mathematics

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Some Types of Identities Involving the Legendre Polynomials

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Received: 10 December 2018; Accepted: 20 January 2019; Published: 22 January 2019

Abstract: In this paper, a new non-linear recursive sequence is firstly introduced. Then, using this sequence, a computational problem involving the convolution of the Legendre polynomial is studied using the basic and combinatorial methods. Finally, we give an interesting identity.

Keywords: Legendre polynomials; recursive sequence; convolution sums; combinatorial method; identity; polynomial congruence

2010 Mathematics Subject Classification: Primary 11E10

1. Introduction

For any $\lambda > 0$, the generating function for the Gegenbauer polynomials $C_n^{\lambda}(x)$ of index λ is

$$\left(1 - 2xt + t^2\right)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x) \cdot t^n, \tag{1}$$

so that $C_n^{\frac{1}{2}}(x) = P_n(x)$, the Legendre polynomial. For example, the first several terms of $P_n(x)$ are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, and the second order non-linear recursive formula is

$$P_{n+1}(x) = \frac{2n+1}{n+1} \cdot xP_n(x) - \frac{n}{n+1} \cdot P_{n-1}(x),\ n \geq 1.$$

It is well known that these polynomials satisfy the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,\ (n = 0, 1, 2, \cdots),$$

with general expression

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\left[(x^2-1)^n\right] = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} \cdot x^{n-2k},\ n \geq 1.$$

The Legendre polynomials $P_n(x)$ are orthogonal polynomials (see [1]), and they play an important role in mathematical theory and application. Therefore, the polynomials $P_n(x)$ attract a large number of mathematical experts and mathematics enthusiasts to study their various properties, and get a series of interesting results. Some theoretical results are as in [2,3], especially the important works [4–6] of T. Kim et al., where they obtained a series of interesting identities involving the Legendre polynomials and their generalization. Some important applications of the Legendre polynomials can also be found in [7–11].

In this paper, we consider the computational problem of the convolution sums

$$C_n^{\frac{k}{2}}(x) = \sum_{a_1+a_2+\cdots+a_k=n} P_{a_1}(x)\cdot P_{a_2}(x)\cdots P_{a_{k-1}}(x)\cdot P_{a_k}(x), \tag{2}$$

where the summation is taken over all k-dimensional nonnegative integer coordinates $(a_1, a_2, \cdots, a_k)$, such that $a_1 + a_2 + \cdots + a_k = n$.

If $k = 2m$ is an even number, then, from the generating function (1) and the definition of the second kind Chebyshev polynomials $U_n(x)$ (see [12]), we have

$$\sum_{n=0}^{\infty} U_n(x)\cdot t^n = \left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^2 = \sum_{n=0}^{\infty}\left(\sum_{a+b=n} P_a(x)P_b(x)\right)\cdot t^n.$$

This time, (2) becomes the convolution sum of the second kind Chebyshev polynomials $U_n(x)$. Related results can be found in [12–17].

When $k = 2m+1$ is an odd number, Yalan Zhou and Xia Wang [18] studied the computational problem of (1), and they used the elementary method and some complex calculation to obtain three identities for (2) with $k = 3,\ 5,\ 7$. In this paper, as a comment on article [18], we will study this problem again and give an effective calculation of formula (1), by using the basic and combinatorial methods. For convenience, we use the Pochhammer symbols, defined by

$$(a)_0 = 1,\ (a)_{n+1} = (a)_n(a+n),\ (a)_n = \prod_{i=1}^{n}(a+i-1)\,.$$

Using this notation, we shall prove the following main result:

Theorem 1. *For any positive integer k and integer $n \geq 0$, we obtain the identity*

$$\begin{aligned}&(2k-1)!! \sum_{a_1+a_2+\cdots+a_{2k+1}=n} P_{a_1}(x)P_{a_2}(x)\cdots P_{a_k}(x)\\ =\;& \sum_{j=1}^{k} C(k,j)\sum_{i=0}^{n}\frac{(n+k+1-i-j)!}{(n-i)!}\cdot\frac{\binom{i+j+k-2}{i}}{x^{k-1+i+j}}\cdot P_{n+k+1-i-j}(x),\end{aligned}$$

where $(2k-1)!! = 1\times 3\times 5\cdots(2k-1) = 2^k\left(\frac{1}{2}\right)_k$, and $C(k,i)$ is a recurrence sequence defined by $C(k,1) = 1$, $C(k+1,k+1) = (2k-1)!!$, and $C(k+1,i+1) = C(k,i+1) + (k-1+i)\cdot C(k,i)$ for all $1 \leq i \leq k-1$.

This theorem represents a complex summation of Legendre polynomials as a linear combination of some $P_n(x)$, and the coefficients $C(k,i)$ are very regular. This is the greatest advantage of the main theorem. Moreover, the recursive formula for the $C(k,i)$ is easy to calculate.

Especially taking $k = 1$ and 2, from the main theorem we can immediately derive the following two corollaries:

Corollary 1. *For any positive integer $n \geq 1$, we have:*

$$\sum_{a+b+c=n} P_a(x)\cdot P_b(x)\cdot P_c(x) = \sum_{i=0}^{n}\frac{n+1-i}{x^{i+1}}\cdot P_{n+1-i}(x).$$

Corollary 2. *For any positive integer $n \geq 1$, we have:*

$$\sum_{a+b+c+d+e=n} P_a(x) \cdot P_b(x) \cdot P_c(x) \cdot P_d(x) \cdot P_e(x)$$
$$= \frac{1}{3} \cdot \sum_{i=0}^{n} \frac{(n+2-i)(n+1-i)(i+1)}{x^{2+i}} \cdot P_{n+2-i}(x)$$
$$+\frac{1}{6} \sum_{i=0}^{n} \frac{(n+1-i)(i+2)(i+1)}{x^{3+i}} \cdot P_{n+1-i}(x).$$

If taking $n = 0$ in theorem, we can also obtain the following two results.

Corollary 3. *For any positive integer k, we have the identity*

$$\sum_{j=1}^{k} C(k,j) \cdot \frac{(k+1-j)!}{x^{j-1}} \cdot P_{k+1-j}(x) = (2k-1)!! \cdot x^k.$$

Corollary 4. *For any positive integer k, we have the polynomial congruence*

$$\sum_{j=1}^{k} C(k,j) \cdot (k+1-j)! \cdot x^{k+1-j} \cdot P_{k+1-j}(x) \equiv 0 \bmod x^{2k}.$$

For clarity, we compute some values of $C(k,i)$ in the following Table 1.

Table 1. Values of $C(k,i)$.

$C(k,i)$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$k=1$	1								
$k=2$	1	1							
$k=3$	1	3	3						
$k=4$	1	6	15	15					
$k=5$	1	10	45	105	105				
$k=6$	1	15	105	420	945	945			
$k=7$	1	21	210	1260	4725	10,395	10,395		
$k=8$	1	28	378	3150	17,325	62,370	135,135	135,135	
$k=9$	1	36	630	6930	51,975	270,270	945,945	2,027,025	2,027,025

Using these data and mathematical induction we can easily verify that

$$C(n+2l,n) = 2^{n-1}\binom{n+l-1}{l}\left(\frac{1}{2}+l\right)_{n-1},$$

$$C(n+2l+1,n) = 2^{n-1}\binom{n+l-1}{l}\left(\frac{3}{2}+l\right)_{n-1}.$$

In terms of double factorials,

$$C(n+2l,n) = \binom{n+l-1}{l}\frac{(2n+2l-3)!!}{(2l-1)!!}$$

and

$$C(n+2l+1,n) = \binom{n+l-1}{l}\frac{(2n+2l-1)!!}{(2l+1)!!}.$$

From these formulae, we may immediately deduce the following interesting result:

Let p be an odd prime p. Then, for any positive integer i with $1 < i \leq p$, we have the congruence

$$C(p,i) \equiv 0 \bmod p.$$

2. Several Simple Lemmas

We give two simple lemmas in this part, which will be used to prove the theorem. We introduce the first lemma:

Lemma 1. *Let $f(t) = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}}$. Then, for any positive integer k, we have the identity*

$$(2k-1)!! \cdot f^{2k+1}(t) = \sum_{i=1}^{k} \frac{C(k,i)}{(x-t)^{k-1+i}} \cdot f^{(k+1-i)}(t),$$

where the definition of $C(k,i)$ is the same as in the theorem, and $f^{(h)}(t)$ denotes the h-order derivative of $f(t)$ with respect to t.

Proof. We use mathematical induction to prove the result. According to the definition and properties of the derivative, we have

$$f'(t) = (x-t) \cdot \left(1 - 2xt + t^2\right)^{-\frac{3}{2}} = (x-t) \cdot f^3(t)$$

and

$$f''(t) = -f^3(t) + 3(x-t)f^2(t)f'(t) = -f^3(t) + 3(x-t)^2 f^5(t),$$

or

$$f^3(t) = \frac{1}{x-t} \cdot f'(t) = \sum_{i=1}^{1} \frac{C(1,i)}{(x-t)^i} \cdot f^{(2-i)}(t) \tag{3}$$

and

$$3 \cdot f^5(t) = \frac{1}{(x-t)^2} \cdot f''(t) + \frac{1}{(x-t)^3} f'(t) = \sum_{i=1}^{2} \frac{C(2,i)}{(x-t)^{1+i}} \cdot f^{(3-i)}(t). \tag{4}$$

That is to say, when $k = 1$ and 2, the result of Lemma 1 is true. Suppose that Lemma 1 is true if $2 \leq k = h$. That is,

$$(2h-1)!! \cdot f^{2h+1}(t) = \sum_{i=1}^{h} \frac{C(h,i)}{(x-t)^{h-1+i}} \cdot f^{(h+1-i)}(t). \tag{5}$$

From (3), (5), and the definitions of $C(k,i)$, we obtain

$$\begin{aligned}
& (2h+1)!! \cdot f^{2h}(t) \cdot f'(t) = (2h+1)!!(x-t) \cdot f^{2h+3}(t) \\
= \ & \sum_{i=1}^{h} \frac{(h-1+i)C(h,i)}{(x-t)^{h+i}} \cdot f^{(h+1-i)}(t) + \sum_{i=1}^{h} \frac{C(h,i)}{(x-t)^{h-1+i}} \cdot f^{(h+2-i)}(t) \\
= \ & \frac{(2h-1)C(h,h)}{(x-t)^{2h}} \cdot f'(t) + \sum_{i=1}^{h-1} \frac{(h-1+i)C(h,i)}{(x-t)^{h+i}} \cdot f^{(h+1-i)}(t) \\
& + \frac{C(h,1)}{(x-t)^h} \cdot f^{(h+1)}(t) + \sum_{i=1}^{h-1} \frac{C(h,i+1)}{(x-t)^{h+i}} \cdot f^{(h+1-i)}(t)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(2h-1)C(h,h)}{(x-t)^{2h}} \cdot f'(t) + \frac{C(h,1)}{(x-t)^h} \cdot f^{(h+1)}(t) \\
&\quad + \sum_{i=1}^{h-1} \frac{C(h,i+1) + (h-1+i)C(h,i)}{(x-t)^{h+i}} \cdot f^{(h+1-i)}(t) \\
&= \frac{(2h-1)C(h,h)}{(x-t)^{2h}} \cdot f'(t) + \frac{C(h,1)}{(x-t)^h} \cdot f^{(h+1)}(t) + \sum_{i=1}^{h-1} \frac{C(h+1,i+1)}{(x-t)^{h+i}} \cdot f^{(h+1-i)}(t) \\
&= \frac{(2h-1)C(h,h)}{(x-t)^{2h}} \cdot f'(t) + \frac{C(h,1)}{(x-t)^h} \cdot f^{(h+1)}(t) + \sum_{i=2}^{h} \frac{C(h+1,i)}{(x-t)^{h-1+i}} \cdot f^{(h+2-i)}(t) \\
&= \sum_{i=1}^{h+1} \frac{C(h+1,i)}{(x-t)^{h-1+i}} \cdot f^{(h+2-i)}(t).
\end{aligned} \tag{6}
$$

From (6), we have

$$(2h+1)!! \cdot f^{2h+3}(t) = \sum_{i=1}^{h+1} \frac{C(h+1,i)}{(x-t)^{h+i}} \cdot f^{(h+2-i)}(t).$$

It is to say that Lemma 1 is also suitable for $k = h+1$. □

Lemma 2. *For any positive integers h and k, we have the power series expansion*

$$\frac{f^{(h)}(t)}{(x-t)^k} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{(n-i+h)!}{(n-i)!} \cdot \binom{i+k-1}{i} \cdot \frac{P_{n-i+h}(x)}{x^{i+k}} \right) t^n$$

for all $|t| < |x|$.

Proof. From the definition of $f(t)$, we have

$$f(t) = \sum_{n=0}^{\infty} P_n(x) \cdot t^n.$$

Then, for any positive integer h, utilizing the properties of the power series, we have

$$
\begin{aligned}
f^{(h)}(t) &= \sum_{n=0}^{\infty} (n+h)(n+h-1)\cdots(n+1) \cdot P_{n+h}(x) \cdot t^n \\
&= \sum_{n=0}^{\infty} \frac{(n+h)!}{n!} \cdot P_{n+h}(x) \cdot t^n.
\end{aligned} \tag{7}
$$

Similarly, for all positive integer $k > 1$ and all $|t| < |x|$, we also have

$$\frac{1}{x-t} = \frac{1}{x} \cdot \sum_{n=0}^{\infty} \frac{t^n}{x^n}$$

and

$$
\begin{aligned}
\frac{1}{(x-t)^k} &= \frac{1}{(k-1)! \cdot x^k} \cdot \sum_{n=0}^{\infty} (n+k-1)(n+k-2)\cdots(n+1) \cdot \frac{t^n}{x^n} \\
&= \frac{1}{x^k} \cdot \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot \frac{t^n}{x^n}.
\end{aligned} \tag{8}
$$

Applying (7), (8), and the properties of the power series, we have

$$\frac{f^{(h)}(t)}{(x-t)^k} = \frac{1}{x^k} \cdot \left(\sum_{n=0}^{\infty} \frac{(n+h)!}{n!} \cdot P_{n+h}(x) \cdot t^n \right) \left(\sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot \frac{t^n}{x^n} \right)$$
$$= \frac{1}{x^k} \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{(n-i+h)!}{(n-i)!} \cdot P_{n-i+h}(x) \cdot \binom{i+k-1}{i} \cdot \frac{1}{x^i} \right) t^n.$$

This proves Lemma 2. □

3. Proof of the Theorem

Now, we will complete the proof of our main result. According to Lemma 1 and the definition of $f(t)$, for any positive integer k, we have the following result from the properties of the power series

$$(2k-1)!!f^{2k+1}(t) = (2k-1)!! \left(\sum_{n=0}^{\infty} P_n(x) \cdot t^n \right)^{2k+1}$$
$$= (2k-1)!! \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_{2k+1}=n} P_{a_1}(x) P_{a_2}(x) \cdots P_{a_k}(x) \right) \cdot t^n. \tag{9}$$

On the other hand, from Lemma 2, we have

$$\sum_{i=1}^{k} \frac{C_k(i)}{(x-t)^{k-1+i}} \cdot f^{(k+1-i)}(t) = \sum_{j=1}^{k} C_k(j) \frac{f^{(k+1-j)}(t)}{(x-t)^{k-1+j}}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{j=1}^{k} C_k(j) \sum_{i=0}^{n} \frac{(n+k+1-i-j)! P_{n+k+1-i-j}(x)}{(n-i)!} \cdot \frac{\binom{i+j+k-2}{i}}{x^{k-1+i+j}} \right) t^n. \tag{10}$$

Then from (9), (10), Lemma 1, and comparing the coefficients of the power series, we have:

$$(2k-1)!! \sum_{a_1+a_2+\cdots+a_{2k+1}=n} P_{a_1}(x) P_{a_2}(x) \cdots P_{a_k}(x)$$
$$= \sum_{j=1}^{k} C(k,j) \sum_{i=0}^{n} \frac{(n+k+1-i-j)!}{(n-i)!} \cdot \frac{\binom{i+j+k-2}{i}}{x^{k-1+i+j}} \cdot P_{n+k+1-i-j}(x).$$

This completes the proof of our main theorem.

4. Conclusions

The main results of this paper include one theorem and four corollaries. The theorem gave an exact expression for the convolution sums (2) with any odd number $k = 2h + 1$. This result is meaningful. It not only reveals the close connection between the Legendre polynomials, but also makes a complex convolution sum (2) able to be expressed as a simple combination of some Legendre polynomials. Especially for $k = 3$ and 5, the corresponding results Corollary 1 and Corollary 2 are easy to understand. These works have good reference for further research on the classical Legendre polynomials and their generalization. In addition, the theorem also shows that the calculation of (2) can be realized by a computer.

Author Contributions: All authors contributed equally to this work.

Funding: This work is supported by the N.S.F. (11771351) of P. R. China, and Northwest University Graduate Innovation and Creativity Funds (YZZ17086).

Acknowledgments: The authors wish to express their gratitude to the reviewers for their helpful comments. In particular, it is pointed out that the specific representation of $C(h,i)$ and the direct verification of congruence are proposed by one of the reviewers.

Conflicts of Interest: The authors declare no conflict of interest.

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