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Some Results on $(s-q)$-Graphic Contraction Mappings in b-Metric-Like Spaces

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Abstract: In this paper we consider $(s-q)$-graphic contraction mapping in b-metric like spaces. By using our new approach for the proof that a Picard sequence is Cauchy in the context of b-metric-like space, our results generalize, improve and complement several approaches in the existing literature. Moreover, some examples are presented here to illustrate the usability of the obtained theoretical results.

Keywords: b-metric space; b-metric-like space; general contractive mappings; graphic contraction mappings

JEL Classification: 47H10; 54H25

1. Introduction and Preliminaries

First, we present some definitions and basic notions of partial-metric, metric-like, b-metric, partial b-metric and b-metric-like spaces as the generalizations of standard metric spaces. After that, we give a process diagram, where arrows stand for generalization relationships.

Definition 1. *[1] Let X be a nonempty set. A mapping $p_{pm} : X \times X \to [0,+\infty)$ is said to be a p-metric if the following conditions hold for all $x,y,z \in X$:*

$(p_{pm}1)$ $x = y$ *if and only if* $p_{pm}(x,x) = p_{pm}(x,y) = p_{pm}(y,y)$;
$(p_{pm}2)$ $p_{pm}(x,x) \leq p_{pm}(x,y)$;
$(p_{pm}3)$ $p_{pm}(x,y) = p_{pm}(y,x)$;
$(p_{pm}4)$ $p_{pm}(x,y) \leq p_{pm}(x,z) + p_{pm}(z,y) - p_{pm}(z,z)$.

Then, the pair (X, p_{pm}) is called a partial metric space.

Definition 2. *[2] Let X be a nonempty set. A mapping $b_{ml} : X \times X \to [0,+\infty)$ is said to be metric-like if the following conditions hold for all $x,y,z \in X$:*

(b_l1) $b_{ml}(x,y) = 0$ *implies* $x = y$;
(b_l2) $b_{ml}(x,y) = b_{ml}(y,x)$;

(b_l3) $\quad b_{ml}(x,z) \leq b_{ml}(x,y) + b_{ml}(y,z)$.
In this case, the pair (X, b_{ml}) *is called a metric-like space.*

Definition 3. *[3,4] Let* X *be a nonempty set and* $s \geq 1$ *a given real number. A mapping* $b : X \times X \to [0, +\infty)$ *is called a b-metric on the set* X *if the following conditions hold for all* $x, y, z \in X$:
$(b1)$ $\quad b(x,y) = 0$ *if and only if* $x = y$;
$(b2)$ $\quad b(x,y) = b(y,x)$;
$(b3)$ $\quad b(x,z) \leq s[b(x,y) + b(y,z)]$.
In this case, the pair (X, b) *is called a b-metric space (with coefficient* $s \geq 1$*).*

Definition 4. *[5,6] Let* X *be a nonempty set and* $s \geq 1$*. A mapping* $b_{pb} : X \times X \to [0, +\infty)$ *is called a partial b-metric on the set* X *if the following conditions hold for all* $x, y, z \in X$:
$\left(b_{pb}1\right)$ $\quad x = y$ *if and only if* $p_{pb}(x,x) = p_{pb}(x,y) = p_{pb}(y,y)$;
$\left(b_{pb}2\right)$ $\quad b_{pb}(x,x) \leq b_{pb}(x,y)$;
$\left(b_{pb}3\right)$ $\quad b_{pb}(x,y) = b_{pb}(y,x)$;
$\left(b_{pb}4\right)$ $\quad b_{pb}(x,y) \leq s\left[b_{pb}(x,z) + b_{pb}(z,y)\right] - b_{pb}(z,z)$.
Then, the pair $\left(X, b_{pb}\right)$ *is called a partial b-metric space.*

Definition 5. *[7] Let* X *be a nonempty set and* $s \geq 1$*. A mapping* $b_{bl} : X \times X \to [0, +\infty)$ *is called b-metric-like on the set* X *if the following conditions hold for all* $x, y, z \in X$:
$(b_{bl}1)$ $\quad b_{bl}(x,y) = 0$ *implies* $x = y$;
$(b_{bl}2)$ $\quad b_{bl}(x,y) = b_{bl}(y,x)$;
$(b_{bl}3)$ $\quad b_{bl}(x,z) \leq s[b_{bl}(x,y) + b_{bl}(y,z)]$.
In this case, the pair (X, b_{bl}) *is called a b-metric-like space with coefficient* $s \geq 1$.

Now, we give the process diagram of the classes of generalized metric spaces that were introduced earlier:

$$\begin{array}{ccccc} \text{Metric space} & \to & \text{Partial metric space} & \to & \text{Metric-like space} \\ \downarrow & & \downarrow & & \downarrow \\ \text{b-Metric space} & \to & \text{Partial b-metric space} & \to & \text{b-Metric-like space} \end{array}$$

For more details on other generalized metric spaces see [8–14].

The next proposition helps us to construct some more examples of b-metric (respectively partial b-metric, b-metric-like) spaces.

Proposition 1. *Let* (X, d) *(resp.* $\left(X, p_{pm}\right), (X, b_{ml})$*) be a metric (resp. partial metric, metric-like) space and* $D(x,y) = (d(x,y))^k$ *(resp.* $P_{pm}(x,y) = \left(p_{pm}(x,y)\right)^k$, $B_{ml}(x,y) = (b_{ml}(x,y))^k$*), where* $k > 1$ *is a real number. Then* D *(resp.* P_{pm}, B_{pm}*) is b-metric (resp. partial b-metric, b-metric-like) with coefficient* $s = 2^{k-1}$.

Proof. The proof follows from the fact that

$$u^k + v^k \leq (u+v)^k \leq (a+b)^k \leq 2^{k-1}\left(a^k + b^k\right),$$

for all nonnegative real numbers a, b, u, v with $u + v \leq a + b$. $\square$

It is clear that each metric-like space, i.e., each partial b-metric space, is a b-metric-like space, while the converse is not true. For more such examples and details see [1,2,5–7,15–27]. Moreover, for various metrics in the context of the complex domain see [28,29].

The definitions of convergent and Cauchy sequences are formally the same in partial metric, metric-like, partial b-metric and b-metric like spaces. Therefore, we give only the definition of convergence and Cauchyness of the sequences in b-metric-like space. Moreover, these two notions are formally the same in metric and b-metric spaces.

Definition 6. *[7] Let $\{x_n\}$ be a sequence in a b-metric-like space (X, b_{bl}) with coefficient s.*

(i) The sequence $\{x_n\}$ is said to be convergent to x if $\lim_{n\to\infty} b_{bl}(x_n, x) = b_{bl}(x, x)$;

(ii) The sequence $\{x_n\}$ is said to be b_{bl}-Cauchy in (X, b_{bl}) if $\lim_{n,m\to\infty} b_{bl}(x_n, x_m)$ exists and is finite;

(iii) One says that a b-metric-like space (X, b_{bl}) is b_{bl}-complete if for every b_{bl}-Cauchy sequence $\{x_n\}$ in X there exists an $x \in X$, such that $\lim_{n,m\to\infty} b_{bl}(x_n, x_m) = b_{bl}(x, x) = \lim_{n\to\infty} b_{bl}(x_n, x)$.

Remark 1. *In a b-metric-like space the limit of a sequence need not be unique and a convergent sequence need not be a b_{bl}-Cauchy sequence (see Example 7 in [18]). However, if the sequence $\{x_n\}$ is b_{bl}-Cauchy with $\lim_{n,m\to\infty} b_{bl}(x_n, x_m) = 0$ in the b_{bl}-complete b-metric-like space (X, b_{bl}) with coefficient $s \geq 1$, then the limit of such a sequence is unique. Indeed, in such a case if $x_n \to x$ $(b_{bl}(x_n, x) \to b_{bl}(x, x))$ as $n \to \infty$ we get that $b_{bl}(x, x) = 0$. Now, if $x_n \to x$ and $x_n \to y$ where $x \neq y$, we obtain that:*

$$\frac{1}{s} b_{bl}(x, y) \leq b_{bl}(x, x_n) + b_{bl}(x_n, y) \to b_{bl}(x, x) + b_{bl}(y, y) = 0 + 0 = 0. \tag{1}$$

From $(b_{bl}1)$ it follows that $x = y$, which is a contradiction. The same is true as well for partial metric, metric like and partial b-metric spaces.

The next definition and the corresponding proposition are important in the context of fixed point theory.

Definition 7. *[30] The self-mappings $f, g : X \to X$ are weakly compatible if $f(g(x)) = g(f(x))$, whenever $f(x) = g(x)$.*

Proposition 2. *[30] Let T and S be weakly compatible self-maps of a nonempty set X. If they have a unique point of coincidence $w = f(u) = g(u)$, then w is the unique common fixed point of f and g.*

In this paper we shall use the following result to prove that certain Picard sequences are Cauchy. The proof is completely identical with the corresponding in [31] (see also [25]).

Lemma 1. *Let $\{x_n\}$ be a sequence in a b-metric-like space (X, b_{bl}) with coefficient $s \geq 1$ such that*

$$b_{bl}(x_n, x_{n+1}) \leq \lambda b_{bl}(x_{n-1}, x_n) \tag{2}$$

for some $\lambda, 0 \leq \lambda < \frac{1}{s}$, and each $n = 1, 2, ...$Then $\{x_n\}$ is a b_{bl}-Cauchy sequence in (X, b_{bl}) such that $\lim_{n,m\to\infty} b_{bl}(x_n, x_m) = 0$.

Remark 2. *It is worth noting that the previous lemma holds in the context of b-metric-like spaces for each $\lambda \in [0, 1)$. For more details see [6,32].*

2. Main Results

In line with Jachymski [33], let (X, b_{bl}) be a b-metric-like space and $\mathcal{D}$ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \mathcal{D}$. We also assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning the distance between its vertices to each edge (see [33]).

By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus, we have

$$E\left(G^{-1}\right) = \{(x,y) \in X \times X : (y,x) \in E(G)\}. \tag{3}$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric under the convention

$$E\left(\widetilde{G}\right) = E(G) \cup E\left(G^{-1}\right). \tag{4}$$

If x and y are vertices in a graph G, then a path in G from x to y of length N $(N \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected.

Recently, some results have appeared providing sufficient conditions for a self mapping of X to be a Picard operator when (X,d) is endowed with a graph. The first result in this direction was given by Jachymski [33]. Moreover, see [34–36].

Definition 8. *[33] We say that a mapping $f : X \to X$ is a Banach G-contraction or simply a G-contraction if f preserves edges of G, i.e.,*

$$\text{for all } x, y \in X : (x,y) \in E(G) \text{ implies } (f(x), f(y)) \in E(G) \tag{5}$$

and f decreases the weights of edges of G as for all $x, y \in X$, there exists $\lambda \in (0,1)$, such that

$$(x,y) \in E(G) \text{ implies } d(f(x), f(y)) \leq \lambda d(x,y). \tag{6}$$

Definition 9. *[37] A mapping $g : X \to X$ is called orbitally continuous, if given $x \in X$ and any sequence $\{k_n\}$ of positive integers,*

$$g^{k_n}(x) \to y \text{ as } n \to \infty \text{ implies } g\left(g^{k_n}(x)\right) \to g(y) \text{ as } n \to \infty. \tag{7}$$

Definition 10. *[33] A mapping $g : X \to X$ is called G-continuous, if for any given $x \in X$ and any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with the properties that for all $n \in \mathbb{N}$ the pair $(x_n, x_{n+1}) \in E(G)$ and that $x_n \to x$ as $n \to \infty$ it follows that $g(x_n) \to g(x)$.*

Definition 11. *[33] A mapping $g : X \to X$ is called orbitally G-continuous, if given $x, y \in X$ and any sequence $\{k_n\}$ of positive integers for all $n \in \mathbb{N}$,*

$$g^{k_n}x \to y \text{ and } \left(g^{k_n}(x), g^{k_n+1}(x)\right) \in E(G) \text{ implies } g\left(g^{k_n}(x)\right) \to g(y) \text{ as } n \to \infty. \tag{8}$$

In this section, we consider self-mappings $f, g : X \to X$ with $f(X) \subset g(X)$. Let $x_0 \in X$ be an arbitrary point, then there exists $x_1 \in X$ such that $z_0 = f(x_0) = g(x_1)$. By repeating this step we can build a sequence $\{z_n\}$ such that $z_n = f(x_n) = g(x_{n+1})$ and the following property:

The property $G_{f,g(x_n)}$. If $\{g(x_n)\}_{n\in\mathbb{N}}$ is a sequence in X such that $(g(x_n), g(x_{n+1})) \in E(G)$ for all $n \geq 1$ and $g(x_n) \to x$, then there is a subsequence $\{g(x_{n_i})\}_{i\in\mathbb{N}}$ of $\{g(x_n)\}_{n\in\mathbb{N}}$ such that $(g(x_{n_i}), x) \in E(G)$ for all $i \geq 1$. Note that the property $G_{f,g(x_n)}$ depends only on the pair of mappings f and g, and does not depend on the sequence $\{x_n\}$. Here, we use notation G_{gf} in the following sense: $x \in X$ belongs to G_{gf} if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_0 = x$, $f(x_{n-1}) = g(x_n)$ for $n \in \mathbb{N}$, and $(g(x_n), g(x_m)) \in E(G)$ for all $m, n \in \mathbb{N}$.

Now, we present the first result of this section.

Theorem 1. *(Hardy-Rogers) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and which satisfy*

$$\begin{aligned} s^q b_{bl}(f(x), f(y)) \leq\ & c_1 b_{bl}(g(x), g(y)) + c_2 b_{bl}(g(x), f(x)) + c_3 b_{bl}(g(y), f(y)) \\ & + c_4 b_{bl}(g(x), f(y)) + c_5 b_{bl}(g(y), f(x)), \end{aligned} \tag{9}$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$ where $q \geq 2, c_i \geq 0, i = 1, \ldots, 5$ and either

$$c_1 + c_2 + c_3 + 2c_4 + 2c_5 < \frac{1}{s} \tag{10}$$

or

$$c_1 + 2c_2 + 2c_3 + c_4 + c_5 < \frac{1}{s}. \tag{11}$$

Suppose that $f(X) \subset g(X)$ and at least one of $f(X), g(X)$ is b_{bl}-complete subspace of (X, b_{bl}). Then:

(i) If the pair (f, g) has property $G_{f,g(x_n)}$ and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E(G)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Proof. (i) Assume that $G_{gf} \neq \emptyset$, there exists $x_0 \in G_{gf}$. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $f(x_0) = g(x_1)$, again we can find $x_2 \in X$ such that $f(x_1) = g(x_2)$. Repeating this step, we can build a sequence $z_n = f(x_n) = g(x_{n+1})$ such that $(z_n, z_m) \in E(G)$. If $z_k = z_{k+1}$ for some $k \in \mathbb{N}$, then $f(x_{k+1}) = g(x_{k+1})$ is a point of coincidence of f and g. Therefore, let $z_n \neq z_{n+1}$ for all $n \in \mathbb{N}$. By Condition (9), we can get that

$$\begin{aligned} b_{bl}(z_n, z_{n+1}) &\leq s^q b_{bl}(z_n, z_{n+1}) = s^q b_{bl}(f(x_n), f(x_{n+1})) \\ &\leq c_1 b_{bl}(g(x_n), g(x_{n+1})) + c_2 b_{bl}(g(x_n), f(x_n)) + c_3 b_{bl}(g(x_{n+1}), f(x_{n+1})) \\ &\quad + c_4 b_{bl}(g(x_n), f(x_{n+1})) + c_5 b_{bl}(g(x_{n+1}), f(x_n)). \end{aligned} \tag{12}$$

Since $z_n = f(x_n) = g(x_{n+1})$ then Condition (12) becomes

$$\begin{aligned} b_{bl}(z_n, z_{n+1}) &\leq c_1 b_{bl}(z_{n-1}, z_n) + c_2 b_{bl}(z_{n-1}, z_n) + c_3 b_{bl}(z_n, z_{n+1}) \\ &\quad + c_4 b_{bl}(z_{n-1}, z_{n+1}) + c_5 b_{bl}(z_n, z_n) \\ &\leq c_1 b_{bl}(z_{n-1}, z_n) + c_2 b_{bl}(z_{n-1}, z_n) + c_3 b_{bl}(z_n, z_{n+1}) + s c_4 b_{bl}(z_{n-1}, z_n) \\ &\quad + s c_4 b_{bl}(z_n, z_{n+1}) + 2 s c_5 b_{bl}(z_{n-1}, z_n), \end{aligned} \tag{13}$$

or equivalently:

$$b_{bl}(z_n, z_{n+1}) \leq \lambda b_{bl}(z_{n-1}, z_n), \tag{14}$$

where $\lambda = \frac{c_1 + c_2 + s c_4 + 2 s c_5}{1 - c_3 - s c_4}$. Since, $c_1 + c_2 + c_3 + s c_4 + 2 s c_5 \leq s c_1 + s c_2 + s c_3 + 2 s c_4 + 2 s c_5 < 1$, it follows that $\lambda < 1$. Therefore, by Remark 2 of Lemma 1, the sequence $z_n = f(x_n) = g(x_{n+1})$ is a b_{bl}-Cauchy sequence. The b_{bl}-completeness of $f(X)$ leads to $u \in f(X) \subset g(X)$ such that $z_n \to u = g(v)$ for some $v \in X$. As $z_0 \in G_{gf}$, this implies that $(z_n, z_m) \in E(G)$ for $n, m = 1, 2, \ldots$ and so $(z_n, z_{n+1}) \in E(G)$.

By property $G_{f,g(x_n)}$, there is a subsequence $\{z_{n_i}\}_{i\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ such that $(z_{n_i}, u) \in E(G)$. Applying $(b_{bl}3)$, we get

$$\begin{aligned}
b_{bl}(f(v), g(v)) &\leq sb_{bl}(f(v), f(x_{n_i})) + sb_{bl}(f(x_{n_i}), g(v)) \\
&\leq s^q b_{bl}(f(v), f(x_{n_i})) + sb_{bl}(f(x_{n_i}), g(v)) \\
&\leq c_1 b_{bl}(g(v), g(x_{n_i})) + c_2 b_{bl}(g(v), f(v)) + c_3 b_{bl}(g(x_{n_i}), f(x_{n_i})) \\
&\leq +c_4 b_{bl}(g(v), f(x_{n_i})) + c_5 b_{bl}(g(x_{n_i}), f(v)) + sb_{bl}(f(x_{n_i}, g(v))) \\
&= c_1 b_{bl}(g(v), z_{n_i-1}) + c_2 b_{bl}(g(v), f(v)) + c_3 b_{bl}(z_{n_i-1}, z_{n_i}) \\
&\quad +c_4 b_{bl}(g(v), z_{n_i}) + c_5 b_{bl}(z_{n_i-1}, f(v)) + sb_{bl}(z_{n_i}, g(v)). \qquad (15)
\end{aligned}$$

Since $b_{bl}(z_{n_i-1}, f(v)) \leq sb_{bl}(z_{n_i-1}, g(v)) + sb_{bl}(g(v), f(v))$, Condition (15) becomes

$$\begin{aligned}
&(1 - c_2 - c_5 s)\, b_{bl}(f(v), g(v)) \\
\leq\ & c_1 b_{bl}(g(v), z_{n_i-1}) + c_3 b_{bl}(z_{n_i-1}, z_{n_i}) + c_4 b_{bl}(g(v), z_{n_i}) \\
& +c_5 s b_{bl}(z_{n_i-1}, g(v)) + sb_{bl}(z_{n_i}, g(v)). \qquad (16)
\end{aligned}$$

Taking the limit in Condition (16) as $i \to \infty$ we obtain that $b_{bl}(f(v), g(v)) = 0$, because $c_2 + c_5 s \leq c_1 s + c_2 s + c_3 s + 2c_4 s + 2c_5 s < 1$. That is, $f(v) = g(v) = u$ is a point of coincidence for the mappings f and g, i.e., (i) is proved in the case if $f(X)$ is b_{bl}-complete. The proof for the case if $g(X)$ is b_{bl}-complete is similar.

(ii) Assume that x and y are two different points of coincidence of f and g with $(x, y) \in E(G)$. This means that there are different points x_1 and y_1 from X such that: $f(x_1) = g(x_1) = x$ and $f(y_1) = g(y_1) = y$. Now, according to Condition (9) we get

$$\begin{aligned}
sb_{bl}(x, y) &\leq s^q b_{bl}(x, y) = s^q b_{bl}(f(x_1), f(y_1)) \\
&\leq c_1 b_{bl}(g(x_1), g(y_1)) + c_2 b_{bl}(g(x_1), f(y_1)) + c_3 b_{bl}(g(y_1), f(y_1)) \\
&\quad +c_4 b_{bl}(g(x_1), f(y_1)) + c_5 b_{bl}(g(y_1), f(x_1)) \\
&= c_1 b_{bl}(x, y) + c_2 b_{bl}(x, y) + c_3 b_{bl}(y, y) \\
&\quad +c_4 b_{bl}(x, y) + c_5 b_{bl}(y, x) \\
&\leq (c_1 + c_2 + 2c_3 s + c_4 + c_5)\, b_{bl}(y, x) \\
&\leq (c_1 s + 2c_2 s + 2c_3 s + c_4 s + c_5 s)\, b_{bl}(y, x) < b_{bl}(y, x). \qquad (17)
\end{aligned}$$

Hence, if $x \neq y$ we get a contradiction.

If f and g are weakly compatible, then by Proposition 2 f and g have a unique common fixed point. □

Example 1. *Let $X = [0, +\infty)$ and $f, g : X \to X$ be the mappings such that*

$$f(x) = e^x - 1 \quad \text{and} \quad g(x) = e^{4x} - 1.$$

Consider b-metric-like space (X, b_{bl}) under the distance $b_{bl}(x, y) = (x + y)^2$ with coefficient $s = 2$, and the graph $G = (V, E)$ with $V = X$ and $E = \{(x, x) : x \in X\} \cup \{(0, x) : x \in X\}$. Assume that $c_1 = \frac{1}{4}$ and $c_2 = c_3 = c_4 = c_5 = \frac{1}{25}$ for which Inequalities (10) and (11) hold. Note that $(g(x), g(y)) \in E$ if and only if $x = y, x \geq 0$ or $x = 0, y > 0$ or $y = 0, x > 0$. For $q = 2$ let us check whether Condition (9) holds in these cases.

Case 1: $x = y, x \geq 0$;

$$\begin{aligned} & c_1 b_{bl}(g(x), g(x)) + c_2 b_{bl}(g(x), f(x)) + c_3 b_{bl}(g(x), f(x)) + c_4 b_{bl}(g(x), f(x)) + c_5 b_{bl}(g(x), f(x)) \\ = \;& c_1 \left(e^{4x} - 1 + e^{4x} - 1\right)^2 + (c_2 + c_3 + c_4 + c_5) \left(e^{4x} - 1 + e^x - 1\right)^2 \\ = \;& 4c_1 \left(e^x - 1\right)^2 \left(e^{3x} + e^{2x} + e^x + 1\right)^2 + (c_2 + c_3 + c_4 + c_5) \left(e^x - 1\right)^2 \left(e^{3x} + e^{2x} + e^x + 2\right)^2 \\ \geq \;& 4c_1 \left(e^x - 1\right)^2 4^2 + (c_2 + c_3 + c_4 + c_5) \left(e^x - 1\right)^2 5^2 = \left(\frac{1}{4} \cdot 64 + \frac{4}{25} \cdot 25\right) \left(e^x - 1\right)^2 \\ > \;& 4 \left(e^x - 1 + e^x - 1\right)^2 = s^q b_{bl}(f(x), f(x)). \end{aligned}$$

Case 2: $x = 0, y > 0$ *(similarly for* $y = 0, x > 0$*)*;

$$\begin{aligned} & c_1 b_{bl}(g(0), g(y)) + c_2 b_{bl}(g(0), f(0)) + c_3 b_{bl}(g(y), f(y)) + c_4 b_{bl}(g(0), f(y)) + c_5 b_{bl}(g(y), f(0)) \\ = \;& c_1 \left(e^{4y} - 1\right)^2 + c_2 (0 + 0)^2 + c_3 \left(e^{4y} - 1 + e^y - 1\right)^2 + c_4 \left(e^y - 1\right)^2 + c_5 \left(e^{4y} - 1\right)^2 \\ = \;& (c_1 + c_5) \left(e^y - 1\right)^2 \left(e^{3y} + e^{2y} + e^y + 1\right)^2 + c_3 \left(e^y - 1\right)^2 \left(e^{3y} + e^{2y} + e^y + 2\right)^2 + c_4 \left(e^y - 1\right)^2 \\ > \;& (c_1 + c_5) \left(e^y - 1\right)^2 4^2 + c_3 \left(e^y - 1\right)^2 5^2 + c_4 \left(e^y - 1\right)^2 = \left(\frac{29}{100} \cdot 16 + \frac{1}{25} \cdot 25 + \frac{1}{25}\right) \left(e^y - 1\right)^2 \\ > \;& 4 \left(e^y - 1\right)^2 = s^q b_{bl}(f(0), f(y)). \end{aligned}$$

Hence, f and g satisfy Condition (9) for all $x, y \in X$ such that $(g(x), g(y)) \in E$.

Moreover, there is $x_1 = \frac{x_0}{4}$ such that $g(x_1) = f(x_0)$, $x_2 = \frac{x_0}{4^2}$ such that $g(x_2) = f(x_1)$, and so on. In this way, we can built the sequence $x_n = \frac{x_0}{4^n}, n \in \mathbb{N}$ such that $g(x_n) = f(x_{n-1})$. For $x_0 \neq 0$ it is clear that $(g(x_n), g(x_m)) \notin E$. For $x_0 = 0$, $x_n = 0, n \in \mathbb{N}$ is obtained. Thus, the constant sequence $x_n = 0$ is only convergent sequence such that $(g(x_n), g(x_m)) = (0, 0) \in E$, and for each subsequence $(g(x_{n_i}))_{i \in \mathbb{N}}$ of $(g(x_n))_{n \in \mathbb{N}}$ holds $(g(x_{n_i}), 0) = (0, 0) \in E$. This means that $x_0 \in G_{gf} \neq \emptyset$ and the pair (f, g) possesses the property $G_{f, g(x_n)}$.

It is obvious that $f(X) \subset g(X)$ and $g(X) = X$ is b_{bl}-complete. Since the mappings f and g are weakly compatible at $x = 0$ ($f(0) = g(0)$ implies $g(f(0)) = f(g(0))$), all conditions of Theorem 1 are satisfied. So, 0 is the unique common fixed point of mappings f and g in X.

Example 2. *Now consider the same b-metric-like space (X, b_{bl}) endowed with the graph G as in Example 1, and the mappings $f, g : X \to X$ such that*

$$f(x) = \begin{cases} e^x - 1\,, & x \neq 0 \\ \quad 1 & , x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} e^{4x} - 1\,, & x \neq 0 \\ \quad 2 & , x = 0 \end{cases}.$$

In this case we have $G_{gf} = \emptyset$. Namely, for $x_0 = 0$, $x_n = \frac{1}{4^n} \ln 2, n \in \mathbb{N}$ is now obtained, and $(g(x_n), g(x_m)) \notin E$. Hence, the conditions of Theorem 1 are not satisfied. Moreover, we can easily see that the mappings f and g have no coincidence point nor common fixed points.

As corollaries of our Theorem 1, we obtain the next results in the context of b-metric-like spaces endowed with a graph:

Corollary 1. *(Jungck) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl}\left(f\left(x\right), f\left(y\right)\right) \leq c_1 b_{bl}\left(g\left(x\right), g\left(y\right)\right) \tag{18}$$

for all $x, y \in X$ with $\left(g\left(x\right), g\left(y\right)\right) \in E\left(G\right)$ when $c_1 < \frac{1}{s}$. Suppose that $f\left(X\right) \subset g\left(X\right)$ and at least one of $f\left(X\right), g\left(X\right)$ is a b_{bl}-complete subspace of $\left(X, b_{bl}\right)$. Then

(i) If the property $G_{f, g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E\left(G\right)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Corollary 2. *(Kannan) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl}\left(f\left(x\right), f\left(y\right)\right) \leq c_2 b_{bl}\left(g\left(x\right), f\left(x\right)\right) + c_3 b_{bl}\left(g\left(y\right), f\left(y\right)\right) \tag{19}$$

for all $x, y \in X$ with $\left(g\left(x\right), g\left(y\right)\right) \in E\left(G\right)$ when

$$c_2 + c_3 < \frac{1}{2s}. \tag{20}$$

Suppose that $f\left(X\right) \subset g\left(X\right)$ and at least one of $f\left(X\right), g\left(X\right)$ is a b_{bl}-complete subspace of $\left(X, b_{bl}\right)$. Then

(i) If the property $G_{f,g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E(G)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Corollary 3. *(Chatterjea) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl}\left(f\left(x\right), f\left(y\right)\right) \leq c_4 b_{bl}\left(g\left(x\right), f\left(y\right)\right) + c_5 b_{bl}\left(g\left(y\right), f\left(x\right)\right), \tag{21}$$

for all $x, y \in X$ with $\left(g\left(x\right), g\left(y\right)\right) \in E\left(G\right)$ when

$$c_4 + c_5 < \frac{1}{2s}. \tag{22}$$

Suppose that $f\left(X\right) \subset g\left(X\right)$ and at least one of $f\left(X\right), g\left(X\right)$ is a b_{bl}-complete subspace of $\left(X, b_{bl}\right)$. Then

(i) If the property $G_{f,g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E(G)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Corollary 4. *(Reich) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl}\left(f\left(x\right), f\left(y\right)\right) \leq c_1 b_{bl}\left(g\left(x\right), g\left(y\right)\right) + c_2 b_{bl}\left(g\left(x\right), f\left(x\right)\right) + c_3 b_{bl}\left(g\left(y\right), f\left(y\right)\right) \tag{23}$$

for all $x, y \in X$ with $\left(g\left(x\right), g\left(y\right)\right) \in E\left(G\right)$ when

$$c_1 + 2c_2 + 2c_3 < \frac{1}{s} \tag{24}$$

Suppose that $f\left(X\right) \subset g\left(X\right)$ and at least one of $f\left(X\right), g\left(X\right)$ is a b_{bl}-complete subspace of $\left(X, b_{bl}\right)$. Then

(i) If the property $G_{f,g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E(G)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Now, we announce our last result in this section in the context of b-metric-like spaces endowed with the graph. The proof is similar enough with the corresponding proof of Theorem 1 and therefore we omit it.

Theorem 2. *(Das-Naik-Ćirić) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$\begin{aligned} s^q b_{bl}(f(x), f(y)) \quad &\leq \quad \lambda \max\{b_{bl}(g(x), g(y)), b_{bl}(g(x), f(x)), b_{bl}(g(y), f(y)), \\ &\qquad b_{bl}(g(x), f(y)), b_{bl}(g(y), f(x))\} \end{aligned} \tag{25}$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$ when $\lambda \in [0, \frac{1}{s})$. Suppose that $f(X) \subset g(X)$ and at least one of $f(X), g(X)$ is a b_{bl}-complete subspace of (X, b_{bl}). Then

(i) If the property $G_{f,g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E(G)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

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