mathcal{B}(\alpha, \beta)$, it follows from the definition that there exist functions p and q in $\mathcal{P}$, with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$ and a function $g \in \mathcal{S}^*$ with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ such that

$$\frac{zf'(z)f^{\alpha-1}(z)}{g^{\alpha}(z)} = p^{\beta}(z) \ \text{ and }\ \frac{zg'(z)}{g(z)} = q(z). \tag{9}$$

Equating coefficients in (9), we obtain

$$a_2 = \frac{1}{\alpha+1}(\alpha b_2 + \beta p_1), \tag{10}$$

$$a_3 = \frac{1}{\alpha+2}\left(\frac{\alpha(\alpha-1)}{2(\alpha+1)^2}b_2^2 + \frac{\alpha\beta(\alpha+3)}{(\alpha+1)^2}p_1 b_2 + \frac{\beta(\beta(\alpha+3)-(\alpha+1)^2)}{2(\alpha+1)^2}p_1^2 + \beta p_2 + \alpha b_3\right), \tag{11}$$

$$b_2 = q_1 \ \text{ and }\ b_3 = \frac{q_2+q_1^2}{2}. \tag{12}$$

Substituting for a_2 and a_3 from (10), (11) in (4) and using the relation (12), we have

$$a_1(\gamma) = \frac{1}{\gamma(\alpha+1)}(\alpha q_1 + \beta p_1) \tag{13}$$

and

$$\begin{aligned}\gamma a_2(\gamma) = {} & \left(\frac{\alpha(\alpha-1)}{2(\alpha+2)(\alpha+1)^2} + \frac{\alpha^2(1-\gamma)}{2\gamma(\alpha+1)^2}\right) q_1^2 + \left(\frac{\alpha\beta(\alpha+3)}{(\alpha+2)(\alpha+1)^2} + \frac{\alpha\beta(1-\gamma)}{\gamma(\alpha+1)^2}\right) p_1 q_1 \\ & + \frac{\beta}{\alpha+2} p_2 + \frac{\alpha}{2(\alpha+2)}(q_2 + q_1^2) + \left(\frac{\beta(\beta(\alpha+3)-(\alpha+1)^2)}{2(\alpha+2)(\alpha+1)^2} + \frac{\beta^2(1-\gamma)}{2\gamma(\alpha+1)^2}\right) p_1^2. \end{aligned} \tag{14}$$

The estimate for $|a_1(\gamma)|$ follows from (13) by applying triangle inequality and using the facts that $|p_1| \le 2$ and $|q_1| \le 2$.

To obtain the estimate on $|a_2(\gamma)|$, we rewrite $a_2(\gamma)$ as

$$\begin{aligned}\gamma a_2(\gamma) = {} & \frac{\alpha}{2(\alpha+2)} q_2 + \left(\frac{\alpha}{2(\alpha+2)} + \frac{\alpha(\alpha-1)}{2(\alpha+2)(\alpha+1)^2} + \frac{\alpha^2(1-\gamma)}{2\gamma(\alpha+1)^2}\right) q_1^2 \\ & + \frac{\beta}{\alpha+2} p_2 + \left(\frac{\beta(\beta(\alpha+3)-(\alpha+1)^2)}{2(\alpha+2)(\alpha+1)^2} + \frac{\beta^2(1-\gamma)}{2\gamma(\alpha+1)^2}\right) p_1^2 \\ & + \left(\frac{\alpha\beta(\alpha+3)}{(\alpha+2)(\alpha+1)^2} + \frac{\alpha\beta(1-\gamma)}{\gamma(\alpha+1)^2}\right) p_1 q_1 \\ = {} & \frac{\alpha}{2(\alpha+2)} q_2 + \frac{\alpha^2(\alpha+\gamma+2)}{2(\alpha+1)^2(\alpha+2)\gamma} q_1^2 + \frac{\alpha\beta(\alpha+\gamma+2)}{(\alpha+1)^2(\alpha+2)\gamma} p_1 q_1 \\ & + \frac{\beta}{\alpha+2} p_2 + \frac{\beta\left(\beta(\alpha+\gamma+2)-(\alpha+1)^2\gamma\right)}{2(\alpha+1)^2(\alpha+2)\gamma} p_1^2 \\ = {} & \frac{\alpha}{2(\alpha+2)}\left(q_2 + \frac{\alpha(\alpha+\gamma+2)}{(\alpha+1)^2\gamma} q_1^2\right) + \frac{\alpha\beta(\alpha+\gamma+2)}{(\alpha+1)^2(\alpha+2)\gamma} p_1 q_1 \\ & + \frac{\beta}{\alpha+2}\left(p_2 + \frac{\beta(\alpha+\gamma+2)-(\alpha+1)^2\gamma}{2(\alpha+1)^2\gamma} p_1^2\right). \end{aligned} \tag{15}$$

Applying triangle inequality in (15) and using $|q_1| \le 2$, we get

$$\begin{aligned}\gamma |a_2(\gamma)| \le {} & \frac{\alpha}{2(\alpha+2)}\left|q_2 + \frac{\alpha(\alpha+\gamma+2)}{(\alpha+1)^2\gamma} q_1^2\right| + \frac{2\alpha\beta(\alpha+\gamma+2)}{(\alpha+1)^2(\alpha+2)\gamma}|p_1| \\ & + \frac{\beta}{\alpha+2}\left|p_2 + \frac{\beta(\alpha+\gamma+2)-(\alpha+1)^2\gamma}{2(\alpha+1)^2\gamma} p_1^2\right|. \end{aligned} \tag{16}$$

It is clear that the coefficient of q_1^2 in the above expression is positive for all $0 < \gamma, 0 < \alpha$ and $0 < \beta \le 1$. Also since $\beta(\alpha+\gamma+2) > 0$, it follows that the coefficient of p_1^2 is also greater than $-1/2$. Using the inequalities in Lemma 1 along with and using $|q_i| \le 2$, we obtain

$$\begin{aligned}\gamma |a_2(\gamma)| \le {} & \frac{\alpha}{2(\alpha+2)}\left(2 + \frac{\alpha(\alpha+\gamma+2)}{(\alpha+1)^2\gamma}|q_1|^2\right) + \frac{2\alpha\beta(\alpha+\gamma+2)}{(\alpha+1)^2(\alpha+2)\gamma}|p_1| \\ & + \frac{\beta}{\alpha+2}\left(2 + \frac{\beta(\alpha+\gamma+2)-(\alpha+1)^2\gamma}{2(\alpha+1)^2\gamma}|p_1|^2\right) \\ \le {} & M + \frac{2\alpha\beta(\alpha+\gamma+2)}{(\alpha+1)^2(\alpha+2)\gamma} s + \frac{\beta(\beta(\alpha+\gamma+2)-(\alpha+1)^2\gamma)}{2(\alpha+2)(\alpha+1)^2\gamma} s^2 =: h(s), \end{aligned} \tag{17}$$

where $s := |p_1| \in [0,2]$ and M is given by

$$\begin{aligned} M &:= \frac{2\alpha^2(\alpha+\gamma+2)}{(\alpha+1)^2(\alpha+2)\gamma} + \frac{2\beta}{\alpha+2} + \frac{\alpha}{\alpha+2} \\ &= \frac{\alpha^3(\gamma+2) + 2\alpha^2((\beta+2)\gamma+2) + \alpha(4\beta\gamma+\gamma) + 2\beta\gamma}{(\alpha+1)^2(\alpha+2)\gamma}. \end{aligned}$$

Now note that $h(0) = M$ and

$$h(2) = \frac{\alpha^3(\gamma+2) + 4\alpha^2(\beta+\gamma+1) + \alpha\left(2\beta^2 + 4\beta(\gamma+2) + \gamma\right) + 2\beta^2(\gamma+2)}{(\alpha+1)^2(\alpha+2)\gamma}.$$

For $\alpha > 0$ and $0 < \beta \le 1$, we have

$$\max\{h(0), h(2)\} = \begin{cases} h(2), & 0 < \gamma \le \frac{(\alpha+2)(2\alpha+\beta)}{\alpha^2-\beta+1} =: \gamma_0, \\ M = h(0), & \gamma \ge \gamma_0. \end{cases} \tag{18}$$

We now consider the case when $s \in (0,2)$. In this case, we see that the unique root of the equation

$$h'(s) = \frac{2\alpha\beta(\alpha+\gamma+2)}{(\alpha+1)^2(\alpha+2)\gamma} + \frac{\beta s\left(\beta(\alpha+\gamma+2) - (\alpha+1)^2\gamma\right)}{(\alpha+1)^2(\alpha+2)\gamma}$$

is given by

$$s_0 := s = \frac{2\alpha(\alpha+\gamma+2)}{(\alpha+1)^2\gamma - \beta(\alpha+\gamma+2)}.$$

To prove the result we now consider the two cases:

$$(a) \quad \gamma > \frac{\alpha^2+\alpha\beta+2\alpha+2\beta}{\alpha^2+\alpha-\beta+1} =: \gamma^* \quad \text{and} \quad (b) \quad 0 < \gamma \le \frac{\alpha^2+\alpha\beta+2\alpha+2\beta}{\alpha^2+\alpha-\beta+1} = \gamma^*.$$

(a) It is easy to see that $s_0 \in (0,2)$ if $\alpha > 0,\ 0 < \beta \le 1$ and $\gamma > \gamma^*$. Further

$$h''(s_0) = \frac{\beta\left(\beta(\alpha+\gamma+2) - (\alpha+1)^2\gamma\right)}{(\alpha+1)^2(\alpha+2)\gamma} < 0.$$

Thus h has a maximum at the point s_0 and so h has maximum at s_0. Now from (18), we see that

$$\begin{aligned} \max\{h(0), h(2), h(s_0)\} &= h(s_0) \\ &= \frac{v}{(\alpha+2)\left((\alpha+1)^2\gamma - \beta(\alpha+\gamma+2)\right)}. \end{aligned} \tag{19}$$

(b) Next when $0 < \gamma \le \gamma^*$, the critical point of h does not belongs to $(0,2)$ and so we consider the end point for the maxima and minima. Since $\gamma^* \le \gamma_0$, it follows that

$$\max\{h(0), h(2)\} = h(2) \ (0 < \gamma \le \gamma^*). \tag{20}$$

Thus (19) and (20) together with (18) give the desired estimate.

For sharpness, we consider the function f defined by

$$zf'(z)f(z)^{\alpha-1} = \frac{z^\alpha}{(1-z)^{2\alpha}}\left(\frac{1+z}{1-z}\right)^\beta. \tag{21}$$

Then it is easy to see that

$$a_2 = \frac{2\alpha + 2\beta}{\alpha + 1} \text{ and } a_3 = \frac{3\alpha^3 + 4\alpha^2(\beta + 2) + \alpha\left(2\beta^2 + 12\beta + 1\right) + 6\beta^2}{(\alpha + 1)^2(\alpha + 2)}$$

these in turn give

$$a_1(\gamma) = \frac{2(\alpha + \beta)}{\gamma(\alpha + 1)} \tag{22}$$

and

$$a_2(\gamma) = \frac{\alpha^3(\gamma + 2) + 4\alpha^2(\beta + \gamma + 1) + \alpha\left(2\beta^2 + 4\beta(\gamma + 2) + \gamma\right) + 2\beta^2(\gamma + 2)}{(\alpha + 1)^2(\alpha + 2)\gamma^2}. \tag{23}$$

Thus (22) shows that the bound for $|a_1(\gamma)|$ is sharp when $0 < \gamma,\ 0 < \alpha,\ 0 < \beta \leq 1$, and that (23) shows that the bound for $|a_2(\gamma)|$ is sharp when $0 < \gamma \leq \gamma^*$.

On the other hand when $\gamma > \gamma^*$ the inequality for $|a_2(\gamma)|$ holds when f is given by

$$zf'(z)f(z)^{\alpha-1} = \frac{z^\alpha}{(1-z)^{2\alpha}}\left(\frac{1 + s_0 z + z^2}{1 - z^2}\right)^\beta.$$

Here

$$a_2 = \frac{2\alpha(\alpha + 1)\gamma}{(\alpha + 1)^2\gamma - \beta(\alpha + \gamma + 2)} \text{ and } a_3 = \frac{8\hat{a} + \beta\hat{b} + 6(\alpha + 1)^2\alpha + 4(\alpha - 1)\alpha}{2(\alpha + 1)^2(\alpha + 2)},$$

where

$$\hat{a} = \frac{(\alpha + 3)\alpha^2\beta(\alpha + \gamma + 2)}{(\alpha + 1)^2\gamma - \beta(\alpha + \gamma + 2)} \text{ and } \hat{b} = 4(\alpha + 1)^2 - \frac{4\alpha^2\left((\alpha + 1)^2 - (\alpha + 3)\beta\right)(\alpha + \gamma + 2)^2}{\left((\alpha + 1)^2\gamma - \beta(\alpha + \gamma + 2)\right)^2}$$

The expressions for a_2 and a_3 along with (4) give

$$a_2(\gamma) = \frac{v}{\gamma(\alpha + 2)\left((\alpha + 1)^2\gamma - \beta(\alpha + \gamma + 2)\right)}.$$

Thus the inequalities are sharp, which complete the proof of the theorem. □

4. A Fekete-Szegö Inequality

For brevity we define σ_i for $i = 1, 2, 3$ and τ_j for $j = 1, 2, 3, 4$ by

$$\sigma_1 := \frac{(\alpha + \beta)[\gamma(\alpha + 3) + (\alpha + 2)(1 - \gamma)] - \gamma(\alpha + 1)^2}{2(\alpha + 2)(\alpha + \beta)},\quad \sigma_2 := \frac{\alpha + \gamma + 2}{2(\alpha + 2)}$$

$$\sigma_3 := \frac{(\alpha + \beta)[\gamma(\alpha + 3) + (\alpha + 2)(1 - \gamma)] + \gamma(\alpha + 1)^2}{2(\alpha + 2)(\alpha + \beta)},$$

$$\tau_1 := \alpha + \Big((\alpha + \gamma + 2) - 2(\alpha + 2)\mu\Big)\frac{2(\alpha + \beta)^2}{\gamma(\alpha + 1)^2},$$

$$\tau_2 := \alpha + 2\beta + \frac{2\alpha^2[\gamma(\alpha + 3) + (\alpha + 2)(1 - \gamma - 2\mu)]}{\gamma(\alpha + 1)^2 - \beta[\gamma(\alpha + 3) + (\alpha + 2)(1 - \gamma - 2\mu)]},\quad \tau_3 := \alpha + 2\beta,$$

and

$$\tau_4 := -\alpha + \Big(2(\alpha + 2)\mu - (\alpha + \gamma + 2)\Big)\frac{2(\alpha + \beta)^2}{\gamma(\alpha + 1)^2}.$$

The following theorem provides sharp upper bound for the Fekete-Szegö functional $M(\gamma)$ for function $f \in \mathcal{B}(\alpha, \beta)$.

Theorem 2. *Let $f \in \mathcal{B}(\alpha,\beta)$ and $a_n(\gamma)$ $(n=1,2)$ be given by (4). Then the following sharp bound holds.*

$$|a_2(\gamma)-\mu a_1^2(\gamma)| \le \begin{cases} \frac{1}{(\alpha+2)\gamma}\tau_1, & \mu \le \sigma_1; \\ \frac{1}{(\alpha+2)\gamma}\tau_2, & \sigma_1 \le \mu \le \sigma_2; \\ \frac{1}{(\alpha+2)\gamma}\tau_3, & \sigma_2 \le \mu \le \sigma_3; \\ \frac{1}{(\alpha+2)\gamma}\tau_4, & \mu \ge \sigma_3. \end{cases}$$

Remark 2. *Note that when $\alpha=\gamma=1$, we obtain the result in [25,27,28]. Also when $\beta=1$, we obtain the result in [26].*

Proof of Theorem 2. Proceeding as in the proof of Theorem 1 and using (13) and (14), we get

$$\begin{aligned}(\alpha+2)\gamma(a_2(\gamma)-\mu a_1^2(\gamma)) = {} & \frac{\alpha}{2}\left(q_2+\frac{\alpha(\gamma(\alpha+3)+(\alpha+2)(1-\gamma-2\mu))}{\gamma(\alpha+1)^2}q_1^2\right) \\ & +\beta\left(p_2+\frac{\gamma(\beta(\alpha+3)-(\alpha+1)^2)+(\alpha+2)(1-\gamma-2\mu)\beta}{2\gamma(\alpha+1)^2}p_1^2\right) \\ & +\frac{\alpha\beta(\gamma(\alpha+3)+(\alpha+2)(1-\gamma-2\mu))}{\gamma(\alpha+1)^2}p_1q_1.\end{aligned}$$

So, with the notation

$$x := \frac{\gamma(\alpha+3)+(\alpha+2)(1-\gamma-2\mu)}{\gamma(\alpha+1)^2},$$

we can write

$$(\alpha+2)\gamma(a_2(\gamma)-\mu a_1^2(\gamma)) = \frac{\alpha}{2}\left(q_2+\alpha x q_1^2\right)+\beta\left(p_2+\frac{1}{2}(\beta x-1)p_1^2\right)+\alpha\beta x p_1 q_1. \tag{24}$$

Since rotations of f also belong to $\mathcal{B}(\alpha,\beta)$, we assume that $a_2(\gamma)-\mu a_1^2(\gamma)$ is positive.

Assume that $q_1 = 2\rho e^{i\phi}$ and $p_1 = 2re^{i\theta}$ with $r,\rho \in [0,1]$ and $\phi,\theta \in [0,2\pi]$. Now using Lemma 1, we have

$$\begin{aligned}\frac{1}{2}\operatorname{Re}\left(q_2+\alpha x q_1^2\right) &= \frac{1}{2}\operatorname{Re}\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}(1+2\alpha x)\operatorname{Re} q_1^2 \\ &\le 1-\rho^2+(1+2\alpha x)\rho^2\cos 2\phi \end{aligned}\tag{25}$$

and

$$\begin{aligned}\operatorname{Re}\left(p_2+\frac{1}{2}(\beta x-1)p_1^2\right) &= \operatorname{Re}\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{2}\beta x\operatorname{Re} p_1^2 \\ &\le 2(1-r^2)+2\beta x r^2\cos 2\theta. \end{aligned}\tag{26}$$

From (24), (25) and (26), we obtain

$$\begin{aligned}\operatorname{Re}(\alpha+2)\gamma(a_2(\gamma)-\mu a_1^2(\gamma)) \le {} & \alpha(1-\rho^2+(1+2\alpha x)\rho^2\cos 2\phi)+2\beta(1-r^2+r^2\beta x\cos 2\theta \\ & +4\alpha\beta x r\rho\cos(\theta+\phi) =: \psi(x).\end{aligned}\tag{27}$$

We shall use the notation $\psi(x)$ when all parameters except x are held constant. Thus we need to find the maximum of the right-hand side of (27).

First, we assume that $\sigma_1 \le \mu \le \sigma_2$ and so we must have $0 \le x \le 1/(\alpha+\beta)$. Under this assumption, we can write (27) as

$$\operatorname{Re}(\alpha+2)\gamma(a_2(\gamma)-\mu a_1^2(\gamma)) \le \alpha(2\alpha x+1)+2\beta(-r^2+r^2\beta x\cos 2\theta+xr). \tag{28}$$

Further computation shows that the expression $\kappa(r) := 2\alpha x r - r^2 + r^2\beta x \cos 2\theta$ has its maximum at $r_0 = r = \alpha x/(1-\beta x \cos 2\theta)$ and

$$\begin{aligned} 2\alpha x r - r^2 + r^2\beta x \cos 2\theta &\leq \kappa(r_0) = \frac{\alpha^2 x^2}{1-\beta x \cos 2\theta} \\ &\leq \frac{\alpha^2 x^2}{1-\beta x}. \end{aligned}$$

Therefore from (28), we get

$$\begin{aligned} \operatorname{Re}(\alpha+2)\gamma(a_2(\gamma) - \mu a_1^2(\gamma)) &\leq (2\alpha x + 1)\alpha + 2\beta\left(1 + \frac{\alpha^2 x^2}{1-\beta x}\right) \\ &= \alpha + 2\beta + \frac{2\alpha^2(\gamma(\alpha+3) + (\alpha+2)(1-\gamma-2\mu))}{\gamma(\alpha+1)^2 - \beta((\alpha+3) + (\alpha+2)(1-\gamma-2\mu))}. \end{aligned}$$

This establishes the second inequality of theorem. Equality occurs only if

$$p_1 = \frac{2\alpha(\gamma(\alpha+3) + (\alpha+2)(1-\gamma-2\mu))}{\gamma(\alpha+1)^2 - \beta((\alpha+3) + (\alpha+2)(1-\gamma-2\mu))}, \; p_2 = q_1 = q_2 = 2,$$

and the corresponding function f is defined by

$$zf'(z)f^{\alpha-1}(z) = \frac{z^\alpha}{(1-z)^{2\alpha}}\left(\frac{1+\chi_0 z + z^2}{1-z^2}\right)^\beta,$$

with

$$\chi_0 = \frac{2\alpha(\gamma(\alpha+3) + (\alpha+2)(1-\gamma-2\mu))}{\gamma(\alpha+1)^2 - \beta((\alpha+3) + (\alpha+2)(1-\gamma-2\mu))}.$$

We now prove the first inequality. Assume that $\mu \leq \sigma_1$. This implies that $x \geq 1/(\alpha+\beta)$. Let $x_0 = x = 1/(\alpha+\beta)$. Then it can be verified that $\psi(x_0) \leq \alpha + 2\beta$. Further, we have

$$\begin{aligned} \psi(x) &= \psi(x_0) + 2(x-x_0)(\alpha^2\rho^2\cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\alpha\beta\rho r\cos(\theta+\phi)) \\ &\leq \psi(x_0) + 2(x-x_0)(\alpha+\beta)^2 \\ &\leq \alpha + (2+\gamma+\alpha-2(\alpha+2)\mu)\frac{2(\alpha+\beta)^2}{\gamma(\alpha+1)^2} \end{aligned}$$

as required. Equality occurs only if $p_1 = p_2 = q_1 = q_2 = 2$, and the corresponding function f is defined by (21).

For $x_1 = x = -1/(\alpha+\beta)$, it is a simple matter to check that $\psi(x_1) \leq \alpha + 2\beta$. Using an argument similar to above, if $x \leq x_1$, which is equivalent to the condition $\mu \geq \sigma_3$, then

$$\begin{aligned} \psi(x) &\leq \psi(x_1) + 2|x - x_1|(\alpha+\beta)^2 \\ &\leq -\alpha + [2(\alpha+2)\mu - (\gamma+\alpha+2)]\frac{2(\alpha+\beta)^2}{\gamma(\alpha+1)^2}. \end{aligned}$$

Equality occurs only if $p_1 = q_1 = 2i$, $p_2 = q_2 = -2$, and the corresponding function f is defined by

$$zf'(z)f^{\alpha-1}(z) = \frac{z^\alpha}{(1-iz)^{2\alpha}}\left(\frac{1+iz}{1-iz}\right)^\beta.$$

Also, for $0 \leq \lambda \leq 1$,

$$\psi(\lambda x_1) = \lambda\psi(x_1) + (1-\lambda)\psi(0) \leq \alpha + 2\beta,$$

so $\psi(x) \le \alpha + 2\beta$ for $x_1 \le x \le 0$, i.e.,

$$\sigma_2 =: \frac{\alpha+\gamma+2}{2(\alpha+2)} \le \mu \le \frac{(\alpha+\beta)(\gamma(\alpha+3)+(\alpha+2)(1-\gamma))+\gamma(\alpha+1)^2}{2(\alpha+2)(\alpha+\beta)} =: \sigma_3.$$

Equality occurs only if $p_1 = q_1 = 0$, $p_2 = q_2 = 2$, and the corresponding function f is defined by

$$zf'(z)f^{\alpha-1}(z) = \frac{z^\alpha}{(1-z^2)^\alpha}\left(\frac{1+z^2}{1-z^2}\right)^\beta.$$

This completes the proof. □

5. Conclusions

In this paper, we have investigated the sharp upper bound for $|a_1(\gamma)|$, $|a_2(\gamma)|$ and $M(\gamma)$ for $f \in \mathcal{B}(\alpha,\beta)$. In general, it is not easy to verify $|a_n(\gamma)| \le b_n(\gamma)$ to hold for many subclasses of normalized univalent functions. However, in this work it has been verified that the inequality $|a_n(\gamma)| \le b_n(\gamma)$ holds for larger values of γ, which is rare for many subclasses of normalized univalent functions. The sharp estimate on the generalized Fekete-Szegö functional is also derived. Special cases are also discussed.

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References

1. Branges, L.D. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152. [CrossRef]
2. Brannan, D.A.; Kirwan, W.E. On some classes of bounded univalent functions. *J. Lond. Math. Soc.* **1969**, *1*, 431–443. [CrossRef]
3. Kaplan, W. Close-to-convex schlicht functions. *Mich. Math. J.* **1952**, *1*, 169–185. [CrossRef]
4. Thomas, D.K. On starlike and close-to-convex univalent functions. *J. Lond. Math. Soc.* **1967**, *42*, 427–435. [CrossRef]
5. Singh, R. On Bazilević functions. *Proc. Am. Math. Soc.* **1973**, *38*, 261–271.
6. Marjono; Sokól J.; Thomas, D.K. The fifth and sixth coefficients for Bazilevič functions $\mathcal{B}_1(\alpha)$. *Mediter. J. Math.* **2017**, *14*, 158. [CrossRef]
7. Cho, N.E.; Kumar, V. On a coefficient conjecture for Bazilevič functions. *Mathematics* preprint.
8. Thomas, D.K. On the coefficients of Bazilević functions with logarithmic growth. *Indian J. Math.* **2015**, *57*, 403–418.
9. Al-Oboudi, F.M. *n*-Bazilevic functions. *Abstr. Appl. Anal.* **2012**, *2012*, 383592. Available online: https://www.hindawi.com/journals/aaa/2012/383592/abs/ (accessed on 2 June 2018). [CrossRef]
10. Bazilević, I.E. On a case integrability in quadratures of the Loewner-Kufarev equation. *Mat. Sb. NS* **1955**, *37*, 471–476.
11. Deng, Q. On the coefficients of Bazilevič functions and circularly symmetric functions. *Appl. Math. Lett.* **2011**, *24*, 991–995. [CrossRef]
12. Girela, D. Logarithmic coefficients of univalent functions. *Ann. Acad. Sci. Fenn. Math.* **2000**, *25*, 337–350.
13. Keogh, F.R.; Miller, S.S. On the coefficients of Bazilevič functions. *Proc. Am. Math. Soc.* **1971**, *30*, 492–496.
14. Kim, Y.C. A note on growth theorem of Bazilevič functions. *Appl. Math. Comput.* **2009**, *208*, 542–546. [CrossRef]
15. Sheil-Small, T. On Bazilević functions. *Quart. J. Math. Oxford Ser.* **1972**, *23*, 135–142. [CrossRef]
16. Thomas, D.K. On a subclass of Bazilević functions. *Int. J. Math. Math. Sci.* **1985**, *8*, 779–783. [CrossRef]

17. El-Yagubi, E.; Darus, M. Bazilević *P*-valent functions associated with generalized hypergeometric functions. *Kragujev. J. Math.* **2015**, *39*, 111–120. [CrossRef]
18. Zamorski, J. On Bazilević schlicht functions. *Ann. Polon. Math.* **1962**, *12*, 83–90. [CrossRef]
19. Littlewood, J. E.; Paley, R.E.A.C. A proof that and odd schlicht function has bounded coefficients. *J. Lond. Math. Soc*. **1932**, *7*, 167–169. [CrossRef]
20. Fekete, M.; Szegö, G. Eine Bermerkung uber ungerade schlichte functions. *J. Lond. Math. Soc.* **1933**, *8*, 85–89. [CrossRef]
21. Hayman, W.K.; Hummel, J.A. Coefficients of powers of univalent functions. *Complex Var. Theory Appl.* **1986**, *7*, 51–70. [CrossRef]
22. Jahangiri, M. On the coefficients of powers of a class of Bazilevič functions. *Indian J. Pure Appl. Math.* **1986**, *17*, 1140–1144.
23. Marjono; Thomas, D.K. A note on the powers of Bazilević functions. *Int. J. Math. Anal.* **2015**, *9*, 2061–2067. [CrossRef]
24. Darus, M.; Thomas, D.K. On the Fekete-Szegö theorem for close-to-convex functions. *Math. Jpn.* **1996**, *44*, 507–511.
25. London, R.R. Fekete-Szegö inequalities for close-to-convex functions. *Proc. Am. Math. Soc*. **1993**, *117*, 947–950.
26. Eenigenburg, P.J.; Silvia, E.M. A coefficient inequality for Bazilevič functions. *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **1973**, *27*, 5–12.
27. Keogh, F.R.; Merkes, E.P. A coefficient inequality for certain classes of analytic functions. *Proc. Am. Math. Soc.* **1969**, *20*, 8–12. [CrossRef]
28. Abdel-Gawad, H.R.; Thomas, D.K. The Fekete-Szegö problem for strongly close-to-convex functions. *Proc. Am. Math. Soc.* **1992**, *114*, 345–349.
29. Farahmand, K.; Jahangiri, J.M. Fekete-Szegö problem and Littlewood-Paley conjecture for powers of close-to-convex functions. *Sci. Math.* **1998**, *1*, 89–95.
30. Carathéodory, G. Uber den Variabilitätsbbereich der Fourieŕ schen Konstanten von positiven harmonischen Funktionen. *Rend. Circ. Mat. Palermo* **1911**, *32*, 193–217. [CrossRef]
31. Duren, P.L. *Univalent Functions*; Springer: New York, NY, USA, 1983.