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Global Asymptotical Stability Analysis for Fractional Neural Networks with Time-Varying Delays

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Abstract: In this paper, the global asymptotical stability of Riemann-Liouville fractional-order neural networks with time-varying delays is studied. By combining the Lyapunov functional function and LMI approach, some sufficient criteria that guarantee the global asymptotical stability of such fractional-order neural networks with both discrete time-varying delay and distributed time-varying delay are derived. The stability criteria is suitable for application and easy to be verified by software. Lastly, some numerical examples are presented to check the validity of the obtained results.

Keywords: time-varying delay; global asymptotical stability; fractional-order; neural networks; linear matrix inequality

1. Introduction

Fractional-order calculus has gained much attention in recent three decades because of its widespread application, such as engineering, diffusion equations, control science, biology, calorifics, and so on [1–5]. As an important branch of fractional-order calculus, stability has been studied by many scholars [6–15]. In [16,17], the Mittag-Leffler stability for fractional nonlinear equation has been discussed. In [18], finite time stability of fractional system has been investigated by delayed Mittag-Leffler type matrix function. Furthermore, in [19,20], the asymptotical stability of fractional systems has been analyzed by using Lyapunov functional method. Some new results were derived for the stability of fractional differential systems with distributed delays in [21,22].

In recent years, neural networks have been widely used in image processing, pattern recognition, associative memory, signal processing, and secure communication, etc. Therefore, the study of neural networks is a hot topic in the theory of fractional differential systems. Some researchers have focused on the research of the fractional neural networks, including stability [21–29]. In [26], the author discussed a delay-dependent condition of uniform stability for fractional neural networks. What is more, the existence and uniqueness of equilibrium solution for the system was proposed. In [27], some criteria for finite-time stability for fractional networks were proved. In [28], a Lyapunov function was established to demonstrate the asymptotical stability for neural networks.

Because of the finite speeds of switching and transmission of signals, almost every neural network has time-varying delays, and time-varying delays also affect the dynamic behavior of the neural networks. Therefore, in the analysis of neural networks, time-varying delays is inevitable. The time delays in neural networks has significant effect on the stability of the system. Some results have been obtained for the stability of fractional-order neural networks with constant time delay, but there is little research results on the stability of neural networks with time-varying delays. Motivated by the above,

this thesis consider the global asymptotical stability of Riemann-Liouville fractional neural networks with time-varying delays as follows:

$${}_{t_0}D_t^\alpha \psi(t) = -A\psi(t) + B\tilde{f}(\psi(t)) + C\tilde{g}(\psi(t - \delta(t))) + D \int_{t-\delta(t)}^t \tilde{h}(\psi(s))ds + I, \tag{1}$$

where $0 < \alpha < 1$, $\psi(t) = [\psi_1(t), \psi_2(t), \dots, \psi_n(t)]^T$ is the neuron state vector; $A = \text{diag}(a_1, a_2, \dots, a_n)$ is a positive diagonal matrix; $B, C, D \in R^{n \times n}$ stand for constant connection weights matrices; $I \in R^n$ denotes an input of neuron; $\tilde{f}(\psi) = [\tilde{f}_1(\psi_1), \tilde{f}_2(\psi_2), \dots, \tilde{f}_n(\psi_n)]^T$, $\tilde{g}(\psi) = [\tilde{g}_1(\psi_1), \tilde{g}_2(\psi_2), \dots, \tilde{g}_n(\psi_n)]^T$, $\tilde{h}(\psi) = [\tilde{h}_1(\psi_1), \tilde{h}_2(\psi_2), \dots, \tilde{h}_n(\psi_n)]^T$ are activation functions with $\tilde{f}(0) = \tilde{h}(0) = \tilde{g}(0) = 0$ and $\delta(t)$ is smooth time-varying delay which follows $0 \leq \delta(t) \leq \delta$, $\dot{\delta}(t) \leq k < 1$. The functions $\tilde{f}_i(\cdot)$, $\tilde{g}_i(\cdot)$, $\tilde{h}_i(\cdot)$, $i = 1, 2, \dots, n$, satisfy the following assumptions

$$(H_1) 0 \leq \frac{\tilde{f}_i(\phi_1) - \tilde{f}_i(\phi_2)}{\phi_1 - \phi_2} \leq l_i, 0 \leq \frac{\tilde{g}_i(\phi_1) - \tilde{g}_i(\phi_2)}{\phi_1 - \phi_2} \leq \sigma_i, 0 \leq \frac{\tilde{h}_i(\phi_1) - \tilde{h}_i(\phi_2)}{\phi_1 - \phi_2} \leq \lambda_i, \forall \phi_1, \phi_2 \in R, \phi_1 \neq \phi_2,$$

where $l_i > 0, \sigma_i > 0, \lambda_i > 0, i = 1, 2, \dots, n$.

Let $\psi^* = (\psi_1^*, \psi_2^*, \dots, \psi_n^*)$ is one equilibrium point of system (1) and shift the equilibrium point $\psi^* = (\psi_1^*, \psi_2^*, \dots, \psi_n^*)$ of system (1) to the origin by the transformation $\varphi(t) = \psi(t) - \psi^*$, then system (1) converts into

$${}_{t_0}D_t^\alpha \varphi(t) = -A\varphi(t) + Bf(\varphi(t)) + Cg(\varphi(t - \delta(t))) + D \int_{t-\delta(t)}^t h(\varphi(s))ds, \tag{2}$$

where $\varphi(t) = [\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)]$ is state vector, and $f(\varphi(t)) = \tilde{f}(\psi(t)) - \tilde{f}(\psi^*)$, $g(\varphi(t)) = \tilde{g}(\psi(t)) - \tilde{g}(\psi^*)$, $h(\varphi(t)) = \tilde{h}(\psi(t)) - \tilde{h}(\psi^*)$.

2. Preliminaries

In this section, the definitions of Riemann-Liouville fractional integral and fractional derivative are described and some related lemmas are presented.

Definition 1. The definition of Riemann-Liouville fractional integral is defined by

$${}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s)ds, \quad (\alpha > 0). \tag{3}$$

Definition 2. The definition of Riemann-Liouville fractional derivative can be written as

$${}_{t_0}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{t_0}^t (t-s)^{n-\alpha-1} f(s)ds, \quad (n-1 \leq \alpha < n), \tag{4}$$

where $\Gamma(\cdot)$ is the Gamma function.

Lemma 1 ([1]). For any $\alpha > \beta > 0$, the following formulas holds if $\varphi(t) \in C^1[t_0, b]$,

$${}_{t_0}D_t^\alpha ({}_{t_0}D_t^{-\beta} \varphi(t)) = {}_{t_0}D_t^{\alpha-\beta} \varphi(t).$$

Lemma 2 ([20]). If $\varphi(t) \in R^n$ is a vector of differential function, then such inequality holds

$$\frac{1}{2} {}_{t_0}D_t^\alpha (\varphi^T(t)P\varphi(t)) \leq \varphi^T(t)P {}_{t_0}D_t^\alpha \varphi(t), \quad (\forall 0 < \alpha < 1, t \geq t_0),$$

where $P \in R^{n \times n}$ is a positive semi-definite and constant matrix, and satisfies $P^T = P$.

Lemma 3. For any $x, y \in R^n, \epsilon > 0$, relationship $2x^T y \leq \epsilon x^T x + \frac{1}{\epsilon} y^T y$ holds.

Lemma 4 ([30]). For any square matrix $K \in R^{n \times n}, K = K^T$, scalar $0 < b_m \leq b_1(t) < b_2(t) \leq b_M$, vector function: $\mu : [b_m, b_M] \rightarrow R^n$ such that the integrations concerned are well defined, then

$$(b_2(t) - b_1(t)) \int_{b_1(t)}^{b_2(t)} \mu^T(\xi) K \mu(\xi) d\xi \geq \left(\int_{b_1(t)}^{b_2(t)} \mu(\xi) d\xi \right)^T K \left(\int_{b_1(t)}^{b_2(t)} \mu(\xi) d\xi \right). \tag{5}$$

Lemma 5 ([31]). For a given matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

where $A_{11} = A_{11}^T, A_{22} = A_{22}^T$, the following three conditions are equivalent:

- (a) $A > 0$;
- (b) $A_{22} > 0, A_{11} - A_{12} A_{22}^{-1} A_{12}^T > 0$;
- (c) $A_{11} > 0, A_{22} - A_{12} A_{11}^{-1} A_{12}^T > 0$.

3. Main Results

In this subsection, we discuss the stability of fractional neural networks with time-varying delays, and some criteria on asymptotical stability are presented.

Theorem 1. Let the following conditions hold:

- 1.1. The conditions H_1 hold.
- 1.2. There exist positive definite diagonal matrices G, H, Q , and a positive definite matrix $P, S = 2PA - \Sigma_1 Q \Sigma_1 - m \Sigma_2 G \Sigma_2 - \delta^{n+1} \Sigma_3 H \Sigma_3, \Sigma_1 = \text{diag}(l_1, l_2, \dots, l_n), \Sigma_2 = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n), \Sigma_3 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and constants $m > 0, n > 0$ such that the inequality

$$\begin{pmatrix} S & B^T P & \eta_1 C^T P & \eta_2 D^T P \\ PB & Q & 0 & 0 \\ \eta_1 PC & 0 & G & 0 \\ \eta_2 PD & 0 & 0 & H \end{pmatrix} > 0, \tag{6}$$

holds, where $\eta_1 = \sqrt{m^{-1}(1-k)^{-1}}, \eta_2 = \sqrt{\delta^{1-n}(1-k)^{-1}}$.

Then the zero solution of system (2) is globally asymptotically stable.

Proof. Consider the following Lyapunov function

$$V(t) = {}_{t_0} D_t^{\alpha-1} \varphi^T(t) P \varphi(t) + m \int_{t-\delta(t)}^t g^T(\varphi(s)) G g(\varphi(s)) ds + \delta^n \int_{t-\delta(t)}^t \int_{\theta}^t h^T(\varphi(s)) H h(\varphi(s)) ds d\theta. \tag{7}$$

Based on Lemmas 1 and 2, calculations of the derivative of $V(t)$ along the trajectories of system (2), one gets

$$\begin{aligned}
 \dot{V}(t) &= {}_{t_0}D_t^\alpha \varphi^T(t)P\varphi(t) + m g^T(\varphi(t))Gg(\varphi(t)) - m(1 - \delta(t))g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))) \\
 &\quad + \delta^n \frac{d}{dt} \left[\int_{t-\delta(t)}^t \rho(t, \theta) d\theta \right] \\
 &\leq 2\varphi^T(t)P {}_{t_0}D_t^\alpha \varphi(t) + m g^T(\varphi(t))Gg(\varphi(t)) - m(1 - k)g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))) \\
 &\quad + \delta^n \frac{d}{dt} \left[\int_{t-\delta(t)}^t \rho(t, \theta) d\theta \right] \\
 &= 2\varphi^T(t)P[-A\varphi(t) + Bf(\varphi(t)) + Cg(\varphi(t - \delta(t))) + D \int_{t-\delta(t)}^t h(\varphi(s)) ds] \\
 &\quad + m g^T(\varphi(t))Gg(\varphi(t)) - m(1 - k)g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))) \\
 &\quad + \delta^n \frac{d}{dt} \left[\int_{t-\delta(t)}^t \rho(t, \theta) d\theta \right] \\
 &\leq 2\varphi^T(t)(-PA)\varphi(t) + 2\varphi^T(t)PBf(\varphi(t)) + 2\varphi^T(t)PCg(\varphi(t - \delta(t))) \\
 &\quad + 2\varphi^T(t)PD \int_{t-\delta(t)}^t h(\varphi(s)) ds + m g^T(\varphi(t))Gg(\varphi(t)) \\
 &\quad - m(1 - k)g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))) \\
 &\quad + \delta^n [\rho(t, t) - (1 - \delta(t))\rho(t, t - \delta(t)) + \int_{t-\delta(t)}^t \dot{\rho}_t(t, \theta) d\theta], \tag{8}
 \end{aligned}$$

where $\rho(t, \theta) = \int_\theta^t h^T(\varphi(s))Gh(\varphi(s)) ds$.

From Lemma 3, for any positive definite diagonal matrices Q , we have

$$\begin{aligned}
 2\varphi^T(t)PBf(\varphi(t)) &= 2\varphi^T(t)PBQ^{-\frac{1}{2}}Q^{\frac{1}{2}}f(\varphi(t)) \\
 &\leq \varphi^T(t)PBQ^{-1}B^T P\varphi(t) + f^T(\varphi(t))Qf(\varphi(t)), \tag{9}
 \end{aligned}$$

$$\begin{aligned}
 2\varphi^T(t)PCg(\varphi(t - \delta(t))) &= 2\varphi^T(t)PCG^{-\frac{1}{2}}G^{\frac{1}{2}}g(\varphi(t - \delta(t))) \\
 &\leq \frac{1}{m(1-k)}\varphi^T(t)PCG^{-1}C^T P\varphi(t) \\
 &\quad + m(1 - k)g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))), \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 2\varphi^T(t)PD \int_{t-\delta(t)}^t h(\varphi(s)) ds &= 2\varphi^T(t)PDH^{-\frac{1}{2}}H^{\frac{1}{2}} \int_{t-\delta(t)}^t h(\varphi(s)) ds \\
 &\leq \frac{1}{\delta^{n-1}(1-k)}\varphi^T(t)PDH^{-1}D^T P\varphi(t) \\
 &\quad + \delta^{n-1}(1 - k) \left(\int_{t-\delta(t)}^t h(\varphi(s)) ds \right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s)) ds \right). \tag{11}
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \dot{V}(t) &\leq \varphi^T(t)[-2PA + PBQ^{-1}B^T P + \frac{1}{m(1 - k)}PCG^{-1}C^T P \\
 &\quad + \frac{1}{\delta^{n-1}(1 - k)}PDH^{-1}D^T P]\varphi(t) + f^T(\varphi(t))Qf(\varphi(t)) + m g^T(\varphi(t))Gg(\varphi(t)) \\
 &\quad + \delta^{n-1}(1 - k) \left(\int_{t-\delta(t)}^t h(\varphi(s)) ds \right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s)) ds \right) \\
 &\quad + \delta^n [0 - (1 - k) \int_{t-\delta(t)}^t h^T(\varphi(s))Hh(\varphi(s)) ds + \delta(t)h^T(\varphi(t))Hh(\varphi(t))].
 \end{aligned}$$

Furthermore, from Lemma 4, we get the following inequality

$$\delta(t) \int_{t-\delta(t)}^t h^T(\varphi(s))Hh(\varphi(s))ds \geq \left(\int_{t-\delta(t)}^t h(\varphi(s))ds\right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s))ds\right),$$

and then we have

$$-\int_{t-\delta(t)}^t h^T(\varphi(s))Hh(\varphi(s))ds \leq -\delta^{-1} \left(\int_{t-\delta(t)}^t h(\varphi(s))ds\right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s))ds\right). \tag{12}$$

From the above provided analysis, we get

$$\begin{aligned} \dot{V}(t) &\leq \varphi^T(t)[-2PA + PBQ^{-1}B^T P + \frac{1}{m(1-k)}PCG^{-1}C^T P \\ &\quad + \frac{1}{\delta^{n-1}(1-k)}PDH^{-1}D^T P]\varphi(t) + f^T(\varphi(t))Qf(\varphi(t)) + m g^T(\varphi(t))Gg(\varphi(t)) \\ &\quad + \delta^{n-1}(1-k) \left(\int_{t-\delta(t)}^t h(\varphi(s))ds\right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s))ds\right) \\ &\quad - \delta^{n-1}(1-k) \left(\int_{t-\delta(t)}^t h(\varphi(s))ds\right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s))ds\right) + \delta^{n+1}h^T(\varphi(t))Hh(\varphi(t)) \tag{13} \\ &\leq \varphi^T(t)[-2PA + PBQ^{-1}B^T P + \frac{1}{m(1-k)}PCG^{-1}C^T P + \frac{1}{\delta^{n-1}(1-k)}PDH^{-1}D^T P \\ &\quad + \Sigma_1 Q \Sigma_1 + m \Sigma_2 G \Sigma_2 + \delta^{n+1} \Sigma_3 H \Sigma_3]\varphi(t) \\ &= -\varphi^T(t)\Lambda\varphi(t), \end{aligned}$$

where $\Lambda = 2PA - PBQ^{-1}B^T P - \frac{1}{m(1-k)}PCG^{-1}C^T P - \frac{1}{\delta^{n-1}(1-k)}PDH^{-1}D^T P - \Sigma_1 Q \Sigma_1 - m \Sigma_2 G \Sigma_2 - \delta^{n+1} \Sigma_3 H \Sigma_3$. According to Lemma 5 and the inequality (6), one get $\Lambda > 0$. So $\dot{V}(t) < 0$. The proof is completed. \square

From the hypothesis of Theorem 1, we know that the matrices Q, G and H are assumed to be positive diagonal matrices. In fact, we can decrease the conservatism of this assumption via some inequality techniques. In the next statement, we also choose the same Lyapunov function that has been constructed in Theorem 1.

Theorem 2. *Let the following conditions hold:*

2.1. *The conditions H_1 hold.*

2.2. *There exist positive definite matrix P, G, H and positive definite diagonal matrices $Q_i = \text{diag}(q_1^i, q_2^i, \dots, q_n^i), i = 1, 2, 3, \Sigma_1 = \text{diag}(l_1, l_2, \dots, l_n), \Sigma_2 = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n), \Sigma_3 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and constants $m > 0, n > 0$ such that the inequality*

$$\Omega = \begin{pmatrix} 2PA - 2\Sigma_1 Q_1 \Sigma_1 & -B^T P & -Q_2 \Sigma_2 & -Q_3 \Sigma_3 & -C^T P & -D^T P \\ -PB & 2Q_1 & 0 & 0 & 0 & 0 \\ -Q_2 \Sigma_2 & 0 & 2Q_2 - mG & 0 & 0 & 0 \\ -Q_3 \Sigma_3 & 0 & 0 & 2Q_3 - \delta^{n+1}H & 0 & 0 \\ -PC & 0 & 0 & 0 & \gamma_1 G & 0 \\ -PD & 0 & 0 & 0 & 0 & \gamma_2 H \end{pmatrix} > 0, \tag{14}$$

holds, where $\gamma_1 = m(1 - k), \gamma_2 = \delta^{n-1}(1 - k)$.

Then the zero solution of system (2) is globally asymptotically stable.

Proof. From Equations (7) and (8), we get the derivative of $V(t)$ along the trajectories of system (2) as follows:

$$\begin{aligned} \dot{V}(t) &\leq 2\varphi^T(t)(-PA)\varphi(t) + 2\varphi^T(t)PBf(\varphi(t)) + 2\varphi^T(t)PCg(\varphi(t - \delta(t))) + 2\varphi^T(t)PD \int_{t-\delta(t)}^t h(\varphi(s))ds \\ &\quad + m g^T(\varphi(t))Gg(\varphi(t)) - m(1-k)g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))) \\ &\quad - \delta^n(1-k) \int_{t-\delta(t)}^t h^T(\varphi(s))Hh(\varphi(s))ds + \delta^{n+1}h^T(\varphi(t))Hh(\varphi(t)) \\ &\leq 2\varphi^T(t)(-PA)\varphi(t) + 2\varphi^T(t)PBf(\varphi(t)) + 2\varphi^T(t)PCg(\varphi(t - \delta(t))) + 2\varphi^T(t)PD \int_{t-\delta(t)}^t h(\varphi(s))ds \\ &\quad + m g^T(\varphi(t))Gg(\varphi(t)) - m(1-k)g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))) \\ &\quad - \delta^{n-1}(1-k) \left(\int_{t-\delta(t)}^t h(\varphi(s))ds \right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s))ds \right) + \delta^{n+1}h^T(\varphi(t))Hh(\varphi(t)) \\ &\quad - 2 \sum_{i=1}^n q_i^1 f_i(\varphi_i)(f_i(\varphi_i) - l_i\varphi_i) - 2 \sum_{i=1}^n q_i^2 g_i(\varphi_i)(g_i(\varphi_i) - \sigma_i\varphi_i) - 2 \sum_{i=1}^n q_i^3 h_i(\varphi_i)(h_i(\varphi_i) - \lambda_i\varphi_i) \\ &\leq 2\varphi^T(t)(-PA + \Sigma_1 Q_1 \Sigma_1)\varphi(t) + 2\varphi^T(t)PBf(\varphi(t)) + 2\varphi^T(t)PCg(\varphi(t - \delta(t))) \\ &\quad + 2\varphi^T(t)PD \int_{t-\delta(t)}^t h(\varphi(s))ds + m g^T(\varphi(t))Gg(\varphi(t)) - m(1-k)g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))) \\ &\quad - \delta^{n-1}(1-k) \left(\int_{t-\delta(t)}^t h(\varphi(s))ds \right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s))ds \right) + \delta^{n+1}h^T(\varphi(t))Hh(\varphi(t)) \\ &\quad - 2f^T(\varphi(t))Q_1 f(\varphi(t)) - 2g^T(\varphi(t))Q_2 g(\varphi(t)) + 2g^T(\varphi(t))Q_2 \Sigma_2 \varphi(t) \\ &\quad - 2h^T(\varphi(t))Q_3 h(\varphi(t)) + 2h^T(\varphi(t))Q_3 \Sigma_3 \varphi(t) \\ &= 2\varphi^T(t)(-PA + \Sigma_1 Q_1 \Sigma_1)\varphi(t) + 2\varphi^T(t)PBf(\varphi(t)) + 2\varphi^T(t)PCg(\varphi(t - \delta(t))) \\ &\quad + 2\varphi^T(t)PD \int_{t-\delta(t)}^t h(\varphi(s))ds + 2g^T(\varphi(t))Q_2 \Sigma_2 \varphi(t) + 2h^T(\varphi(t))Q_3 \Sigma_3 \varphi(t) - 2f^T(\varphi(t))Q_1 f(\varphi(t)) \\ &\quad + g^T(\varphi(t))(mG - 2Q_2)g(\varphi(t)) + h^T(\varphi(t))(\delta^{n+1}H - 2Q_3)h(\varphi(t)) \\ &\quad - m(1-k)g^T(\varphi(t - \delta(t)))Gg(\varphi(t - \delta(t))) - \delta^{n-1}(1-k) \left(\int_{t-\delta(t)}^t h(\varphi(s))ds \right)^T H \left(\int_{t-\delta(t)}^t h(\varphi(s))ds \right) \\ &= -\zeta^T(t)\Omega\zeta(t), \end{aligned}$$

where $\zeta^T(t) = [\varphi^T(t), f^T(\varphi(t)), g^T(\varphi(t)), h^T(\varphi(t)), g^T(\varphi(t - \delta(t))), (\int_{t-\delta(t)}^t h(\varphi(s))ds)^T]$. So, $\dot{V}(t) < 0$ for all $\varphi(t) \neq 0$ because $\Omega > 0$ and $\dot{V}(t) = 0$ if and only if all components of $\zeta^T(t)$ equal to zero. But then, $V(t)$ is radially unbounded when $\|\varphi(t)\| \rightarrow \infty$. This implies that the zero solution of system (2) is globally asymptotically stable. The proof is completed. \square

4. Illustrative Examples

In the following subsection, we present two simple examples to check the usefulness of the results by LMI method.

Example 1. Considering the fractional neural networks as

$${}_t_0 D_t^\alpha \psi(t) = -A\psi(t) + B\tilde{f}(\psi(t)) + C\tilde{g}(\psi(t - \delta(t))) + D \int_{t-\delta(t)}^t \tilde{h}(\psi(s))ds + I, \tag{15}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}, C = \begin{pmatrix} 0.5 & -0.1 \\ -0.1 & 0.3 \end{pmatrix}, D = \begin{pmatrix} 0.2 & -0.3 \\ 0 & -0.3 \end{pmatrix}, I = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and activation functions $\tilde{f}_i(\psi_i) = \tilde{g}_i(\psi_i) = \tilde{h}_i(\psi_i) = \tanh(\psi_i), i = 1, 2, \delta(t) = \frac{1}{2}\sin t, m = n = 1, 0 < \alpha < 1$. Then the LMI (6) has the following feasible solution

$$P = \begin{pmatrix} 1.0793 & -0.0878 \\ -0.0878 & 1.2253 \end{pmatrix}, Q = \begin{pmatrix} 0.5145 & 0 \\ 0 & 0.6298 \end{pmatrix},$$

$$G = \begin{pmatrix} 0.6639 & 0 \\ 0 & 0.6401 \end{pmatrix}, H = \begin{pmatrix} 0.7764 & 0 \\ 0 & 0.8057 \end{pmatrix},$$

which implies that the zero solution of the system (15) is globally asymptotically stable.

Example 2. Considering the fractional neural networks as

$${}_{t_0}D_t^\alpha \psi(t) = -A\psi(t) + B\tilde{f}(\psi(t)) + C\tilde{g}(\psi(t - \delta(t))) + D \int_{t-\delta(t)}^t \tilde{h}(\psi(s))ds + I, \quad (16)$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}, C = \begin{pmatrix} 0.5 & -0.1 \\ -0.1 & 0.3 \end{pmatrix}, D = \begin{pmatrix} 0.2 & -0.3 \\ 0 & -0.3 \end{pmatrix}, I = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and activation functions $\tilde{f}_i(\psi_i) = \tilde{g}_i(\psi_i) = \tilde{h}_i(\psi_i) = \tanh(\psi_i), i = 1, 2, \delta(t) = \frac{1}{2}\sin t, m = n = 1, 0 < \alpha < 1$. Then the LMI (14) has the following feasible solution

$$P = \begin{pmatrix} 0.8384 & -0.0006 \\ -0.0006 & 0.8386 \end{pmatrix}, G = \begin{pmatrix} 0.4505 & -0.0001 \\ -0.0001 & 0.4501 \end{pmatrix}, H = \begin{pmatrix} 0.7037 & -0.0000 \\ -0.0000 & 0.7012 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0.3006 & 0 \\ 0 & 0.3072 \end{pmatrix}, Q_2 = \begin{pmatrix} 0.4577 & 0 \\ 0 & 0.4587 \end{pmatrix}, Q_3 = \begin{pmatrix} 0.3560 & 0 \\ 0 & 0.3564 \end{pmatrix},$$

which implies that the zero solution of the system (16) is globally asymptotically stable.

5. Conclusions

This paper has analyzed the global asymptotical stability for the fractional time-varying delay neural networks with the Riemann-Liouville derivative. Two sufficient conditions on global asymptotical stability are given by the Lyapunov functional functions method and linear matrix inequality techniques. The conditions of the two theorems in this paper are different. In Theorem 1, the conclusion requires that G and H are positive definite diagonal matrices, while in Theorem 2, the conditions of the Theorem require G and H to be positive definite matrices, without requiring them to be diagonal. The conditions of Theorem 2 are weakened. The criterion obtained in this paper can be processed by computer program without any other calculation, which makes it easy to verify the condition. Finally, we have shown two simple examples to illustrate the usefulness of this method.

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