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On Neutral Functional Differential Inclusions involving Hadamard Fractional Derivatives

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Abstract: We prove the existence of solutions for neutral functional differential inclusions involving Hadamard fractional derivatives by applying several fixed point theorems for multivalued maps. We also construct examples for illustrating the obtained results.

Keywords: functional fractional differential inclusions; Hadamard fractional derivative; existence; fixed point

1. Introduction

Fractional calculus has emerged as an important area of investigation in view of the application of its tools in scientific and engineering disciplines. Examples include bio-medical sciences, ecology, finance, reaction-diffusion systems, wave propagation, electromagnetics, viscoelasticity, material sciences, and so forth. Fractional-order operators give rise to more informative and realistic mathematical models in contrast to their integer-order counterparts. It has been due to the non-local nature of fractional-order operators, which enables us to gain insight into the hereditary behavior (past history) of the associated phenomena. For examples and recent development of the topic, see References [1,2] and the references cited therein.

Differential inclusions—known as generalization of differential equations and inequalities—are found to be of great utility in the study of dynamical systems, stochastic processes, optimal control theory, and so forth. One can find a detailed account of the topic in Reference [3]. In recent years, an overwhelming interest in the subject of fractional-order differential equations and inclusions has been shown, for instance, see References [4–14] and the references cited therein.

In Reference [15], the authors obtained some existence results for sequential neutral differential equations involving Hadamard derivatives:

$$\begin{cases} \mathcal{D}^{\alpha}[\mathcal{D}^{\beta}y(t)-g(t,y_t)]=f(t,y(t)), \ \ t\in J:=[1,b], \\ y(t)=\phi(t), \ t\in[1-r,1], \qquad \mathcal{D}^{\beta}y(1)=\eta\in\mathbb{R}, \end{cases} \tag{1}$$

where $\mathcal{D}^{\alpha},\mathcal{D}^{\beta}$ are the Hadamard fractional derivatives of order $0<\alpha,\beta<1$, respectively and $f,g:J\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $J\subseteq\mathbb{R}$ and $\phi\in C([1-r,1],\mathbb{R})$.

In this paper, we cover the multivalued case of problem (1) and investigate the Hadamard type neutral fractional differential inclusions given by

$$\begin{cases} \mathcal{D}^{\alpha}[\mathcal{D}^{\beta}y(t)-g(t,y_t)]\in F(t,y(t)), \ \ t\in J:=[1,b], \\ y(t)=\phi(t), \ t\in[1-r,1], \qquad \mathcal{D}^{\beta}y(1)=\eta\in\mathbb{R}, \end{cases} \tag{2}$$

where $F: J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ represents the family of all nonempty subsets of $\mathbb{R}$, and the other quantities in (2) are the same as taken in (1). Here y_t is an element of the Banach space $C_r := C([-r,0],\mathbb{R})$ equipped with norm $\|\phi\|_C := \sup\{|\phi(\theta)| : -r\leq\theta\leq 0\}$, and is defined by $y_t(\theta)=y(t+\theta), \ \ \theta\in[-r,0]$, where y is a function defined on $[1-r,b]$ and $t\in J$. The standard fixed point theorems for multivalued maps are applied to establish the existence results for the problem (2).

The remaining content of the paper is composed as follows. In Section 2, we describe the necessary background material needed for our work. Section 3 deals with the main theorems. In Section 4, we construct illustrative examples for the obtained results.

2. Preliminaries

Let us begin this section with some necessary definitions of fractional calculus [1].

Definition 1. *For a function $h:[1,\infty)\to\mathbb{R}$, the Hadamard derivative of fractional order χ is defined by*

$$D^{\chi}h(t)=\frac{1}{\Gamma(n-\chi)}\left(t\frac{d}{dt}\right)^{n}\int_1^t\left(\log\frac{t}{s}\right)^{n-\chi-1}\frac{h(s)}{s}ds, \ \ n=[\chi]+1,$$

where $[\chi]$ denotes the integer part of the real number χ and $\log(\cdot)=\log_e(\cdot)$.

Definition 2. *The Hadamard fractional integral of order χ for a function h is defined as*

$$I^{\chi}h(t)=\frac{1}{\Gamma(\chi)}\int_1^t\left(\log\frac{t}{s}\right)^{\chi-1}\frac{h(s)}{s}ds, \ \ \chi>0,$$

provided the integral exists.

Now we state a known result [15], which plays a key role in the forthcoming analysis.

Lemma 1 (Lemma 2.3 in [15]). *The function y is a solution of the problem*

$$\begin{cases} D^{\alpha}[D^{\beta}y(t)-g(t,y_t)]=f(t,y_t), \ t\in J:=[1,b], \\ y(t)=\phi(t), \ t\in[1-r,1], \\ D^{\beta}y(1)=\eta\in\mathbb{R}, \end{cases} \tag{3}$$

if and only if

$$y(t)=\begin{cases} \phi(t), & \text{if } t\in[1-r,1], \\ \Big\{\phi(1)+(\eta-g(1,\phi))\dfrac{(\log t)^{\beta}}{\Gamma(\beta+1)}+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds & \\ +\dfrac{1}{\Gamma(\alpha+\beta)}\displaystyle\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds\Big\}, & \text{if } t\in J. \end{cases} \tag{4}$$

3. Existence Results

For a normed space $(X, \|\cdot\|)$, we define $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, $\mathcal{P}_{cl,b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$, $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$ and $\mathcal{P}_{b,cl,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded, closed and convex}\}$. In passing, we remark that a closed and bounded set in a metric space is not necessarily compact in general; however, it is true that a set in a metric space of real or complex numbers is compact if and only if it is closed and bounded.

For each $y \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{\xi \in L^1(J, \mathbb{R}) : \xi(t) \in F(t, y(t)) \text{ on } J\}.$$

Denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ endowed with the norm $\|y\| := \sup\{|y(t)| : t \in J\}$. $L^1(J, \mathbb{R})$ represents the space of functions $y : J \to \mathbb{R}$ such that $\|y\|_{L^1} = \int_1^b |y(t)| dt$.

Our first existence result deals with the case when F has convex values and is based on nonlinear alternative for Kakutani maps [16] with the assumption that the multivalued map F is Carathéodory.

Definition 3 (Granas, Dugundji [16])**.** *A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t, x)$ *is measurable for each* $x \in \mathbb{R}$*;*
(ii) $x \longmapsto F(t, x)$ *is upper semicontinuous for almost all* $t \in J$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \le \varphi_\rho(t)$$

for all $x \in \mathbb{R}$ with $\|x\| \le \rho$ and for almost everywhere $t \in J$.

Theorem 1. *Assume that:*

(H_0) *there exists a non-negative constant $k < \Gamma(\alpha+1)(\log b)^{-\alpha}$ such that*

$$|g(t, u_1) - g(t, u_2)| \le k\|u_1 - u_2\|_C, \quad \text{for } t \in J \text{ and every } u_1, u_2 \in C_r.$$

(H_1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is L^1-Carathéodory;*
(H_2) *there exists a continuous non-decreasing function $\Phi : [0, \infty) \to (0, \infty)$ and a function $p \in C(J, \mathbb{R}^+)$ such that*

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \le p(t)\Phi(\|x\|) \text{ for each } (t, x) \in J \times \mathbb{R};$$

(H_3) *there exists a constant $\omega > 0$ such that*

$$\frac{\left(1 - \dfrac{k(\log b)^\alpha}{\Gamma(\alpha+1)}\right)\omega}{\|\phi\|_C + (|\eta| + k\|\phi\|_C + g_0)\dfrac{(\log b)^\beta}{\Gamma(\beta+1)} + \dfrac{g_0(\log b)^\alpha}{\Gamma(\alpha+1)} + \dfrac{\Phi(\omega)\|p\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}} > 1,$$

where $g_0 = |g(1, 0)|$.

Then the problem (2) has at least one solution on $[1 - r, b]$.

Proof. Let us first transform the problem (2) into a fixed point problem by introducing an operator $\mathcal{V} : C([1-r,b],\mathbb{R}) \longrightarrow \mathcal{P}(C([1-r,b],\mathbb{R}))$ by

$$\mathcal{V}(y) = \left\{ \begin{array}{l} h \in C([1-r,b],\mathbb{R}) : \\ h(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r,1], \\ \Big\{\phi(1) + (\eta - g(1,\phi))\dfrac{(\log t)^\beta}{\Gamma(\beta+1)} & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds & \\ +\dfrac{1}{\Gamma(\alpha+\beta)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi(s)}{s}ds\Big\}, & \text{if } t \in J, \end{cases} \end{array} \right\} \tag{5}$$

for $\xi \in S_{F,x}$. It is obvious by Lemma 1 that the fixed points of the operator $\mathcal{V}$ are solutions of the problem (2).

We verify the hypothesis of nonlinear alternative for Kakutani maps [16] in several steps.

Step 1. *$\mathcal{V}(y)$ is convex for each $y \in C([1-r,b],\mathbb{R})$.* It directly follows from the fact that $S_{F,y}$ is convex (F has convex values).

Step 2. *$\mathcal{V}$ maps bounded sets (balls) into bounded sets in $C([1-r,b],\mathbb{R})$.* Let $B_\zeta = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} \le \zeta\}$ be a bounded set in $C([1-r,b],\mathbb{R})$. Then, for each $h \in \mathcal{B}(y), y \in B_\zeta$, there exists $\xi \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= \phi(1) + (\eta - g(1,\phi))\frac{(\log t)^\beta}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds \\ &\quad+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi(s)}{s}ds. \end{aligned}$$

Then, for $t \in J$, we have

$$\begin{aligned} |h(t)| &\le \|\phi\|_C + (|\eta| + k\|\phi\|_C + g_0)\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{k\|y\|_{[1-r,b]} + g_0}{\Gamma(\alpha+1)}(\log b)^\alpha \\ &\quad+\frac{\Phi(\|y\|_{[1-r,b]})\|p\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}. \end{aligned}$$

Thus,

$$\|h\| \le \|\phi\|_C + (|\eta| + k\|\phi\|_C + g_0)\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{k\zeta + g_0}{\Gamma(\alpha+1)}(\log b)^\alpha + \frac{\Phi(\zeta)\|p\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}.$$

Step 3. *$\mathcal{V}$ maps bounded sets into equicontinuous sets of $C([1-r,b],\mathbb{R})$.*

Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $y \in B_\zeta$. Then, for each $h \in \mathcal{B}(y)$, we obtain

$$\begin{aligned} |h(t_2) - h(t_1)| &\le \frac{|\eta| + k\|\phi\|_C + g_0}{\Gamma(\beta+1)}\Big[(\log t_2)^\beta - (\log t_1)^\beta\Big] \\ &\quad+\frac{k\zeta + g_0}{\Gamma(\alpha+\beta)}\int_1^{t_1}\Big|\left(\log\frac{t_2}{s}\right)^{\alpha+\beta-1} - \left(\log\frac{t_1}{s}\right)^{\alpha+\beta-1}\Big|\frac{ds}{s} \\ &\quad+\frac{k\zeta + g_0}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha+\beta-1}\frac{ds}{s} \\ &\quad+\frac{\Phi(\zeta)\|p\|}{\Gamma(\alpha+\beta)}\int_1^{t_1}\Big|\left(\log\frac{t_2}{s}\right)^{\alpha+\beta-1} - \left(\log\frac{t_1}{s}\right)^{\alpha+\beta-1}\Big|\frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\Phi(\zeta)\|p\|}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha+\beta-1}\frac{ds}{s}\\
\leq\ &\frac{|\eta|+k\|\phi\|_C+g_0}{\Gamma(\beta+1)}\left[(\log t_2)^{\beta}-(\log t_1)^{\beta}\right]\\
&+\left\{\frac{k\zeta+g_0}{\Gamma(\alpha+\beta)}+\frac{\Phi(\zeta)\|p\|}{\Gamma(\alpha+\beta+1)}\right\}\left[\left|(\log t_2)^{\alpha+\beta}-(\log t_1)^{\alpha+\beta}\right|+|\log t_2/t_1|^{\alpha+\beta}\right],
\end{aligned}
$$

which tends to zero as $t_2 - t_1 \to 0$ independently of $y \in B_\zeta$. For the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$, the equicontinuity can be established in a similar manner. Thus, by Arzelá-Ascoli theorem [17], we deduce that $\mathcal{V} : C([1-r,b],\mathbb{R}) \to \mathcal{P}(C([1-r,b],\mathbb{R}))$ is completely continuous.

Now we show that $\mathcal{V}$ has a closed graph. Then it will follow by the Proposition 1.2 in Reference [18] that $\mathcal{V}$ is upper semi-continuous, as it is already proved to be completely continuous.

Step 4. $\mathcal{V}$ has a closed graph. We need to show that $h_* \in \mathcal{V}(y_*)$ when $y_n \to x_*, h_n \in \mathcal{V}(y_n)$ and $h_n \to h_*$. Associated with $h_n \in \mathcal{V}(y_n)$, there exists $\xi_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$
\begin{aligned}
h_n(t) &= \phi(1)+(\eta-g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi_n(s)}{s}ds.
\end{aligned}
$$

Thus it suffices to show that there exists $\xi_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$
\begin{aligned}
h_*(t) &= \phi(1)+(\eta-g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi_*(s)}{s}ds.
\end{aligned}
$$

Let us introduce the linear operator $\Theta : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ given by

$$
\begin{aligned}
\xi \mapsto \Theta(\xi)(t) &= \phi(1)+(\eta-g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi(s)}{s}ds.
\end{aligned}
$$

Notice that $\|h_n(t) - h_*(t)\| \to 0$, as $n \to \infty$. Therefore, it follows from a result dealing with the closed graph operators derived in Reference [19] that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,y_n})$. Since $y_n \to y_*$, we have

$$
\begin{aligned}
h(t) &= \phi(1)+(\eta-g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi_*(s)}{s}ds,
\end{aligned}
$$

for some $\xi_* \in S_{F,y_*}$.

Step 5. We can find an open set $U \subseteq C([1-r,b],\mathbb{R})$ with $y \notin \nu\mathcal{V}(y)$ for any $\nu \in (0,1)$ and all $y \in \partial U$.

Let $\nu \in (0,1)$ and $y \in \nu\mathcal{V}(y)$. Then there exists $\xi \in L^1(J,\mathbb{R})$ with $\xi \in S_{F,y}$ such that for $t \in J$,

$$
|y(t)| \leq \|\phi\|_C+(|\eta|+k\|\phi\|_C+g_0)\frac{(\log b)^{\beta}}{\Gamma(\beta+1)}+\frac{k\|y\|_{[1-r,b]}+g_0}{\Gamma(\alpha+1)}(\log b)^{\alpha}
$$

$$+\frac{\Phi(\|y\|_{[1-r,b]})\|p\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}, \ t \in J,$$

which implies that

$$\begin{aligned}\|y\|_{[1-r,b]}\left\{1-\frac{k(\log b)^{\alpha}}{\Gamma(\alpha+1)}\right\} \ &\leq \ \|\phi\|_C + (|\eta| + k\|\phi\|_C + g_0)\frac{(\log b)^{\beta}}{\Gamma(\beta+1)} \\ &\quad + \frac{g_0(\log b)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Phi(\|y\|_{[1-r,b]})\|p\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}.\end{aligned}$$

Consequently

$$\frac{\left(1-\frac{k(\log b)^{\alpha}}{\Gamma(\alpha+1)}\right)\|y\|_{[1-r,b]}}{\|\phi\|_C + (|\eta| + k\|\phi\|_C + g_0)\frac{(\log b)^{\beta}}{\Gamma(\beta+1)} + \frac{g_0(\log b)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Phi(\|y\|_{[1-r,b]})\|p\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}} \leq 1.$$

By (H_3), there exists a real number ω such that $\|y\|_{[1-r,b]} \neq \omega$. Let us consider an open set

$$U = \{y \in C([1-r,b],\mathbb{R})) : \|y\|_{[1-r,b]} < \omega\},$$

with $\overline{U} = U \cup \partial U$. Notice that $\mathcal{V} : \overline{U} \to \mathcal{P}(C([1-r,b],\mathbb{R}))$ is compact and upper semi-continuous multivalued map with convex closed values. The choice of U implies that there does not exist any $y \in \partial U$ satisfying $y \in \nu\mathcal{V}(y)$ for some $\nu \in (0,1)$. In consequence, we deduce from the nonlinear alternative for Kakutani maps [16] that $\mathcal{V}$ has a fixed point $y \in \overline{U}$ which corresponds to a solution to the problem (2). This finishes the proof. □

In the following result, we make use of the nonlinear alternative for contractive maps ([20] Corollary 3.8) to show the existence of solutions for the problem (2).

Lemma 2. *(Nonlinear alternative [20]) Let D be a bounded neighborhood of $0 \in X$, where X is a Banach space. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ be multivalued operators such that (a) Z_1 is contraction, and (b) Z_2 is upper semi-continuous and compact. Then, if $G = Z_1 + Z_2$, either (i) G has a fixed point in $\overline{D}$ or (ii) there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

Theorem 2. *If the conditions $(H_0) - (H_3)$ of Theorem 1 hold, then there exists at least one solution for the problem (2) on $[1-r,b]$.*

Proof. In order to verify the hypotheses of Lemma 2, we introduce the operator $\Psi_1 : C([1-r,b],\mathbb{R}) \longrightarrow C([1-r,b],\mathbb{R})$ by

$$\Psi_1 y(t) = \begin{cases} 0, & \text{if } t \in [1-r,1], \\ (\eta - g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds, & \text{if } t \in J. \end{cases} \tag{6}$$

and the multivalued operator $\Psi_2 : C([1-r,b],\mathbb{R}) \longrightarrow \mathcal{P}(C([1-r,b],\mathbb{R}))$ by

$$\Psi_2 y(t) = \left\{\begin{array}{l} h \in C([1-r,b],\mathbb{R}) : \\ h(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r,1], \\ \phi(1) + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi(s)}{s}ds, & \text{if } t \in J, \end{cases} \end{array}\right\} \tag{7}$$

for $\xi \in S_{F,y}$. Observe that $\mathcal{V} = \Psi_1 + \Psi_2$, where $\mathcal{V}$ is defined by (5). In the first step, it will be established that the operators Ψ_1 and Ψ_2 define the multivalued operators $\Psi_1, \Psi_2 : B_\theta \to \mathcal{P}_{cp,c}(C([1-r,b],\mathbb{R}))$, where $B_\theta = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} \le \theta\}$ is a bounded set in $C([1-r,b],\mathbb{R})$. Let us show that Ψ_2 is compact-valued on B_θ. Observe that the operator Ψ_2 is equivalent to the composition $\mathcal{L} \circ S_F$, where $\mathcal{L}$ is the continuous linear operator on $L^1(J,\mathbb{R})$ into $C([1-r,b],\mathbb{R})$, defined by

$$\mathcal{L}(v)(t) = \phi(1) + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} \frac{v(s)}{s} ds.$$

Let $y \in B_\theta$ be arbitrary and let $\{\xi_n\}$ be a sequence in $S_{F,y}$. Then it follows by the definition of $S_{F,y}$ that $\xi_n(t) \in F(t, y(t))$ for almost all $t \in J$. As $F(t, y(t))$ is compact for all $t \in J$, we have a convergent subsequence of $\{\xi_n(t)\}$ (we denote it by $\{\xi_n(t)\}$ again) that converges in measure to some $\xi(t) \in S_{F,y}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(\xi_n)(t) \to \mathcal{L}(\xi)(t)$ pointwise on J.

The convergence will be uniform once it is shown that $\{\mathcal{L}(\xi_n)\}$ is an equicontinuous sequence. For $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned} |\mathcal{L}(\xi_n)(t_2) - \mathcal{L}(\xi_n)(t_1)| &\le \frac{\Phi(\theta)\|p\|}{\Gamma(\alpha+\beta)} \int_1^{t_1} \left| \left(\log\frac{t_2}{s}\right)^{\alpha+\beta-1} - \left(\log\frac{t_1}{s}\right)^{\alpha+\beta-1} \right| \frac{ds}{s} \\ &\quad + \frac{\Phi(\theta)\|p\|}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\alpha+\beta-1} \frac{ds}{s} \\ &\le \frac{\Phi(\theta)\|p\|}{\Gamma(\alpha+\beta+1)} \left[\left| (\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta} \right| + |\log t_2/t_1|^{\alpha+\beta} \right] \to 0, \end{aligned}$$

as $t_2 \to t_1$, which shows that the sequence $\{\mathcal{L}(\xi_n)\}$ is equicontinuous. As a consequence of the Arzelá-Ascoli theorem, there exists a uniformly convergent subsequence of $\{\xi_n\}$ (we denote it again by $\{\xi_n\}$) such that $\mathcal{L}(\xi_n) \to \mathcal{L}(\xi)$. Noting that $\mathcal{L}(\xi) \in \mathcal{L}(S_{F,y})$, we deduce that $\mathcal{B}(y) = \mathcal{L}(S_{F,y})$ is compact for all $y \in B_\theta$. So $\Psi_2(y)$ is compact.

Now, we show that $\Psi_2(y)$ is convex for all $y \in C([1-r,b],\mathbb{R})$. Let $h_1, h_2 \in \Psi_2(y)$. We select $\xi_1, \xi_2 \in S_{F,y}$ such that

$$h_i(t) = \phi(1) + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} \frac{\xi_i(s)}{s} ds, \quad i = 1, 2,$$

for almost all $t \in J$. Then

$$[\lambda h_1 + (1-\lambda) h_2](t) = \phi(1) + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} \frac{[\lambda\xi_1(s) + (1-\lambda)\xi_2(s)]}{s} ds,$$

where $0 \le \lambda \le 1$. Since $S_{F,y}$ is convex (as F has convex values), $\lambda\xi_1(s) + (1-\lambda)\xi_2(s) \in S_{F,y}$. Thus $\lambda h_1 + (1-\lambda) h_2 \in \Psi_2(y)$, which shows that Ψ_2 is convex-valued.

On the other hand, it is easy to show that Ψ_1 is compact and convex-valued. Next we prove that Ψ_1 is a contraction on $C([1-r,b],\mathbb{R})$. For $y, z \in C([1-r,b],\mathbb{R})$, we have

$$\begin{aligned} |\Psi_1(y)(t) - \Psi_1(z)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{|g(s,y_s) - g(s,z_s)|}{s} ds \\ &\le \frac{k}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{\|y_s - z_s\|_C}{s} ds \\ &\le \frac{k(\log t)^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{[1-r,b]}, \end{aligned}$$

which implies that $\|\Psi_1(y) - \Psi_1(z)\|_{[1-r,b]} \le \frac{k(\log b)^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{[1-r,b]}$. By the assumption (H_0), we conclude that Ψ_1 is a contraction.

As in the proof of Theorem 1, it can easily be shown that the operator Ψ_2 is compact and upper semi-continuous.

In view of the foregoing steps, we deduce that Ψ_1 and Ψ_2 satisfy the hypothesis of Lemma 2. So, from the conclusion of Lemma 2, either condition (i) or condition (ii) holds. We show that conclusion (ii) is not possible. If $y \in \lambda\Psi_1(y) + \lambda\Psi_2(y)$ for $\lambda \in (0,1)$, then there exist $\xi \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda\Bigg(\phi(1) + (\eta - g(1,\phi))\frac{(\log t)^\beta}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi(s)}{s}ds\Bigg), \quad t \in J. \end{aligned}$$

By our assumptions, we can obtain

$$\begin{aligned} |y(t)| &\le \|\phi\|_C + [|\eta| + k\|\phi\|_C + g_0]\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{k\|y\|_{[1-r,b]} + g_0}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} p(s)\Phi(\|y_s\|_C)\frac{ds}{s} \\ &\le \|\phi\|_C + [|\eta| + k\|\phi\|_C + g_0]\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{k\|y\|_{[1-r,b]} + g_0}{\Gamma(\alpha+1)}(\log b)^\alpha \\ &\quad + \frac{\|p\|\Phi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}. \end{aligned}$$

Thus

$$\frac{\left(1 - \frac{k(\log b)^\alpha}{\Gamma(\alpha+1)}\right)\|y\|_{[1-r,b]}}{\|\phi\|_C + [|\eta| + k\|\phi\|_C + g_0]\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{g_0(\log b)^\alpha}{\Gamma(\alpha+1)} + \frac{\Phi(\|y\|_{[1-r,b]})\|p\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}} \le 1. \tag{8}$$

If condition (ii) of Lemma 2 is satisfied, then there exists $\lambda \in (0,1)$ and $y \in \partial B_\omega$ with $y = \lambda\mathcal{V}(y)$. Then, y is a solution of (2) with $\|y\|_{[1-r,b]} = \omega$. Now, by the inequality (8), we get

$$\frac{\left(1 - \frac{k(\log b)^\alpha}{\Gamma(\alpha+1)}\right)\omega}{\|\phi\|_C + [|\eta| + k\|\phi\|_C + g_0]\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{g_0(\log b)^\alpha}{\Gamma(\alpha+1)} + \frac{\Phi(\omega)\|p\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}} \le 1,$$

which contradicts (H_3). Hence, $\mathcal{V}$ has a fixed point on $[1-r,b]$ by Lemma 2, which implies that the problem (2) has a solution. The proof is complete. □

Our next result deals with the non-convex valued map in the problem (2) and is based on Covitz and Nadler's fixed point theorem [21] (*If* $N : X \to \mathcal{P}_{cl}(X)$ *is a contraction, then* $FixN \neq \emptyset$, *where* X *is a metric space*).

For a metric space (X,d) induced from the normed space $(X; \|\cdot\|)$, it is argued in Reference [22] that $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space, where $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ is defined by $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$.

Definition 4 (Granas, Dugundji [16]). *A multivalued operator* $N : X \to \mathcal{P}_{cl}(X)$ *is called*

(a) $\gamma-$*Lipschitz if and only if there exists* $\gamma > 0$ *such that*

$$H_d(N(x), N(y)) \le \gamma d(x,y) \text{ for each } x, y \in X;$$

(b) *a contraction if and only if it is* $\gamma-$*Lipschitz with* $\gamma < 1$.

Theorem 3. *Assume that (H_0) and the following conditions hold:*

(A_1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, y) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $y \in \mathbb{R}$.

(A_2) $H_d(F(t, y), F(t, \bar{y})) \leq m(t)|y - \bar{y}|$ *for almost all* $t \in J$ *and* $y, \bar{y} \in \mathbb{R}$ *with* $m \in C(J, \mathbb{R}^+)$ *and* $d(0, F(t, 0)) \leq m(t)$ *for almost all* $t \in J$.

Then there exists at least one solution for the problem (2) on J, provided that

$$\delta := \frac{k}{\Gamma(\alpha+1)}(\log b)^{\alpha} + \frac{\|m\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta} < 1. \tag{9}$$

Proof. Observe that the set $S_{F,y}$ is nonempty for each $y \in C(J, \mathbb{R})$ by the assumption (A_1). Therefore F has a measurable selection (see Theorem III.6 [23]). Next we consider the operator $\mathcal{V}$ given by (5) and verify that it satisfies the hypothesis of the Covitz and Nadler theorem [21]. We show that $\mathcal{V}(y) \in \mathcal{P}_{cl}(C(J, \mathbb{R}))$ for each $y \in C(J, \mathbb{R})$. Let $\{v_n\}_{n \geq 0} \in \mathcal{F}(y)$ be such that $v_n \to v$ $(n \to \infty)$ in $C(J, \mathbb{R})$. Then $v \in C(J, \mathbb{R})$ and we can find $\xi_n \in S_{F, y_n}$ such that, for each $t \in J$,

$$\begin{aligned} v_n(t) &= \phi(1) + (\eta - g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi_n(s)}{s}ds. \end{aligned}$$

Since F has compact values, we pass onto a subsequence (if necessary) such that ξ_n converges to ξ in $L^1(J, \mathbb{R})$. So $\xi \in S_{F,y}$ and for each $t \in J$, we have

$$\begin{aligned} u_n(t) \to \xi(t) &= \phi(1) + (\eta - g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi(s)}{s}ds. \end{aligned}$$

Hence, $v \in \mathcal{V}(y)$.

Next we prove that there exists $0 < \delta < 1$ (δ is defined by (9)) such that

$$H_d(\mathcal{V}(y), \mathcal{V}(\bar{y})) \leq \delta\|y - \bar{y}\| \text{ for each } y, \bar{y} \in C^2(J, \mathbb{R}).$$

Let $y, \bar{y} \in C^2(J, \mathbb{R})$ and $h_1 \in \mathcal{V}(y)$. Then there exists $\xi_1(t) \in F(t, y(t))$ such that, for each $t \in J$,

$$\begin{aligned} h_1(t) &= \phi(1) + (\eta - g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi_1(s)}{s}ds. \end{aligned}$$

By (A_2), we have

$$H_d(F(t, y), F(t, \bar{y})) \leq m(t)|y(t) - \bar{y}(t)|.$$

So, there exists $v \in F(t, \bar{y}(t))$ such that

$$|\xi_1(t) - v(t)| \leq m(t)|y(t) - \bar{y}(t)|, \quad t \in J.$$

Define $V : J \to \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{v \in \mathbb{R} : |\xi_1(t) - v(t)| \leq m(t)|y(t) - \bar{y}(t)|\}.$$

By Proposition III.4 in Reference [23], it follows that the multivalued operator $V(t) \cap F(t, \bar{y}(t))$ is measurable. So we can find a measurable selection $\xi_2(t)$ for V. So $\xi_2(t) \in F(t, \bar{y}(t))$ and satisfying $|\xi_1(t) - \xi_2(t)| \le m(t)|y(t) - \bar{y}(t)|$ for each $t \in J$.

For each $t \in J$, we define

$$\begin{aligned} h_2(t) &= \phi(1) + (\eta - g(1,\phi))\frac{(\log t)^\beta}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,\bar{y}_s)}{s}ds \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi_2(s)}{s}ds. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|g(s,y_s) - g(s,\bar{y}_s)|}{s}ds \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{|\xi_1(s) - \xi_2(s)|}{s}ds \\ &\le \frac{k\|y-\bar{y}\|_{[1-r,b]}}{\Gamma(\alpha+1)}(\log b)^\alpha + \frac{\|m\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}\|y-\bar{y}\|_{[1-r,b]}. \end{aligned}$$

Hence

$$\|h_1 - h_2\| \le \left\{\frac{k}{\Gamma(\alpha+1)}(\log b)^\alpha + \frac{\|m\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}\right\}\|y-\bar{y}\|_{[1-r,b]}.$$

On the other hand, interchanging the roles of y and $\bar{y}$ leads to

$$H_d(\mathcal{F}(y), \mathcal{F}(\bar{y})) \le \left\{\frac{k}{\Gamma(\alpha+1)}(\log b)^\alpha + \frac{\|m\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}\right\}\|y-\bar{y}\|_{[1-r,b]}.$$

So $\mathcal{V}$ is a contraction. Therefore, from the conclusion of Covitz and Nadler theorem [21], the operator $\mathcal{V}$ has a fixed point y which is indeed a solution of the problem (2). This finishes the proof. $\square$

Finally, we prove an existence result by applying the multivalued version of Krasnoselskii's fixed point theorem [24], which is stated below.

Lemma 3 (Krasnoselskii [24])**.** *Let X be a Banach space, $Y \in \mathcal{P}_{b,cl,c}(X)$ and $W_1, W_2 : Y \to \mathcal{P}_{cp,c}(X)$ be multivalued operators satisfying the conditions: (i) $W_1y + W_2y \subset Y$ for all $y \in Y$; (ii) W_1 is contraction; and (iii) W_2 is upper semicontinuous and compact. Then there exists $y \in Y$ such that $y \in W_1y + W_2y$.*

Theorem 4. *Suppose that (H_0), (H_1) and the following assumption are satisfied*

(B_1) *there exists a function $q \in C([1,b], \mathbb{R}^+)$ such that*

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,u)\} \le q(t), \text{ for each } (t,u) \in [1,b] \times C_r.$$

Then there exists at least one solution for the problem (2) on $[1-r,b]$.

Proof. Let us consider the operators Ψ_1 and Ψ_2 defined by (6) and (7) respectively. As in Theorem 2, one can show that $\Psi_1, \Psi_2 : B_\theta \to \mathcal{P}_{cp,c}(C([1-r,b],\mathbb{R}))$ are indeed multivalued operators, where $B_\theta = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} \le \theta\}$ is a bounded set in $C([1-r,b],\mathbb{R})$. Moreover, Ψ_1 is a contraction on $C([1-r,b],\mathbb{R})$ and Ψ_2 is upper semi-continuous and compact.

Next we show that $\Psi_1(y)+\Psi_2(y)\subset B_\theta$ for all $y\in B_\theta$. Let $y\in B_\theta$ and suppose that

$$\theta\left(1-\frac{k(\log b)^\alpha}{\Gamma(\alpha+1)}\right)>\|\phi\|_C+\frac{[|\eta|+k\|\phi\|_C+g_0](\log b)^\beta}{\Gamma(\beta+1)}+\frac{g_0(\log b)^\alpha}{\Gamma(\alpha+1)}+\frac{\|q\|(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

For $h\in\Psi_1,\Psi_2$ and $\xi\in S_{F,y}$, we have

$$\begin{aligned} h(t) &= \phi(1)+(\eta-g(1,\phi))\frac{(\log t)^\beta}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y_s)}{s}ds \\ &\quad +\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\xi(s)}{s}ds,\quad t\in J. \end{aligned}$$

With the given assumptions, one can obtain

$$\begin{aligned} |h(t)| &\le \|\phi\|_C+[|\eta|+k\|\phi\|_C+g_0]\frac{(\log b)^\beta}{\Gamma(\beta+1)}+\frac{k\|y\|_{[1-r,b]}+g_0}{\Gamma(\alpha+1)}(\log b)^\alpha \\ &\quad +\frac{\|q\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}. \end{aligned}$$

Thus

$$\|h\| \le \|\phi\|_C+[|\eta|+k\|\phi\|_C+g_0]\frac{(\log b)^\beta}{\Gamma(\beta+1)}+\frac{k\theta+g_0}{\Gamma(\alpha+1)}(\log b)^\alpha+\frac{\|q\|}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}<\theta,$$

which means that $\Psi_1(y)+\Psi_2(y)\subset B_\theta$ for all $y\in B_\theta$.

Thus, the operators Ψ_1 and Ψ_2 satisfy the hypothesis of Lemma 3 and hence its conclusion implies that $y\in\mathcal{A}(y)+\mathcal{B}(y)$ in B_θ. Therefore the problem (2) has a solution in B_θ and the proof is finished. □

4. Examples

In this section, we demonstrate the application of our main results by considering the following Hadamard type neutral fractional differential inclusions:

$$D^{1/4}\left(D^{2/3}y(t)-g(t,y_t)\right)\in F(t,y_t),\quad t\in J=[1,e], \tag{10}$$

$$y(t)=\phi(t),\quad t\in[1/2,1],\quad D^{2/3}y(1)=1/4. \tag{11}$$

Here $\alpha=1/4,\beta=2/3,r=1/2,b=e$,

$$F(t,y_t)=\left[\frac{\sqrt{3+\ln t}}{4}\sin(y_t),\ \frac{\sqrt{3}|y_t|^3}{8(1+|y_t|^3)}\sin(\pi t/2e)+\frac{1}{16}\right],$$

$$g(t,y_t)=\frac{1}{4+\ln t}\tan^{-1}(y_t)+\sin(\pi t/2),\quad \phi(t)=\frac{1}{16\sqrt{\frac{3}{4}+t^2}}.$$

With the given data, it is easy to see that (H_0) is satisfied with $k<\Gamma(5/4)$, (H_2) is satisfied with $p(t)=\sqrt{3+\ln t}/4, \|p\|=1/2, \Phi(\|u\|_C)=\|u\|_C$ and (H_3) holds true for $M>7.05996548$ $(M_0=1.46447352, g_0=1)$ with a particular choice of $k=1/4$. Thus all the conditions of Theorem 1 hold true. Hence the problem (10) and (11) has at least one solution on $[1/2,e]$ by the conclusion of Theorem 1. In a similar manner, one can check that the hypotheses of Theorem 2 hold with $M>1.71978641$ and consequently the conclusion of Theorem 2 applies to the problem (10) and (11).

In order to illustrate Theorem 3, let us take

$$F(t, y_t) = \left[0, \frac{\sqrt{15 + (\ln t)^2}}{8} \frac{|y_t|}{(1 + |y_t|)} + \frac{1}{4}\right] \tag{12}$$

in (10). Then $\|m\| = 1/2$ and from the condition (9), $\delta \approx 0.74950391 < 1$. Clearly the hypothesis of Theorem 3 is satisfied. Therefore, there exists at least one solution for the problem (10) and (11) with $F(t, y_t)$ given by (12) on $[1/2, e]$.

5. Conclusions

In this paper, we have derived several existence results for an initial value problem of neutral functional Hadamard-type fractional differential inclusions. In our first result (Theorem 1), we apply a nonlinear alternative for Kakutani multivalued maps to prove the existence of solutions for the problem at hand when the multivalued map F is assumed to be convex-valued. The nonlinear alternative for contractive maps is applied to prove the existence of solutions for the given problem in Theorem 2. In Theorem 3, we show the existence of solutions for the given problem involving non-convex valued maps with the aid of Covitz and Nadler's fixed point theorem. Our final existence result (Theorem 4) relies on the multivalued version of Krasnoselskii's fixed point theorem. In the nutshell, we have presented a comprehensive study of neutral functional Hadamard-type fractional differential inclusions by making use of different tools of fixed point theory for multivalued maps. In our future work, we plan to investigate the existence of solutions to an initial value problem for neutral functional fractional differential inclusions involving a combination of Caputo and Hadamard fractional derivatives.

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