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Some Identities Involving the Fubini Polynomials and Euler Polynomials

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Abstract: In this paper, we first introduce a new second-order non-linear recursive polynomials $U_{h,i}(x)$, and then use these recursive polynomials, the properties of the power series and the combinatorial methods to prove some identities involving the Fubini polynomials, Euler polynomials and Euler numbers.

Keywords: Fubini polynomials; Euler polynomials; recursive polynomials; combinatorial method; power series identity

MSC: 11B39, 11B50

1. Introduction

For any real number x and y, the two variable Fubini polynomials $F_n(x,y)$ are defined by means of the following (see [1,2])

$$\frac{e^{xt}}{1-y\left(e^t-1\right)}=\sum_{n=0}^{\infty}\frac{F_n(x,y)}{n!}\cdot t^n. \tag{1}$$

The first several terms of $F_n(x,y)$ are $F_0(x,y)=1$, $F_1(x,y)=x+y$, $F_2(x,y)=x^2+2xy+2y^2+y$, $\cdots$. Taking $x=0$, then $F_n(0,y)=F_n(y)$ (see [1]) are called the Fubini polynomials. If $y=-\frac{1}{2}$, then $F_n\left(x,-\frac{1}{2}\right)=E_n(x)$, the Euler polynomials, $E_0(x)=1$, $E_1(x)=x-\frac{1}{2}$, $E_2(x)=x^2-x$, and

$$E_n(x)=\sum_{i=0}^{n}(-1)^i\cdot\binom{n}{i}\cdot x^i\cdot E_{n-i}(x),\ n=0,1,2,\cdots.$$

If $x=0$, then $E_n(0)=E_n$ are the famous Euler numbers. $E_0=1$, $E_1=-\frac{1}{2}$, $E_2=0$, $E_3=\frac{1}{4}$, $E_4=0$, $E_5=-\frac{1}{2}$, $E_6=0$, and $E_{2n}=0$ for all positive integer n.

These polynomials appear in combinatorial mathematics and play a very important role in the theory and application of mathematics, thus many number theory and combination experts have studied their properties, and obtained a series of interesting results. For example, Kim and others proved a series of identities related to $F_n(x,y)$ (see [2–4]), one of which is

$$F_n(x,y)=\sum_{l=0}^{n}\binom{n}{l}x^l\cdot F_{n-l}(y),\ n\geq 0.$$

T. Kim et al. [5] also studied the properties of the Fubini polynomials $F_n(y)$, and proved the identity

$$F_n(y) = \sum_{k=0}^{n} S_2(n,k)\, k!\, y^k,\ (n \geq 0),$$

where $S_2(n,k)$ are the Stirling numbers of the second kind.

Zhao and Chen [6] proved that, for any positive integers n and k, one has the identity

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \cdots \frac{F_{a_k}(y)}{(a_k)!} \\ = \quad & \frac{1}{(k-1)!(y+1)^{k-1}} \cdot \frac{1}{n!} \sum_{i=0}^{k-1} C(k-1,i) F_{n+k-1-i}(y), \end{aligned} \tag{2}$$

where the summation is taken over all k-dimensional nonnegative integer coordinates $(a_1, a_2, \cdots, a_h)$ such that $a_1 + a_2 + \cdots + a_h = n$. The sequence $\{C(k,i)\}$ is defined as follows: For any positive integer k and integers $0 \leq i \leq k$, $C(k,0) = 1$, $C(k,k) = k!$ and $C(k+1,i+1) = C(k,i+1) + (k+1)C(k,i)$, for all $0 \leq i < k$.

Some other papers related to Fubini polynomials and Euler numbers can be found elsewhere [7–19], and we do not repeat them here.

In this paper, as a note of [6], we study a similar calculating problem of Equation (2) for two variable Fubini polynomials $F_n(x,y)$. We also introduce a new second order non-linear recursive polynomials, and then use this polynomials to give a new expression for the summation

$$W(h,n,x) = \sum_{a_1+a_2+\cdots+a_{h+1}=n} \frac{F_{a_1}(x,y)}{a_1!} \cdot \frac{F_{a_2}(x,y)}{a_2!} \cdots \frac{F_{a_{h+1}}(x,y)}{a_{h+1}!}.$$

That is, we prove the following:

Theorem 1. *Let h be a positive integer. Then, for any integer $n \geq 0$, we have the identity*

$$W(h,n,x) = \frac{1}{(y+1)^h \cdot h! \cdot n!} \cdot \sum_{k=0}^{h} \sum_{i=0}^{n} U_{h,k}(x) \cdot x^i \cdot h^i \cdot \binom{n}{i} \cdot F_{n-i+k}(x,y),$$

where $U_{h,k}(x)$ is a second order non-linear recurrence polynomial defined by $U_{h,h}(x) = 1$, and $U_{h+1,0}(x) = (h+1-x)U_{h,0}(x)$, and $U_{h+1,k+1}(x) = (h+1-x)U_{h,k+1}(x) + U_{h,k}(x)$ for all integers $h \geq 1$ and k with $0 \leq k \leq h-1$.

It is clear that our theorem is a generalization of Equation (2). If taking $y = -\frac{1}{2}$, $n = 0$, $x = 0$ and $x = 1$ in this theorem, respectively, and noting that $U_{h,0}(1) = 0$, $E_0(1) = 1$ and $E_n(1) = -E_n$ for all $n \geq 1$, we can deduce the following five corollaries:

Corollary 1. *For any positive integer $h \geq 1$, we have the identity*

$$\sum_{k=0}^{h} U_{h,k}(x) \cdot E_k(x) = \frac{h!}{2^h}.$$

Corollary 2. *For any positive integer $h \geq 1$ and real x, we have the identity*

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} \frac{E_{a_1}(x)}{a_1!} \cdot \frac{E_{a_2}(x)}{a_2!} \cdots \frac{E_{a_{h+1}}(x)}{a_{h+1}!} = \frac{2^h}{h! \cdot n!} \cdot \sum_{k=0}^{h} \sum_{i=0}^{n} U_{h,k}(x) \cdot x^i \cdot h^i \cdot \binom{n}{i} \cdot E_{n-i+k}(x).$$

Corollary 3. *For any positive integer $h \geq 1$, we have the identity*

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} \frac{E_{a_1}}{a_1!} \cdot \frac{E_{a_2}}{a_2!} \cdots \frac{E_{a_{h+1}}}{a_{h+1}!} = \frac{2^h}{h! \cdot n!} \cdot \sum_{k=0}^{h} U_{h,k}(0) \cdot \binom{n}{i} \cdot E_{n+k}.$$

Corollary 4. *For any positive integer $h \geq 1$, we have the identity*

$$\frac{h!}{2^h} + \sum_{k=1}^{h} U_{h,k}(1) \cdot E_k = 0.$$

From Equation (2) with $y = -\frac{1}{2}$ and Corollary 3 we can deduce the identities $U_{h,i}(0) = C(h, h-i)$ for all nonnegative integers $0 \leq i \leq h$.

On the other hand, from the definition of $U_{h,k}(1)$, we can easily prove that the sequence $U_{h,k}(1)$ are the coefficients of the polynomial $F(x) = \prod_{i=1}^{h-1}(x+i)$. That is,

$$F(x) = (x+1)(x+2)\cdots(x+h-1) = \sum_{i=0}^{h-1} U_{h,i+1}(1) \cdot x^i.$$

Thus, if $h = p$ is an odd prime, then using the elementary number theory methods we deduce the following:

Corollary 5. *Let p be an odd prime. Then, for any positive integer $2 \leq k \leq p-1$, we have the congruence*

$$U_{p,k}(1) \equiv 0 \bmod p.$$

Taking $h = p$, noting that $U_{p,p}(1) = 1$, $E_1 = -\frac{1}{2}$ and $U_{p,1}(1) = (p-1)! \equiv -1 \bmod p$, and then combining Corollaries 4 and 5, we have the following:

Corollary 6. *Let p be an odd prime. Then, we have the congruence*

$$2E_p + 1 \equiv 0 \bmod p.$$

This congruence is also recently obtained by Hou and Shen [12] using the different methods.

2. Several Simple Lemmas

In this section, we give several necessary lemmas in the proof process of our theorem. First, we have the following:

Lemma 1. *Let function $f(t) = \frac{e^{xt}}{1-y(e^t-1)}$. Then, for any positive integer h, real numbers x and t, we have the identity*

$$(y+1)^h \cdot h! \cdot f^{h+1}(t) = e^{hxt} \cdot \sum_{i=0}^{h} U_{h,i}(x) \cdot f^{(i)}(t),$$

where $U_{h,i}(x)$ is defined as in the theorem, and $f^{(h)}(t)$ denotes the h-order derivative of $f(t)$ with respect to variable t.

Proof. We can prove this Lemma 1 by mathematical induction. First, from the properties of the derivative, we have

$$f'(t) = \frac{xe^{xt}}{1-y\left(e^t-1\right)} + \frac{y \cdot e^t \cdot e^{xt}}{\left(1-y\left(e^t-1\right)\right)^2} = xf(t) - f(t) + \frac{(y+1) \cdot e^{xt}}{\left(1-y\left(e^t-1\right)\right)^2}$$

or

$$(y+1)f^2(t) = e^{xt}\left[f'(t) + (1-x)f(t)\right] = e^{xt}\cdot\sum_{i=0}^{1} U_{1,i}(x)\cdot f^{(i)}(t).$$

That is, Lemma 1 is correct for $h=1$. □

Assuming that Lemma 1 is correct for $1 \le h = k$, i.e.,

$$(y+1)^k \cdot k! \cdot f^{k+1}(t) = e^{kxt}\cdot\sum_{i=0}^{k} U_{k,i}(x)\cdot f^{(i)}(t). \tag{3}$$

Then, from Equation (3) and the definitions of $U_{k,i}(x)$ and derivative, we have

$$\begin{aligned}
& e^{xt}\cdot(y+1)^k\cdot(k+1)!\cdot f^k(t)\cdot f'(t) \\
= \;& (y+1)^k(k+1)!\cdot f^k(t)\left((y+1)f^2(t) + (x-1)\cdot e^{xt}\cdot f(t)\right) \\
= \;& (y+1)^{k+1}(k+1)!\cdot f^{k+2}(x) + (k+1)!\cdot(x-1)\cdot e^{xt}\cdot(y+1)^k\cdot f^{k+1}(t) \\
= \;& e^{(k+1)xt}\cdot\left(xk\cdot\sum_{i=0}^{k} U_{k,i}(x)\cdot f^{(i)}(t) + \sum_{i=0}^{k} U_{k,i}(x)\cdot f^{(i+1)}(t)\right)
\end{aligned}$$

or

$$\begin{aligned}
& (y+1)^{k+1}\cdot(k+1)!\cdot f^{k+2}(t) = e^{(k+1)xt}\cdot\sum_{i=0}^{k} xk\cdot U_{k,i}(x)\cdot f^{(i)}(t) \\
& +e^{(k+1)xt}\cdot\left(\sum_{i=0}^{k} U_{k,i}(x)\cdot f^{(i+1)}(t) + \sum_{i=0}^{k}(k+1)(1-x)U_{k,i}(x)\cdot f^{(i)}(t)\right) \\
= \;& e^{(k+1)xt}\cdot\left(\sum_{i=0}^{k} U_{k,i}(x)\cdot f^{(i+1)}(t) + \sum_{i=0}^{k}(k+1-x)U_{k,i}(x)\cdot f^{(i)}(t)\right) \\
= \;& e^{(k+1)xt}\cdot\left(\sum_{i=0}^{k-1} U_{k,i}(x)\cdot f^{(i+1)}(t) + \sum_{i=1}^{k}(k+1-x)U_{k,i}(x)\cdot f^{(i)}(t)\right) \\
& +e^{(k+1)xt}\cdot\left(U_{k,k}(x)\cdot f^{(k+1)}(t) + (k+1-x)\cdot U_{k,0}(x)\cdot f(t)\right) \\
= \;& e^{(k+1)xt}\cdot\left(\sum_{i=0}^{k-1} U_{k,i}(x)\cdot f^{(i+1)}(t) + \sum_{i=0}^{k-1}(k+1-x)U_{k,i+1}(x)\cdot f^{(i+1)}(t)\right) \\
& +e^{(k+1)xt}\cdot\left(U_{k+1,k+1}(x)\cdot f^{(k+1)}(t) + (k+1-x)\cdot U_{k,0}(x)\cdot f(t)\right) \\
= \;& e^{(k+1)xt}\cdot\sum_{i=0}^{k-1} U_{k+1,i+1}(x)\cdot f^{(i+1)}(t) \\
& +e^{(k+1)xt}\cdot\left(U_{k+1,k+1}(x)\cdot f^{(k+1)}(t) + U_{k+1,0}(x)\cdot f(t)\right) \\
= \;& e^{(k+1)xt}\cdot\sum_{i=1}^{k} U_{k+1,i}(x)\cdot f^{(i)}(t) \\
& +e^{(k+1)xt}\cdot\left(U_{k+1,k+1}(x)\cdot f^{(k+1)}(t) + U_{k+1,0}(x)\cdot f(t)\right) \\
= \;& e^{(k+1)xt}\cdot\sum_{i=0}^{k+1} U_{k+1,i}(x)\cdot f^{(i)}(t).
\end{aligned}$$

which means Lemma 1 is also correct for $h = k+1$.

This proves Lemma 1 by mathematical induction.

Lemma 2. *For any positive integers h and k, we have the power series expansion*

$$e^{xht} \cdot f^{(k)}(t) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} x^i \cdot h^i \cdot \binom{n}{i} \cdot \frac{F_{n-i+k}(x,y)}{n!} \right) \cdot t^n.$$

Proof. For any positive integer k, from Equation (1) and the properties of the power series, we have

$$f^{(k)}(t) = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1) \cdot \frac{F_{n+k}(x,y)}{(n+k)!} \cdot t^n = \sum_{n=0}^{\infty} \frac{F_{n+k}(x,y)}{n!} \cdot t^n. \tag{4}$$

On the other hand, we have

$$e^{xht} = \sum_{n=0}^{\infty} \frac{x^n \cdot h^n}{n!} \cdot t^n. \tag{5}$$

Thus, from Equations (4) and (5) and the multiplicative properties of the power series, we have

$$\begin{aligned} e^{xht} \cdot f^{(k)}(t) &= \left(\sum_{n=0}^{\infty} \frac{x^n \cdot h^n}{n!} \cdot t^n \right) \left(\sum_{n=0}^{\infty} \frac{F_{n+k}(x,y)}{n!} \cdot t^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} x^i \cdot h^i \cdot \frac{\binom{n}{i}}{n!} \cdot F_{n-i+k}(x,y) \right) \cdot t^n. \end{aligned}$$

This proves Lemma 2. □

3. Proof of the Theorem

In this section, we complete the proof of our theorem. In fact from Equation (1) and Lemmas 1 and 2, we have

$$\begin{aligned} & (y+1)^h \cdot h! \cdot f^{h+1}(t) = (y+1)^h \cdot h! \cdot \left(\sum_{n=0}^{\infty} \frac{F_n(x,y)}{n!} \cdot t^n \right)^{h+1} \\ = \; & (y+1)^h \cdot h! \cdot \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_{h+1}=n} \frac{F_{a_1}(x,y)}{a_1!} \frac{F_{a_2}(x,y)}{a_2!} \cdots \frac{F_{a_{h+1}}(x,y)}{a_{h+1}!} \right) \cdot t^n \\ = \; & \sum_{k=0}^{h} U_{h,k}(x) \cdot e^{hxt} \cdot f^{(k)}(t) \\ = \; & \sum_{k=0}^{h} U_{h,k}(x) \left(\sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} x^i \cdot h^i \cdot \binom{n}{i} \cdot \frac{F_{n-i+k}(x,y)}{n!} \right) \cdot t^n \right) \\ = \; & \sum_{n=0}^{\infty} \left(\sum_{k=0}^{h} U_{h,k}(x) \sum_{i=0}^{n} x^i \cdot h^i \cdot \binom{n}{i} \cdot \frac{F_{n-i+k}(x,y)}{n!} \right) \cdot t^n. \end{aligned} \tag{6}$$

Comparing the coefficients of the power series in Equation (6), we may immediately deduce the identity

$$\begin{aligned} & (y+1)^h \cdot \sum_{a_1+a_2+\cdots+a_{h+1}=n} \frac{F_{a_1}(x,y)}{a_1!} \cdot \frac{F_{a_2}(x,y)}{a_2!} \cdots \frac{F_{a_{h+1}}(x,y)}{a_{h+1}!} \\ = \; & \frac{1}{h! \cdot n!} \cdot \sum_{k=0}^{h} \sum_{i=0}^{n} U_{h,k}(x) \cdot x^i \cdot h^i \cdot \binom{n}{i} \cdot F_{n-i+k}(x,y). \end{aligned}$$

This completes the proof of our theorem.

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