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Article

On the High-Power Mean of the Generalized Gauss Sums and Kloosterman Sums

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Abstract: The main aim of this paper is to use the properties of the trigonometric sums and character sums, and the number of the solutions of several symmetry congruence equations to research the computational problem of a certain sixth power mean of the generalized Gauss sums and generalized Kloosterman sums, and to give two exact computational formulae for them.

Keywords: symmetry congruence equation; generalized Gauss sums; generalized Kloosterman sums; sixth power mean; computational formula

MSC: 11L03; 11L07

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1. Introduction

Let $q \geq 3$ be a positive integer, m and n be integers. Then for any positive integers $r > s \geq 1$ and Dirichlet character χ mod q, the generalized Gauss sums $G(m, n, r, s, \chi; q)$ is defined as

$$G(m, n, r, s, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^r + na^s}{q}\right),$$

where $e(y) = e^{2\pi i y}$.

It is well known that this sum plays an extremely essential role in the research of analytic number theory, and plenty of classical problems in analytic number theory are closely related to it. For instance, if $q = p$ is an odd prime, $r = p - 2$ and $s = 1$, then

$$G(m, n, p-2, 1, \chi; p) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{m\bar{a} + na}{q}\right)$$

becomes the well-known Kloosterman sum, where $a \cdot \bar{a} \equiv 1 \bmod p$, see H. Iwaniec's book [1] or Zhang Wenpeng's papers [2–5]. Therefore, any substantial advance in the study of $G(m, n, r, s, \chi; q)$ will certainly promote the development of multiplicative number theory and analytic number theory. Due to these reasons, a number of scholars have research the properties of $G(m, n, r, s, \chi; q)$, and obtained various vital results. For instance, Zhang Han and Zhang Wenpeng [6] proved that for any odd prime p, one has the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid p - 1, \\ 2p^3 - 7p^2 & \text{if } 3 \mid p - 1, \end{cases}$$

where n represents any integer with $(n, p) = 1$.

Zhang Wenpeng and Han Di [7] acquired the identity

$$\sum_{a=1}^{p-1}\left|\sum_{n=0}^{p-1} e\left(\frac{n^3+an}{p}\right)\right|^6 = 5p^4-8p^3-p^2,$$

where p denotes an odd prime with $3 \nmid (p-1)$.

Duan Ran and Zhang Wenpeng [8] proved that for any prime p with $3 \nmid (p-1)$, and any Dirichlet character $\lambda \bmod p$, one has the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 3p^3-8p^2 & \text{if } \lambda = \left(\frac{*}{p}\right), \\ 2p^3-7p^2 & \text{if } \lambda \neq \chi_0, \left(\frac{*}{p}\right), \\ 2p^3-3p^2-3p-1 & \text{if } \lambda = \chi_0, \end{cases}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol and χ_0 is the principal character $\bmod p$.

Several other outcomes related to exponential sums and Kloostermann sums can also be found in the references [9–16]. These contents will not be repeated here.

In this paper, we mainly take into account the computational problems of the $2k$-th power mean of the generalized Gauss sum and generalized Kloosterman sum. In other words,

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3+a}{p}\right)\right|^{2k} \tag{1}$$

and

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma+\bar{a}}{p}\right)\right|^{2k}, \tag{2}$$

where $k \geq 3$ represents an integer.

Regarding $2k$-th power means (1) and (2) with $k \geq 3$, up to now no one seems to research them, and we have not seen any relevant papers before. These problems make a lot of sense. In addition to reflecting the value distribution properties of generalized Gauss sums and Kloosterman sums themselves, they also have many important applications in analytic number theory. For example, Zhang Yitang's very important works [17] on the gaps between primes is obtained based on the sieve methods and some special mean value estimate for Kloosterman sums. Therefore, this research is necessary, which is also the original intention of our paper.

The main objective of this paper is to apply the properties of the trigonometric sums and character sums, and the number of the solutions of several congruence equations to research the computational problem of (1) and (2) for $k = 3$, and give two exact computational formulae for them. In other words, we are going to prove the following:

Theorem 1. *For any odd prime p with $3 \nmid (p-1)$, we have the identity*

$$\begin{aligned}\sum_{\chi \bmod p} \sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3+a}{p}\right)\right|^6 &= \sum_{\chi \bmod p} \sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^3+ma}{p}\right)\right|^6 \\ &= p(p-1)\left(6p^3-28p^2+39p+5\right).\end{aligned}$$

Theorem 2. *For any odd prime p, we have the identity*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma+\overline{a}}{p}\right) \right|^6 = p(p-1)\left(5p^3 - 18p^2 + 14p + 5\right).$$

According to these results, we may instantly deduce the following two corollaries:

Corollary 1. *For any odd prime p with $3 \nmid (p-1)$, there exists an integer $1 \le m \le p-1$ and a non-principal character $\chi \bmod p$ such that the following inequality holds,*

$$\left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3+a}{p}\right) \right| > 6^{\frac{1}{6}} \cdot p^{\frac{1}{2}} - \frac{1}{\sqrt{p}}.$$

Corollary 2. *For any odd prime p, there exists an integer $1 \le m \le p-1$ and a non-principal character $\chi \bmod p$ and we can get the inequality*

$$\left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma+\overline{a}}{p}\right) \right| > 5^{\frac{1}{6}} \cdot p^{\frac{1}{2}} - \frac{1}{\sqrt{p}}.$$

Open problems: in Theorem 1, we merely discussed the special case $3 \nmid (p-1)$. Suppose that $3 \mid (p-1)$, the situation will be more complicated. We have not found an effective approach to calculate it up to now, so it is also an open problem.

For $3 \nmid (p-1)$ and $k = 4$, applying our method, we do not seem to be able to acquire an exact computational formula for (1) or (2). These are also two thoughtful problems. Interested readers can give them a try.

2. Some Lemmas

In this part, we are going to introduce four uncomplicated lemmas. These are actually gained by a certain decomposition in the process of theorem proving. Of course, plenty of elementary number theory knowledge required for the following arguments can be found in reference [18], we will not repeat them here. At first, we have the following:

Lemma 1. *Let p be an odd prime with $3 \nmid (p-1)$, then we have the identity*

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3 \equiv d^3+e^3+1 (\bmod p) \\ abc \equiv de (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 = p^3 - 3p^2 + 5p - 5.$$

Proof. According to the properties of the reduced residue system $\bmod p$, we are aware of that if d and e pass through a reduced residue system $\bmod p$ respectively, then da and eb also pass through a reduced residue system $\bmod p$ for all $1 \le a \le p-1$ and $1 \le b \le p-1$. Since $3 \nmid (p-1)$, if a passes through a reduced residue system $\bmod p$, then a^3 also passes through a reduced residue system $\bmod p$. Combining these properties, we obtain

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3 \equiv d^3+e^3+1 (\bmod p) \\ abc \equiv de (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a^3+b^3+c^3 \equiv a^3d^3+b^3e^3+1 (\bmod p) \\ c \equiv de (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1$$

$$\begin{aligned}
&= \sum_{\substack{a=1 \\ a^3+b^3+d^3e^3\equiv a^3d^3+b^3e^3+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a+b+de\equiv ad+be+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} 1 + 2 \sum_{\substack{a=1 \\ b(e-1)\equiv e-1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{e=2}^{p-1} + \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1} 1 \\
&= (p-1)^2 + 2(p-1)(p-2) + \sum_{\substack{a=0 \\ a+b\equiv de-1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1} 1 - \sum_{\substack{b=1 \\ b\equiv de-1(\bmod p)}}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1} 1 \\
&= (p-1)(3p-5) + (p-1)(p-2)^2 - \Big((p-2)^2 - (p-2)\Big) \\
&= p^3 - 3p^2 + 5p - 5.
\end{aligned}$$

This proves Lemma 1. □

Lemma 2. *Let p be an odd prime, then we have the identity*

$$\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = p^3 - 3p^2 + 5p - 5.$$

Proof. According to the method of proving Lemma 1, we can easily deduce that

$$\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a+b+de\equiv ad+be+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = p^3 - 3p^2 + 5p - 5.$$

This proves Lemma 2. □

Lemma 3. *Let p is an odd prime with $3 \nmid (p-1)$, then we have the identity*

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+e^3+1(\bmod p) \\ abc\equiv de(\bmod p) \\ a+b+c\equiv d+e+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = 7p^2 - 31p + 44.$$

Proof. First note that the conditions $a^3+b^3+c^3 \equiv d^3+e^3+1(\bmod p)$, $a+b+c \equiv d+e+1(\bmod p)$ and $abc \equiv de(\bmod p)$ are equivalent to $(a+b+c)^3 - a^3 - b^3 - c^3 \equiv (d+e+1)^3 - (d^3+e^3+1)(\bmod p)$, $a+b+c \equiv d+e+1(\bmod p)$ and $abc \equiv de(\bmod p)$. These conditions are equivalent to $a^2(b+c) + b^2(a+c) + c^2(a+b) \equiv d^2(e+1) + e^2(d+1) + (d+e)(\bmod p)$, $a+b+c \equiv d+e+1(\bmod p)$ and $abc \equiv de(\bmod p)$, they are equivalent to $a^2(d+e+1-a) + b^2(d+e+1-b) + c^2(d+e+1-c) \equiv d^2(a+b+c-d) + e^2(a+b+c-e) + (a+b+c-1)(\bmod p)$, $a+b+c \equiv d+e+1(\bmod p)$ and $abc \equiv de(\bmod p)$, or they are equivalent to $(a+b+c)(a^2+b^2+c^2-d^2-e^2-1) \equiv 0(\bmod p)$, $a+b+c \equiv d+e+1(\bmod p)$ and $abc \equiv de(\bmod p)$.

It is clear that $(a+b+c)(a^2+b^2+c^2-d^2-e^2-1) \equiv 0(\bmod p)$ is equivalent to $a+b+c \equiv 0(\bmod p)$ or $a^2+b^2+c^2-d^2-e^2-1 \equiv 0(\bmod p)$.

First case: if $a+b+c \equiv 0(\bmod p)$, then from the properties of the reduced residue system mod p and note that if $(3, p-1) = 1$ and c pass through a reduced residue system mod p, then c^3 also pass through a reduced residue system mod p, so we have

$$\underset{\substack{a+b+c\equiv d+e+1\equiv 0(\bmod p)\\ abc\equiv de(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 = \underset{\substack{c(a+b+1)\equiv d+e+1\equiv 0(\bmod p)\\ abc^3\equiv de(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 = \underset{\substack{a+b+1\equiv d+e+1\equiv 0(\bmod p)\\ abc\equiv de(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1$$

$$= \underset{a+b+1\equiv d+e+1\equiv 0(\bmod p)}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 = \left(\underset{a+b+1\equiv 0(\bmod p)}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}} 1 \right)^2 = (p-2)^2. \tag{3}$$

Second case: if $a^2+b^2+c^2-d^2-e^2-1 \equiv 0(\bmod p)$, then we have

$$\underset{\substack{a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p)\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 = \underset{\substack{ab+ac+bc\equiv de+d+e(\bmod p)\\ a+b+c\equiv d+e+1(\bmod p\\ abc\equiv de(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 = \underset{\substack{ab+(a+b)c\equiv abde+ad+be(\bmod p)\\ a+b+c\equiv ad+be+1(\bmod p)\\ c\equiv de(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1$$

$$= \underset{\substack{(a-1)(b-1)(de-1)\equiv 0(\bmod p)\\ a+b+de\equiv ad+be+1(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 = 1 + \underset{\substack{(d-1)(e-1)\equiv 0(\bmod p)\\ (de-1,\ p)=1}}{\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 + 2 \underset{(d-1)(b-\overline{d})\equiv 0(\bmod p)}{\sum_{b=2}^{p-1}\sum_{d=1}^{p-1}} 1$$

$$+2 \underset{\substack{(e-1)(b-d)\equiv 0(\bmod p)\\ (de-1,\ p)=1}}{\sum_{b=2}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 + \underset{\substack{a+b\equiv ad+be(\bmod p)\\ de\equiv 1(\bmod p)}}{\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 \tag{4}$$

$$= 1 + 2(p-2) + 4(p-2) + 2(p-2)(2p-5) + \underset{(a+b)e\equiv a+be^2(\bmod p)}{\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{e=1}^{p-1}} 1$$

$$= 1 + 4(p-2)(p-1) + \underset{(e-1)(eb-a)\equiv 0(\bmod p)}{\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{e=1}^{p-1}} 1$$

$$= 1 + 4(p-2)(p-1) + 2(p-2)^2 - (p-2) = 1 + 3(p-2)(2p-3).$$

Third case: if $a+b+c \equiv d+e+1 \equiv 0(\bmod p)$ and $a^2+b^2+c^2 \equiv d^2+e^2+1(\bmod\ p)$, then we have

$$\underset{\substack{a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p)\\ a+b+c\equiv d+e+1\equiv 0 \bmod p\\ abc\equiv de(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 = \underset{\substack{(a-1)(b-1)(de-1)\equiv 0(\bmod p)\\ a+b+de\equiv ad+be+1\equiv 0(\bmod p)}}{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 = \underset{2+de\equiv d+e+1\equiv 0(\bmod p)}{\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1$$

$$+2 \underset{2+b\equiv d+b\overline{d}+1\equiv 0(\bmod p)}{\sum_{b=2}^{p-1}\sum_{d=1}^{p-1}} 1 + 2 \underset{\substack{1+b+de\equiv d+be+1\equiv 0(\bmod p)\\ (de-1,\ p)=1}}{\sum_{b=2}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}} 1 + \underset{a+b+1\equiv ad+b\overline{d}+1\equiv 0(\bmod p)}{\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{d=1}^{p-1}} 1 \tag{5}$$

$$= 2 + 4 + 2\,(p-4+p-5) + p-4+p-5 = 3(2p-7).$$

Summarizing (3)–(5) we have the identity

$$\begin{aligned}
\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+e^3+1(\bmod p) \\ abc\equiv de(\bmod p) \\ a+b+c\equiv d+e+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 &= \sum_{\substack{a=1 \\ (a+b+c)(a^2+b^2+c^2-d^2-e^2-1)\equiv 0(\bmod p) \\ abc\equiv de(\bmod p) \\ a+b+c\equiv d+e+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= \sum_{\substack{a=1 \\ abc\equiv de(\bmod p) \\ a+b+c\equiv d+e+1\equiv 0(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 + \sum_{\substack{a=1 \\ a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p) \\ abc\equiv de(\bmod p) \\ a+b+c\equiv d+e+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 - \sum_{\substack{a=1 \\ a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p) \\ abc\equiv de(\bmod p) \\ a+b+c\equiv d+e+1\equiv 0(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= (p-2)^2+1+3(p-2)(2p-3)-3(2p-7) = 7p^2-31p+44.
\end{aligned}$$

This proves Lemma 3. □

Lemma 4. *Let p be an odd prime, then we have the identity*

$$\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p) \\ \overline{a}+\overline{b}+\overline{c}\equiv \overline{d}+\overline{e}+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = 6p^2-21p+19.$$

Proof. It is obvious that the conditions $a+b+c\equiv d+e+1(\bmod p)$, $\overline{a}+\overline{b}+\overline{c}\equiv \overline{d}+\overline{e}+1(\bmod p)$ and $abc\equiv de(\bmod p)$ are equivalent to $a+b+c\equiv d+e+1 \bmod p$, $ab+ac+bc\equiv de+d+e(\bmod p)$ and $abc\equiv de(\bmod p)$. These conditions are equivalent to $a+b+c\equiv d+e+1 \bmod p$, $ab+ac+bc+1\equiv abc+a+b+c(\bmod p)$ and $abc \equiv de(\bmod p)$, or they are equivalent to $(a-1)(b-1)(c-1)\equiv 0(\bmod p)$ and $a+b+c\equiv d+e+1(\bmod p)$ and $abc\equiv de(\bmod p)$. From these we can easily obtain

$$\begin{aligned}
\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p) \\ \overline{a}+\overline{b}+\overline{c}\equiv \overline{d}+\overline{e}+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 &= \sum_{\substack{a=1 \\ (a-1)(b-1)(c-1)\equiv 0(\bmod p) \\ abc\equiv de(\bmod p) \\ a+b+c\equiv d+e+1(\bmod p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= \sum_{\substack{d=1 \\ d+e\equiv 2(\bmod p) \\ de\equiv 1(\bmod p)}}^{p-1}\sum_{e=1}^{p-1} 1 + 3\sum_{\substack{c=2 \\ c+1\equiv d+e(\bmod p) \\ c\equiv de(\bmod p)}}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 + 3\sum_{\substack{b=2 \\ b+c\equiv d+e(\bmod p) \\ bc\equiv de(\bmod p)}}^{p-1}\sum_{c=2}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= 1+3\sum_{\substack{c=2 \\ de+1\equiv d+e(\bmod p) \\ c\equiv de(\bmod p)}}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 + 3\sum_{\substack{b=2 \\ b^2+de\equiv (d+e)b(\bmod p) \\ bc\equiv de(\bmod p)}}^{p-1}\sum_{c=2}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= 1+3\sum_{\substack{c=2 \\ (d-1)(e-1)\equiv 0 \bmod p \\ c\equiv de(\bmod p)}}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 + 3\sum_{\substack{b=2 \\ (b-d)(b-e)\equiv 0(\bmod p) \\ bc\equiv de(\bmod p)}}^{p-1}\sum_{c=2}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= 1+6(p-2)+3\Big(2(p-2)^2-(p-2)\Big) = 6p^2-21p+19.
\end{aligned}$$

This finishes the proof of Lemma 4. □

3. Proofs of the Theorems

Proof of Theorem 1. It is not difficult to complete the proofs of our theorems. At first, we are going to prove Theorem 1. Suppose that $3 \nmid (p-1)$, then from Lemma 1, Lemma 3, the trigonometric identity

$$\sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) = \begin{cases} p, & \text{if } p \mid m; \\ 0, & \text{if } p \nmid m \end{cases} \tag{6}$$

and the orthogonality of characters $\bmod\ p$, we obtain

$$\begin{aligned}
& \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3+a}{p}\right) \right|^6 \\
= \;& p(p-1) \sum_{\substack{a=1 \\ a^3+b^3+c^3 \equiv d^3+e^3+f^3 (\bmod p) \\ abc \equiv def (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} e\left(\frac{a+b+c-d-e-f}{p}\right) \\
= \;& p(p-1) \sum_{\substack{a=1 \\ a^3+b^3+c^3 \equiv d^3+e^3+1 (\bmod p) \\ abc \equiv de (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} e\left(\frac{f(a+b+c-d-e-1)}{p}\right) \\
= \;& p^2(p-1) \sum_{\substack{a=1 \\ a^3+b^3+c^3 \equiv d^3+e^3+1 (\bmod p) \\ abc \equiv de (\bmod p) \\ a+b+c \equiv d+e+1 (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 - p(p-1) \sum_{\substack{a=1 \\ a^3+b^3+c^3 \equiv d^3+e^3+1 (\bmod p) \\ abc \equiv de (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 \\
= \;& p^2(p-1)\left(7p^2-31p+44\right) - p(p-1)\left(p^3-3p^2+5p-5\right) \\
= \;& p(p-1)\left(6p^3-28p^2+39p+5\right).
\end{aligned}$$

This finishes of the proof of Theorem 1.

Proof of Theorem 2. According to (6), Lemma 2 and Lemma 4, we obtain

$$\begin{aligned}
& \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma+\bar{a}}{p}\right) \right|^6 \\
= \;& p(p-1) \sum_{\substack{a=1 \\ a+b+c \equiv d+e+f (\bmod p) \\ abc \equiv def (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} e\left(\frac{\bar{a}+\bar{b}+\bar{c}-\bar{d}-\bar{e}-\bar{f}}{p}\right) \\
= \;& p(p-1) \sum_{\substack{a=1 \\ a+b+c \equiv d+e+1 (\bmod p) \\ abc \equiv de (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} e\left(\frac{f\left(\bar{a}+\bar{b}+\bar{c}-\bar{d}-\bar{e}-1\right)}{p}\right) \\
= \;& p^2(p-1) \sum_{\substack{a=1 \\ a+b+c \equiv d+e+1 (\bmod p) \\ abc \equiv de (\bmod p) \\ \bar{a}+\bar{b}+\bar{c} \equiv \bar{d}+\bar{e}+1 (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 - p(p-1) \sum_{\substack{a=1 \\ a+b+c \equiv d+e+1 (\bmod p) \\ abc \equiv de (\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 \\
= \;& p^2(p-1)\left(6p^2-21p+19\right) - p(p-1)\left(p^3-3p^2+5p-5\right) \\
= \;& p(p-1)\left(5p^3-18p^2+14p+5\right).
\end{aligned}$$

Proof of Corollary 1. Note that if $m = 0$, then from the properties of the classical

Gauss sums, we deduce that

$$\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{0\cdot a^3+a}{p}\right)\right| = \left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a}{p}\right)\right| = \begin{cases} \sqrt{p}, & \text{if } \chi\neq\chi_0; \\ 1, & \text{if } \chi=\chi_0. \end{cases} \tag{7}$$

Hence, from (7) we can easily obtain

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{0\cdot a^3+a}{p}\right)\right|^6 = p^3(p-2)+1 = (p-1)\left(p^3-p^2-p-1\right). \tag{8}$$

Combining (8) and Theorem 1, we obtain

$$\begin{aligned} &\sum_{\chi \bmod p}\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a}{p}\right)\right|^6 \\ = \;& p(p-1)\left(6p^3-28p^2+39p+5\right)-(p-1)\left(p^3-p^2-p-1\right) \\ = \;& (p-1)\left(6p^4-29p^3+40p^2+6p+1\right). \end{aligned} \tag{9}$$

Taking

$$\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{m_1a^3+a}{p}\right)\right| = \max_{\substack{\chi \bmod p \\ 1\leq m\leq p-1}}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a}{p}\right)\right| \tag{10}$$

and combining (9) and (10), we may instantly deduce that

$$\begin{aligned} &(p-1)^2\cdot\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{m_1a^3+a}{p}\right)\right|^6 \geq \sum_{\chi \bmod p}\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a}{p}\right)\right|^6 \\ = \;& (p-1)\left(6p^4-29p^3+40p^2+6p+1\right). \end{aligned}$$

It is clear that this inequality implies

$$\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{m_1a^3+a}{p}\right)\right| \geq \left(\frac{6p^4-29p^3+40p^2+6p+1}{p-1}\right)^{\frac{1}{6}} > 6^{\frac{1}{6}}\cdot p^{\frac{1}{2}}-\frac{1}{\sqrt{p}}.$$

This finishes of the proof of Corollary 1.
Proof of Corollary 2. We can also deduce Corollary 2 easily. In other words, we have the inequality

$$\max_{\substack{\chi \bmod p \\ 1\leq m\leq p-1}}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma+\bar{a}}{p}\right)\right| > 5^{\frac{1}{6}}\cdot p^{\frac{1}{2}}-\frac{1}{\sqrt{p}}.$$

The proofs of our results have been completed.

4. Conclusions

The main results of this paper are two theorems and two corollaries. The theorems give two exact computational formulae for the sixth power mean of the generalized Gauss sums and generalized Kloosterman sums. The corollaries give two sharper lower bound estimates for the generalized Gauss sums and generalized Kloosterman sums. In addition, as some notes of our results, we also proposed two thoughtful open problems. These results profoundly reveal the law of the value distribution of the generalized Gauss sums and generalized Kloosterman sums, which can also be used for references in the study of similar problems.

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