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Non-Unique Fixed Point Results in Extended *B*-Metric Space †

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Abstract: In this paper, we investigate the existence of fixed points that are not necessarily unique in the setting of extended b-metric space. We state some examples to illustrate our results.

Keywords: b-metric; extend b-metric space; nonunique fixed point

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1. Introduction and Preliminaries

Metric fixed point theory was initiated by the elegant results of Banach, the contraction mapping principle, and all researchers in this area agree on this. He formulated that every contraction in a complete metric space possesses a unique fixed point. Researchers have generalized this result by refining the contraction condition and/or by changing the metric space with a refined abstract space. One interesting generalization of metric space is b-metric space, formulated recently by Czerwik [1]. Following this result on b-metric space, several authors have reported a number of fixed point results in the framework of b-metric space (see, e.g., [2–9] and related references therein).

Throughout this manuscript, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ represents the positive integers. Further, $\mathbb{R}$ represents the real numbers and $\mathbb{R}_0^+ := [0, \infty)$.

Definition 1 (Czerwik [1])**.** *For a non-empty set X, a function $m_b : X \times X \to \mathbb{R}_0^+$ is said to be b-metric if it satisfies the following conditions:*

(m_b1) $\quad m_b(x,y) = 0$ *if and only if* $x = y$.
(m_b2) $\quad m_b(x,y) = m_b(y,x)$ *for all* $x, y \in X$.
(m_b3) $\quad m_b(x,y) \leq s[m_b(x,z) + m_b(z,y)]$ *for all* $x, y, z \in X$, *where* $s \geq 1$.

In addition, the pair (X, m_b) is called a b-metric space, in short, bMS.

One of the standard examples of b-metric is the following:

Example 1. *Let $X = \mathbb{R}$ be the set of real numbers and $m_b(x,y) = (x-y)^2$. Then m_b is a b-metric on $\mathbb{R}$ with $s = 2$. It is clear that m_b is not a metric on $\mathbb{R}$.*

Remark 1. *It is worth mentioning that, for $s = 1$, the b-metric becomes a usual metric.*

Recently, Kamran et al. [10] introduced a new type of generalized b-metric space. Furthermore, they observed the analog of a Banach contraction mapping principle in the framework of this new space.

Definition 2. *[10] Let $\theta : X \times X \to [1, \infty)$ be a mapping. For a non-empty set X, a function $d_\theta : X \times X \to [0, \infty)$ is said to be an extended b-metric if it satisfies the following state of affairs*

$(d_\theta 1)$ $d_\theta(\xi, \eta) = 0$ *if and only if* $\xi = \eta$,
$(d_\theta 2)$ $d_\theta(\xi, \eta) = d_\theta(\eta, \xi)$, *and*
$(d_\theta 3)$ $d_\theta(\xi, \zeta) \le \theta(\xi, \zeta)\,[d_\theta(\xi, \eta) + d_\theta(\eta, \zeta)]$,

for all $\xi, \eta, \zeta \in X$. In addition, the pair (X, d_θ) is called an extended b-metric space, in short extended-bMS.

Remark 2. *If $\theta(\xi, \eta) = s$, constant, for $s \ge 1$, then it coincides with the standard definition of b-metric space.*

Example 2. *Let $\theta : X \times X \to [1, \infty)$ be a mapping defined as $\theta(x, y) = x^2 + y^2 + 2$. For a set $X = \{a, b, c\}$, we set the mapping $d_\theta : X \times X \to [0, \infty)$ as follows:*

$$d_\theta(a, b) = d_\theta(b, a) = 5,\ d_\theta(a, c) = d_\theta(c, a) = 3,\ d_\theta(b, c) = d_\theta(c, b) = 1,$$

$$d_\theta(a, a) = d_\theta(b, b) = d_\theta(c, c) = 0.$$

Obviously, $(d_\theta 1)$ and $(d_\theta 2)$ hold. For $(d_\theta 3)$, we have

$$5 = d_\theta(a, b) \le \theta(a, b)(d_\theta(a, c) + d_\theta(c, b)) = (a^2 + b^2 + 2) \cdot 4$$

$$3 = d_\theta(a, c) \le \theta(a, c)(d_\theta(a, b) + d_\theta(b, c)) = (a^2 + c^2 + 2) \cdot 6$$

$$1 = d_\theta(b, c) \le \theta(b, c)(d_\theta(b, a) + d_\theta(a, c)) = (b^2 + c^2 + 2) \cdot 8.$$

In conclusion, for any $\xi, \eta, \zeta \in X$, the third axiom

$$d_\theta(\xi, \zeta) \le \theta(\xi, \zeta)\,[d_\theta(\xi, \eta) + d_\theta(\eta, \zeta)]$$

is satisfied. Accordingly, (X, d_θ) is an extended b-metric space. Notice also that the standard triangle inequality is not satisfied for the following case

$$5 = d_\theta(a, b) > 4 = d_\theta(a, c) + d_\theta(c, b).$$

Hence, (X, d) does not form a standard metric space.

In an extended-bMS, it is possible to obtain an analogy of basic topological notions, such as convergence, Cauchy sequences, and completeness. For more details, see, e.g., [10].

Definition 3. *[10] Let (X, d_θ) be an extended-bMS.*

- (i) *We say that a sequence ξ_n in X converges to $\xi \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(\xi_n, \xi) < \epsilon$, for all $n \ge N$, and it is denoted as $\lim_{n \to \infty} \xi_n = \xi$.*
- (ii) *We say that a sequence ξ_n in X is Cauchy if, for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(\xi_m, \xi_n) < \epsilon$, for all $m, n \ge N$.*

In what follows, we recollect the notion of completeness:

Definition 4. *[10]. An extended-bmetric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.*

Lemma 1. *[10] Suppose that the pair (X, d_θ) is a complete extended-bMS, where d_θ is continuous. Then every convergent sequence has a unique limit.*

Theorem 1. *[10] Suppose that the pair (X, d_θ) is an complete extended-bMS, where d_θ is continuous. If a self-mapping $T : X \to X$ satisfies*

$$d_\theta(T\xi, T\eta) \leq k d_\theta(\xi, \eta) \tag{1}$$

for all $\xi, \eta \in X$, where $k \in [0,1)$ is such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(\xi_n, \xi_m) < \frac{1}{k}$, where $\xi_n = T^n \xi_0$, $n = 1, 2, ...$, then T has precisely one fixed point u. Moreover, for each $\eta \in X$, $T^n \eta \to u$.

For our purposes, we need to recall the following definitions and results.

Definition 5. *[11] Suppose that the pair (X, d_θ) is an extended-bMS For a self-mapping $T : X \to X$, for each $\xi \in X$ and $n \in \mathbb{N}$, we define*

$$\mathcal{O}(\xi; n) = \{\xi, T\xi, ..., T^n\xi\} \text{ and } \mathcal{O}(\xi; \infty) = \{\xi, T\xi, ..., T^n\xi, ...\}.$$

We say that the set $\mathcal{O}(\xi; \infty)$ is the orbit of T.

Definition 6. *Suppose that the pair (X, d_θ) is an extended-bMS. A self-mapping $T : X \to X$ is called orbitally continuous if $\lim_{i\to\infty} T^{n_i}(\xi) = \zeta$ for some $\zeta \in X$ implies that $\lim_{i\to\infty} T(T^{n_i}(\xi)) = T\zeta$. Moreover, if every Cauchy sequence of the form $\{T^{n_i}(\xi)\}_{i=1}^{\infty}$, $\xi \in X$ converges in (X, d_θ), then we say that an extended-bMS (X, d_θ) is called T-orbitally complete.*

Remark 3. *It is evident that the orbital continuity of T yields orbital continuity of any iterative power of T, that is, orbital continuity of T^m for any $m \in \mathbb{N}$.*

Definition 7. *[12] Suppose that T is a self-mapping on a non-empty set X. Let $\alpha : X \times X \to [0, \infty)$ be a mapping. Then T is called an α-orbital admissible if, for all $\xi \in X$, we have*

$$\alpha(\xi, T\xi) \geq 1 \Rightarrow \alpha(T\xi, T^2\xi) \geq 1. \tag{2}$$

Remark 4. *We note that any α-admissible mapping is also an α-orbital admissible mapping. (see, e.g., [12]).*

2. Main Results

Throughout the paper, we shall assume that d_θ is a continuous functional.

Lemma 2. *[13] Let (X, d_θ) be an extended b-metric space. If there exists $q \in [0, 1)$ such that the sequence $\{x_n\}$, for an arbitrary $x_0 \in X$, satisfies $\lim\limits_{n,m\to\infty} \theta(x_n, x_m) < \dfrac{1}{q}$, and*

$$0 < d_\theta(x_n, x_{n+1}) \leq q d_\theta(x_{n-1}, x_n) \tag{3}$$

for any $n \in \mathbb{N}$, then the sequence $\{x_n\}$ is Cauchy in X.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a given sequence. By employing Inequality (3), recursively, we derive that

$$0 < d_\theta(x_n, x_{n+1}) \leq q^n d_\theta(x_0, x_1). \tag{4}$$

Since $q \in [0, 1)$, we find that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0. \tag{5}$$

On the other hand, by $(d_\theta 3)$, together with triangular inequality, for $p \geq 1$, we derive that

$$\begin{aligned}
d_\theta(x_n, x_{n+p}) &\leq \theta(x_n, x_{n+p}) \cdot [d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+p})] \\
&\leq \theta(x_n, x_{n+p}) d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+p}) d_\theta(x_{n+1}, x_{n+p}) \\
&\leq \theta(x_n, x_{n+p}) q^n d_\theta(x_0, x_1) + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) [d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_{n+p})] \\
&\leq \theta(x_n, x_{n+p}) \cdot q^n d_\theta(x_0, x_1) + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) \cdot q^{n+1} d_\theta(x_0, x_1) + \ldots + \\
&\qquad + \theta(x_n, x_{n+p}) \cdot \ldots \cdot \theta(x_{n+p-1}, x_{n+p}) \cdot k^{n+p-1} d_\theta(x_0, x_1) \\
&= d_\theta(x_0, x_1) \sum_{i=1}^{n+p-1} q^i \prod_{j=1}^{i} \theta(x_{n+j}, x_{n+p}).
\end{aligned} \tag{6}$$

Notice that the inequality above is dominated by $\sum_{i=1}^{n+p-1} q^i \prod_{j=1}^{i} \theta(x_{n+j}, x_{n+p}) \leq \sum_{i=1}^{n+p-1} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p})$.

On the other hand, by employing the ratio test, we conclude that the series $\sum_{i=1}^{\infty} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p})$ converges some $S \in (0, \infty)$. Indeed, $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = \lim_{i\to\infty} q\theta(x_i, x_{i+p}) < 1$, which is why we obtain the desired result. Thus, we have

$$S = \sum_{i=1}^{\infty} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}) \text{ with the partial sum } S_n = \sum_{i=1}^{n} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}).$$

Consequently, we observe for $n \leq 1, p \leq 1$ that

$$d_\theta(x_n, x_{n+p}) \leq q^n d_\theta(x_0, x_1) \left[S_{n+p-1} - S_{n-1}.\right] \tag{7}$$

Letting $n \to \infty$ in Equation (7), we conclude that the constructive sequence $\{x_n\}$ is Cauchy in the extended b—metric space (X, d_θ). □

Lemma 3. *Let $T : X \to X$ be an α-orbital admissible mapping and $x_n = Tx_{n-1}, n \in \mathbb{N}$. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then we have*

$$\alpha(x_{n-1}, x_n) \geq 1 \text{ for all } n \in \mathbb{N}_0.$$

Proof. By assumption, there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. On account of the definition of $\{x_n\} \subset X$ and owing to the fact that T is α-orbital admissible, we derive

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, T^2x_0) = \alpha(x_1, x_2) \geq 1.$$

Recursively, we have

$$\alpha(x_{n-1}, x_n) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{8}$$

□

Theorem 2. *Suppose that T is an orbitally continuous self-mapping on the T-orbitally complete extended-bMS (X, d_θ). Assume that there exists $k \in [0, 1)$ and $a \geq 1$ such that*

$$\alpha(x, y) \min\{d_\theta(Tx, Ty), d_\theta(x, Tx), d_\theta(y, Ty)\} - a \min\{d_\theta(x, Ty), d_\theta(Tx, y)\} \leq kd_\theta(x, y)) \tag{9}$$

for all $x, y \in X$. Furthermore, we presume that

(i) *T is α-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) $\lim_{n,m\to\infty} \theta(x_n, x_m) < \dfrac{1}{k}$.

Then, for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n \quad \forall\ n \in \mathbb{N}_0. \tag{10}$$

If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}_0$, then $x^* = x_{n_0}$ forms a fixed point for T that the proof finishes. Hence, from now on, we assume that

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N}_0. \tag{11}$$

On account of the assumptions (i) and (ii), together with Lemma (3), Inequality (8) is yielded, that is,

$$\alpha(x_{n-1}, x_n) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{12}$$

By replacing $x = x_{n-1}$ and $y = x_n$ in Inequality (9) and taking Equation (12) into account, we find that

$$\begin{aligned}
&\min\{d_\theta(Tx_{n-1}, Tx_n), d_\theta(x_{n-1}, Tx_{n-1}), d_\theta(x_n, Tx_n)\} \\
&\qquad -a\min\{d_\theta(Tx_n, x_{n-1}), d_\theta(Tx_{n-1}, x_n)\} \\
&\leq \alpha(x_{n-1}, x_n)\min\{d_\theta(Tx_{n-1}, Tx_n), d_\theta(x_{n-1}, Tx_{n-1}), d_\theta(x_n, Tx_n)\} - \\
&\qquad -a\min\{d_\theta(Tx_n, x_{n-1}), d_\theta(Tx_{n-1}, x_n)\} \\
&\leq kd_\theta(x_{n-1}, x_n)
\end{aligned} \tag{13}$$

or,

$$\min\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} \leq kd_\theta(x_{n-1}, x_n). \tag{14}$$

Since $k \in [0, 1)$, the case $d_\theta(x_{n-1}, x_n) \leq kd_\theta(x_{n-1}, x_n)$ is impossible. Thus, we conclude that

$$d(x_n, x_{n+1}) \leq kd_\theta(x_{n-1}, x_n). \tag{15}$$

On account of Lemma 2, we find that the sequence $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d_θ), the sequence x_n converges to some point $u \in X$ as $n \to \infty$. Owing to the construction $x_n = T^n x_0$ and the fact that (X, d_θ) is T-orbitally complete, there is $u \in X$ such that $x_n \to u$. Since T, is orbital continuity, we deduce that $x_n \to Tu$. Accordingly, we conclude that $u = Tu$. □

Example 3. *Let $X = \{1, 2, 3, 4\}$ be endowed with extended b-metric $d_\theta : X \times X \to [0, \infty)$, defined by $d_\theta(x, y) = (x - y)^2$, where $\theta : X \times X \to [1, \infty)$, $\theta(x, y) = x + y + 1$. Let $k = \frac{1}{4}$, $a = 4$ and $T : X \to X$ such that*

$$T1 = T3 = 1,\ T2 = 4,\ T4 = 3.$$

Define also $\alpha, \beta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 0 & \text{if}, (x, y) \in \{(3, 4), (4, 3)\} \\ 1 & \text{otherwise.} \end{cases}$$

Let us first notice that for any $x \in \{1,2,3,4\}$, the sequence $\{T^n x\}$ tends to 1 when $n \to \infty$. For this reason, we can conclude that the mapping T is orbitally continuous and $\lim_{n,m\to\infty} \theta(T^n x, T^m x) = 3 < 4 = \frac{1}{k}$, so (iii) is satisfied. It can also be easily verified that T is orbital admissible.

If $x = 1$ or $y = 1$, then $d(1, T1) = 0$ so Inequality (9) holds. We have to consider the following cases.

Case 1. For $x = 2$ and $y = 3$, we have

$$d_\theta(2,3) = 1,\ d_\theta(T2,T3) = 9,\ d_\theta(2,T2) = 4,\ d_\theta(3,T3) = 4,\ d_\theta(2,T3) = 1,\ d_\theta(3,T2) = 1$$

and Inequality (9) yields

$$0 = \min\{9,4,4\} - 4 \cdot \min\{1,1\} \le \frac{1}{4} = \frac{1}{4} \cdot d_\theta(2,3).$$

Case 2. For $x = 2$ and $y = 4$, we have

$$d_\theta(2,4) = 4,\ d_\theta(T2,T4) = 1,\ d_\theta(2,T2) = 4,\ d_\theta(4,T4) = 1,\ d_\theta(2,T4) = 1,\ d_\theta(4,T2) = 0$$

and

$$1 = \min\{1,4,1\} - 4 \cdot \min\{1,0\} \le 1 = \frac{1}{4} \cdot d_\theta(2,4).$$

Case 3. For $x = 3$ and $y = 4$, because $\alpha(3,4) = 0$, Inequality (9) holds.

Therefore, all the conditions of Theorem 2 are satisfied and T has a fixed point, $x = 1$.

In Theorem 2, if we presume that $\alpha(x,y) = 1$ and $\theta(x,y) = 1$, then we deduce the renowned non-unique fixed point theorem of Ćirić [14] as follows:

Corollary 1. **[Ćirić [14]]** Suppose that T is an orbitally continuous self-map on the T-orbitally complete standard metric space (X,d). We presume that there is a $k \in [0,1)$ such that

$$\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} - \min\{d(x,Ty), d(Tx,y)\} \le kd(x,y)$$

for all $x, y \in X$. Then, for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Theorem 3. *Suppose that T is an orbitally continuous self-map on the T-orbitally complete extended-bMS (X,d). We presume that there exists $k \in [0,1)$ such that*

$$\alpha(x,y)\Gamma(x,y) \le kd_\theta(x,y) \tag{16}$$

for all $x, y \in X$, where

$$\begin{aligned} \Gamma(x,y) &= \frac{P(x,y) - Q(x,y)}{R(x,y)} \\ P(x,y) &= \min\{d_\theta(Tx,Ty)d_\theta(x,y), d_\theta(x,Tx)d_\theta(y,Ty)\} \\ Q(x,y) &= \min\{d_\theta(x,Tx)d_\theta(x,Ty), d_\theta(y,Ty)d_\theta(Tx,y)\} \\ R(x,y) &= \min\{d_\theta(x,Tx), d_\theta(y,Ty)\} \end{aligned}$$

where $R(x,y) \neq 0$. Furthermore, we assume that

(i) *T is α-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
(iii) $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$.

Then, for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. As a first step, we construct an iterative sequence $\{x_n\}$ as in the proof of Theorem 2. For this purpose, we take an arbitrary initial value $x \in X$ and define the following recursion:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{17}$$

We also suppose that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}, \tag{18}$$

as is discussed in the proof of Theorem 2.

For $x = x_{n-1}$ and $y = x_n$, Inequality (16) becomes (taking into account Lemma (3))

$$\Gamma(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)\Gamma(x_{n-1}, x_n) \leq kd_\theta(x_{n-1}, x_n) \tag{19}$$

where

$$\begin{aligned}
P(x_{n-1}, x_n) &= \min\{d_\theta(Tx_{n-1}, Tx_n)d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, Tx_{n-1})d_\theta(x_n, Tx_n)\} \\
&= d_\theta(x_n, x_{n+1})d_\theta(x_{n-1}, x_n) \\
Q(x_{n-1}, x_n) &= \min\{d_\theta(x_{n-1}, Tx_{n-1})d_\theta(x_{n-1}, Tx_n), d_\theta(x_n, Tx_n)d_\theta(Tx_{n-1}, x_n)\} = 0 \\
R(x_{n-1}, x_n) &= \min\{d_\theta(x_{n-1}, Tx_{n-1}), d_\theta(x_n, Tx_n)\} = \min\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1}) \\
\Gamma(x_{n-1}, x_n) &= \frac{P(x_{n-1}, x_n) - Q(x_{n-1}, x_n)}{R(x_{n-1}, x_n)}.
\end{aligned}$$

We obtain that

$$\frac{d_\theta(x_n, x_{n+1})d_\theta(x_{n-1}, x_n)}{\min\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\}} \leq kd_\theta(x_{n-1}, x_n). \tag{20}$$

If $R(x_{n-1}, x_n) = d_\theta(x_n, x_{n+1})$, then Inequality (20) turns into

$$d_\theta(x_{n-1}, x_n) \leq kd_\theta(x_{n-1}, x_n) < d_\theta(x_{n-1}, x_n) \tag{21}$$

which is a contraction, since $k \in [0, 1)$. Consequently, we deduce that

$$d_\theta(x_n, x_{n+1}) \leq kd_\theta(x_{n-1}, x_n). \tag{22}$$

Applying Equation (22) recurrently, we find that

$$d_\theta(x_n, x_{n+1}) \leq kd_\theta(x_{n-1}, x_n)) \leq k^2 d_\theta(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n d_\theta(x_0, x_1). \tag{23}$$

The rest of the proof is a verbatim restatement of the related lines in the proof of Theorem 2. □

Theorem 4. *Suppose that T is an orbitally continuous self-map on the T-orbitally complete extended-bMS (X, d_theta). We presumed that there exists $k \in [0, 1)$ and $a > 0$ such that*

$$\alpha(x, y)P(x, y) - aQ(x, y) \leq kR(x, y) \tag{24}$$

for all $x, y \in X$, where

$$\begin{aligned}
P(x, y) &= \min\{d_\theta(Tx, Ty), d_\theta(x, y), d_\theta(x, Tx), d_\theta(y, Ty)\} \\
Q(x, y) &= \min\{d_\theta(x, Ty), d_\theta(Tx, y)\} \\
R(x, y) &= \min\{d_\theta(x, y), d_\theta(x, Tx)\}
\end{aligned}$$

with $R(x, y) \neq 0$. We also assume that

(*i*) *T is α-orbital admissible;*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$.

Then, for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. Basically, we shall use the same technique that was used in the proof of Theorem 2. We built a recursive $\{x_n\}$,

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N} \tag{25}$$

for an arbitrary initial value $x \in X$. Regarding the discussion in the proof of Theorem 2, we presume that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{26}$$

For $x = x_{n-1}$ and $y = x_n$, Inequality (24) becomes (taking into account Lemma 3)

$$\begin{aligned} P(x_{n-1}, x_n) - aQ(x_{n-1}, x_n) &\leq \alpha(x_{n-1}, x_n)P(x_{n-1}, x_n) - aQ(x_{n-1}, x_n) \\ &\leq kR(x_{n-1}, x_n) \end{aligned} \tag{27}$$

where

$$\begin{aligned} P(x_{n-1}, x_n) &= \min\{d_\theta(Tx_{n-1}, Tx_n), d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, Tx_{n-1}), d_\theta(x_n, Tx_n)\} \\ &= \min\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} \\ Q(x_{n-1}, x_n) &= \min\{d_\theta(x_{n-1}, Tx_n), d_\theta(Tx_{n-1}, x_n)\} = 0 \\ R(x_{n-1}, x_n) &= \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, Tx_{n-1})\} = d_\theta(x_{n-1}, x_n). \end{aligned}$$

Thus, Inequality (27) becomes

$$\min\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} \quad \leq kd_\theta(x_{n-1}, x_n). \tag{28}$$

If $\min\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} = d_\theta(x_{n-1}, x_n)$, then Inequality (28) turns into

$$d_\theta(x_{n-1}, x_n) \leq kd_\theta(x_{n-1}, x_n) < d_\theta(x_{n-1}, x_n), \tag{29}$$

a contraction, since $k \in [0, 1)$. Accordingly, we conclude that

$$d_\theta(x_n, x_{n+1}) \leq kd_\theta(x_{n-1}, x_n). \tag{30}$$

Recursively, we derive that

$$d_\theta(x_n, x_{n+1}) \leq kd_\theta(x_{n-1}, x_n)) \leq k^2 d_\theta(x_{n-2}, x_{n-1})) \leq \cdots \leq k^n d_\theta(x_0, x_1). \tag{31}$$

By following the related lines in the proof of Theorem 2, we complete the proof. □

Theorem 5. *Assume that T is an orbitally continuous self-mapping on the T-orbitally complete extended-bMS (X, d). We also presumed that there exists $k \in [0, 1)$ and $a > 0$ such that*

$$\alpha(x, y)m(x, y) - n(x, y) \leq kd_\theta(x, Tx)d_\theta(y, Ty) \tag{32}$$

for all $x, y \in X$, where

$$\begin{aligned} n(x, y) &= \min\{[d_\theta(Tx, Ty)]^2, d_\theta(x, y)d_\theta(Tx, Ty), [d_\theta(y, Ty)]^2\} \\ m(x, y) &= \min\{d_\theta(x, Tx)d_\theta(y, Ty), d_\theta(x, Ty)d_\theta(Tx, y)\}. \end{aligned}$$

Assume the following:

(i) *T is α-orbital admissible;*

(*ii*) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

(*iii*) $\lim\limits_{n,m\to\infty} \theta(x_n, x_m) < \dfrac{1}{k}$.

Then, for each $x_0 \in X$, *the sequence* $\{T^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of* T.

Proof. As a first step, we shall construct an recursive sequence $\{x_n = Tx_{n-1}\}_{n\in\mathbb{N}}$, for an arbitrary initial value $x_0 := x \in X$, as in the proof of Theorem 2. By following the same steps in the proof of Theorem 2, we deduce that adjacent terms of the sequence $\{x_n\}$ should be chosen distinct, that is,

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

For $x = x_{n-1}$ and $y = x_n$, using (i) and Lemma (3) Inequality (32) infer that

$$m(x_{n-1}, x_n) - n(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n) m(x_{n-1}, x_n) - n(x_{n-1}, x_n) \leq k d_\theta(x_{n-1}, Tx_{n-1}) d_\theta(x_n, Tx_n), \tag{33}$$

where

$$\begin{aligned}
m(x_{n-1}, x_n) &= \min\{[d_\theta(Tx_{n-1}, Tx_n)]^2, d_\theta(x_{n-1}, x_n) d_\theta(Tx_{n-1}, Tx_n), [d_\theta(x_n, Tx_n)]^2\} \\
&= \min\{[d_\theta(x_n, x_{n+1})]^2, d_\theta(x_{n-1}, x_n) d_\theta(x_n, x_{n+1}), [d_\theta(x_n, x_{n+1})]^2\} \\
&= \min\{[d_\theta(x_n, x_{n+1})]^2, d_\theta(x_{n-1}, x_n) d_\theta(x_n, x_{n+1})\}, \\
n(x_{n-1}, x_n) &= \min\{d_\theta(x_{n-1}, Tx_{n-1}) d_\theta(x_n, Tx_n), d_\theta(x_{n-1}, Tx_n) d_\theta(Tx_{n-1}, x_n)\} \\
&= \min\{d_\theta(x_{n-1}, x_n) d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_{n+1}) d_\theta(x_n, x_n)\} = 0.
\end{aligned}$$

The case $m(x_{n-1}, x_n) = d_\theta(x_{n-1}, x_n) d_\theta(x_n, x_{n+1})$, is not possible because in this situation inequality (33) becomes

$$d_\theta(x_{n-1}, x_n) d_\theta(x_n, x_{n+1}) \leq k d_\theta(x_{n-1}, x_n) d_\theta(x_n, x_{n+1}) < d_\theta(x_{n-1}, x_n) d_\theta(x_n, x_{n+1}), \tag{34}$$

which is a contradiction. Consequently, we derive

$$[d(x_n, x_{n+1})]^2 \leq k(d(x_{n-1}, x_n) d(x_n, x_{n+1})), \tag{35}$$

which yields

$$d(x_n, x_{n+1}) < k d(x_{n-1}, x_n). \tag{36}$$

Iteratively, we get that

$$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n (d(x_0, x_1)).$$

A verbatim repetition of the related lines in the proof of Theorem 2 completes the proof. □

Theorem 6. *Assume* T *is an orbitally continuous self-mapping on the* T*-orbitally complete extended-bMS* (X, d_θ). *We presumed there exists* $k \in [0, 1)$ *and* $a \geq 0$ *such that*

$$\alpha(x, y) K(x, y) - a Q(x, y) \leq k S(x, y), \tag{37}$$

for all distinct $x, y \in X$ *where*

$$\begin{aligned}
K(x, y) &= \min\{d_\theta(Tx, Ty), d_\theta(y, Ty)\}, \\
Q(x, y) &= \min\{d_\theta(x, Ty), d_\theta(y, Tx)\}, \\
S(x, y) &= \max\{d_\theta(x, y), d_\theta(x, Tx), d_\theta(y, Ty)\}.
\end{aligned}$$

If the following three conditions are fulfilled,

(*i*) *T is α-orbital admissible;*
(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(*iii*) $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$,

then, for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. Let $x_0 \in X$. Starting from this arbitrary initial value, we construct the iterative sequence $x_n = Tx_{n-1_{n\in\mathbb{N}}}$. As discussed in the proof of Theorem 2, we can suppose that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{38}$$

On the other hand, from (i) and Lemma 3, we have that $\alpha(x_{n-1}, x_n) \geq 1$, so, for $x = x_{n-1}$ and $y = x_n$, Inequality (37) implies that

$$K(x_{n-1}, x_n) - aQ(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)K(x_{n-1}, x_n) - aQ(x_{n-1}, x_n) \leq kS(x_{n-1}, x_n) \tag{39}$$

where

$$\begin{aligned}
K(x_{n-1}, x_n) &= \min\{d_\theta(Tx_{n-1}, Tx_n), d_\theta(x_n, Tx_n)\} = \min\{d_\theta(x_n, x_{n+1}), d_\theta(x_n, x_{n+1})\} = d_\theta(x_n, x_{n+1}) \\
Q(x_{n-1}, x_n) &= \min\{d_\theta(x_{n-1}, Tx_n), d_\theta(x_n, Tx_{n-1})\} = 0 \\
S(x_{n-1}, x_n) &= \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, Tx_{n-1}), d_\theta(x_n, Tx_n)\} = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\}.
\end{aligned}$$

Obviously, since $k \in [0, 1)$, the case $S(x_{n-1}, x_n) = d(x_n, x_{n+1})$ is impossible. More precisely, Inequality (39) turns into

$$K(x_{n-1}, x_n) = d(x_n, x_{n+1}) \leq kd(x_n, x_{n+1}) < d(x_n, x_{n+1})$$

which is a contradiction. Hence, Inequality (39) yields that

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) < d(x_{n-1}, x_n) \text{ and } d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

for all $n \in \mathbb{N}$. A verbatim restatement of the related lines in the proof of Theorem 2 completes the proof. □

3. Conclusions

We note that several consequences can be observed from the main results in distinct aspects. For example, taking $\theta(x, y) = s \geq 1$ implies corresponding fixed point results in the context of b-metric space. In addition, standard versions of the given results follow when we take $\theta(x, y) = 1$. Notice also that, as in [15], by assigning $\alpha(x, y)$ in a proper way, we can conclude results in the frame of "partially ordered spaces" and for "cyclic contraction".

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References

1. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inf. Univ. Ostrav.* **1993**, *1*, 5–11.

2. Afshari, H.; Aydi, H.; Karapınar, E. Existence of Fixed Points of Set-Valued Mappings in B-Metric Spaces. *East Asian Math. J.* **2016**, *32*, 319–332. [CrossRef]
3. Aksoy, U.; Karapınar, E.; Erhan, Y.M. Fixed points of generalized alpha-admissible contractions on b-metric spaces with an application to boundary value problems. *J. Nonlinear Convex Anal.* **2016**, *17*, 1095–1108.
4. Aydi, H.; Bota, M.; Karapınar, E.; Mitrović, S. A fixed point theorem for set-valued quasi-contractions in b-metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 88. [CrossRef]
5. Aydi, H.; Bota, M.; Karapınar, E.; Moradi, S. A common fixed point for weak ϕ-contractions in b-metric spaces. *Fixed Point Theory* **2012**, *13*, 337–346.
6. Boriceanu, M. Strict fixed point theorems for multivalued operators in b-metric spaces. *Int. J. Mod. Math.* **2009**, *4*, 285–301.
7. Bota, M.; Karapinar, E. A note on "Some results on multi-valued weakly Jungck mappings in b-metric space". *Cent. Eur. J. Math.* **2013**, *11*, 1711–1712. [CrossRef]
8. Bota, M.; Karapınar, E.; Mleşniţe, O. Ulam-Hyers stability for fixed point problems via α-ϕ-contractive mapping in b-metric spaces. *Abstr. Appl. Anal.* **2013**, *2013*, 855293. [CrossRef]
9. Bota, M.; Chifu, C.; Karapinar, E. Fixed point theorems for generalized (alpha-psi)-Ciric-type contractive multivalued operators in b-metric spaces. *J. Nonlinear Sci. Appl.* **2016**, *9*, 1165–1177. [CrossRef]
10. Kamran, T.; Samreen, M.; Ain, O.U. A Generalization of b-Metric Space and Some Fixed Point Theorems. *Mathematics* **2017**, *5*, 19. [CrossRef]
11. Ćirić, L.B. A generalization of Banach contraction principle. *Proc. Am. Math. Soc.* **1974**, *45*, 267–273. [CrossRef]
12. Popescu, O. Some new fixed point theorems for α-Geraghty-contraction type maps in metric spaces. *Fixed Point Theory Appl.* **2014**, *190*, 1–12. [CrossRef]
13. Alqahtani, B.; Fulga, A.; Karapınar, E. Contractions with rational expression in the extended b-metric space. in press.
14. Ćirić, L.B. On some maps with a nonunique fixed point. *Publ. Inst. Math.* **1974**, *17*, 52–58.
15. Karapinar, E.; Samet, B. Generalized Alpha-Psi Contractive Type Mappings and Related Fixed Point Theorems with Applications. *Abstr. Appl. Anal.* **2012**, *2012*, 793486. [CrossRef]