

## Article

# Existence and Stability Results for a Fractional Order Differential Equation with Non-Conjugate Riemann-Stieltjes Integro-Multipoint Boundary Conditions

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**Abstract:** We discuss the existence and uniqueness of solutions for a Caputo-type fractional order boundary value problem equipped with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions on an arbitrary domain. Modern tools of functional analysis are applied to obtain the main results. Examples are constructed for the illustration of the derived results. We also investigate different kinds of Ulam stability, such as Ulam-Hyers stability, generalized Ulam-Hyers stability, and Ulam-Hyers-Rassias stability for the problem at hand.

**Keywords:** Caputo fractional derivative; nonlocal; integro-multipoint boundary conditions; existence; uniqueness; Ulam-Hyers stability

**MSC:** 34A08; 34B10; 34B15

## 1. Introduction

Fractional calculus played a pivotal role in improving the mathematical modeling of many real-world problems. The extensive application of fractional order (differential and integral) operators indeed reflects the popularity of this branch of mathematical analysis. In contrast to the integer order operators, such operators are nonlocal in nature and do have the capacity to trace the history of the phenomenon under investigation. A detailed account of the use of fractional calculus tools can be found in several scientific disciplines such as, chaos and fractional dynamics [1], evolution in honeycomb lattice via fractional Schrödinger equation [2], financial economics [3], ecology [4], bio-engineering [5], etc. For theoretical development and further application of the topic, see the texts [6–9].

During the past two decades, the study of fractional order boundary value problems has been one of the hot topics of scientific research. Several researchers contributed to the development of this class of problems by producing a huge number of articles, special issues, monographs, etc. Now the literature on the topic contains a variety of existence and uniqueness results, and analytic and numerical methods of solutions for these problems. In particular, there has been shown a great interest in the formulation and investigation of fractional order boundary value problems involving non-classical (nonlocal and integral) boundary conditions. The nonlocal boundary conditions are found to be of great utility in modeling the changes happening within the domain of the given scientific phenomena, while the concept of integral boundary conditions is applied to model the physical problems, such as

blood flow problems on arbitrary structures and ill-posed backward problems. For some recent works on fractional order differential equations involving Riemann-Liouville, Caputo, and Hadamard type fractional derivatives, equipped with classical, nonlocal, and integral boundary conditions, we refer the reader to a series of papers [10–28] and the references cited therein.

In this paper, we study the existence of solutions for a nonlinear Liouville-Caputo-type fractional differential equation on an arbitrary domain:

$${}^c D^q x(t) = f(t, x(t)), \quad 3 < q \leq 4, \quad t \in [a, b], \quad (1)$$

supplemented with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions of the form:

$$x(a) = \sum_{i=1}^{n-2} \alpha_i x(\eta_i) + \int_a^b x(s) dA(s), \quad x'(a) = 0, \quad x(b) = 0, \quad x'(b) = 0, \quad (2)$$

where  ${}^c D^q$  denotes the Caputo fractional derivative of order  $q$ ,  $a < \eta_1 < \eta_2 < \dots < \eta_{n-2} < b$ ,  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is a given continuous function,  $A$  is a function of bounded variation, and  $\alpha_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n-2$ .

The main emphasis in the present work is to introduce non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions and develop the existence theory for a Caputo-type fractional order boundary value problem equipped with these conditions on an arbitrary domain. Conjugate conditions on the body/fluid interface provide continuity of the thermal fields by specifying the equalities of temperatures and heat fluxes of a body and a flow at the vicinity of interface. The results obtained in this paper may have potential applications in diffraction-free and self-healing optoelectronic devices. Moreover, propagation properties for fractional Schrödinger equation similar to our results are well known theoretically [29].

The rest of the paper is organized as follows. An auxiliary result related to the linear variant of the problems (1) and (2), which plays a key role in the forthcoming analysis, is presented in Section 2. Some basic ideas of fractional calculus are also given in this section. In Section 3, we obtain some existence results for the given problem, while Section 4 contains a uniqueness result for the problem at hand. Ulam stability of different kinds for the problem (1) and (2) is studied in Section 5.

## 2. Preliminary Material

We begin this section with some basic definitions of fractional calculus [6]. Later we prove an auxiliary lemma, which plays a key role in defining a fixed-point problem associated with the given problem.

**Definition 1.** Let  $g$  be a locally integrable real-valued function on  $-\infty \leq a < t < b \leq +\infty$ . The Riemann-Liouville fractional integral  $I_a^p$  of order  $p \in \mathbb{R}$  ( $p > 0$ ) for the function  $g$  is defined as

$$I_a^p g(t) = (g * K_p)(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} g(s) ds,$$

where  $K_p(t) = \frac{t^{p-1}}{\Gamma(p)}$ ,  $\Gamma$  denotes the Euler gamma function.

**Definition 2.** The Caputo derivative of fractional order  $p$  for an  $(m-1)$ -times absolutely continuous function  $g : [a, \infty) \rightarrow \mathbb{R}$  is defined as

$${}^c D^p g(t) = \frac{1}{\Gamma(m-p)} \int_a^t (t-\bar{t})^{m-p-1} g^{(m)}(\bar{t}) d\bar{t}, \quad m-1 < p \leq m, \quad m = [p] + 1,$$

where  $[p]$  denotes the integer part of the real number  $p$ .

**Lemma 1.** [6] The general solution of the fractional differential equation  ${}^c D^q x(t) = 0$ ,  $m - 1 < q < m$ ,  $t \in [a, b]$  is

$$x(t) = \omega_0 + \omega_1(t-a) + \omega_2(t-a)^2 + \dots + \omega_{m-1}(t-a)^{m-1},$$

where  $\omega_i \in \mathbb{R}$ ,  $i = 0, 1, \dots, m-1$ . Furthermore,

$$I^q {}^c D^q x(t) = x(t) + \sum_{i=0}^{m-1} \omega_i(t-a)^i.$$

**Lemma 2.** Let

$$\gamma_1 = \frac{-A_1(b-a)^2}{3} + \frac{2A_3}{3(b-a)} - A_2 \neq 0. \quad (3)$$

For  $\hat{f} \in C([a, b], \mathbb{R})$ , the unique solution of the linear equation

$${}^c D^q x(t) = \hat{f}(t), \quad 3 < q \leq 4, \quad t \in [a, b], \quad (4)$$

supplemented with the boundary conditions (2) is given by

$$\begin{aligned} x(t) = & \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} \hat{f}(s) ds - g_1(t) \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} \hat{f}(s) ds - g_2(t) \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} \hat{f}(s) ds \\ & + g_3(t) \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} \hat{f}(u) du \right) dA(s), \end{aligned} \quad (5)$$

where

$$\begin{aligned} g_1(t) &= \lambda_1 - \frac{(t-a)^2 A_1}{\gamma_1} + (t-a)^3 \lambda_4, \quad g_2(t) = \lambda_2 + \frac{(t-a)^2 \gamma_2}{\gamma_1} + (t-a)^3 \lambda_5, \\ g_3(t) &= \lambda_3 + \frac{(t-a)^2}{\gamma_1} + (t-a)^3 \lambda_6, \end{aligned} \quad (6)$$

$$\gamma_2 = \frac{A_3 + (b-a)^3 A_1}{3(b-a)^2}, \quad (7)$$

$$\begin{aligned} \lambda_1 &= 1 + \frac{(b-a)^2 A_1}{3\gamma_1}, \quad \lambda_2 = \frac{-(b-a)}{3} - \frac{(b-a)^2 \gamma_2}{3\gamma_1}, \quad \lambda_3 = -\frac{(b-a)^2}{3\gamma_1}, \\ \lambda_4 &= \frac{2A_1}{3(b-a)\gamma_1}, \quad \lambda_5 = \frac{\gamma_1 - 2(b-a)\gamma_2}{3(b-a)^2\gamma_1}, \quad \lambda_6 = \frac{-2}{3(b-a)\gamma_1}, \end{aligned} \quad (8)$$

$$\begin{aligned} A_1 &= 1 - \sum_{i=1}^{n-2} \alpha_i - \int_a^b dA(s), \quad A_2 = \sum_{i=1}^{n-2} \alpha_i (\eta_i - a)^2 + \int_a^b (s-a)^2 dA(s), \\ A_3 &= \sum_{i=1}^{n-2} \alpha_i (\eta_i - a)^3 + \int_a^b (s-a)^3 dA(s). \end{aligned} \quad (9)$$

**Proof.** Applying the integral operator  $I^q$  to both sides of (4) and using Lemma 1, we get

$$x(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} \hat{f}(s) ds + c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3, \quad (10)$$

where  $c_i \in \mathbb{R}$ ,  $i = 0, 1, 2, 3$  are unknown arbitrary constants. Differentiating (10) with respect to  $t$ , we have

$$x'(t) = \int_a^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \hat{f}(s) ds + c_1 + 2c_2(t-a) + 3c_3(t-a)^2. \quad (11)$$

Using the boundary conditions (2) in (10) and (11), we obtain  $c_1 = 0$  and

$$c_0 + (b-a)^2 c_2 + (b-a)^3 c_3 = I_1, \quad (12)$$

$$2(b-a)c_2 + 3(b-a)^2c_3 = I_2, \quad (13)$$

$$A_1c_0 - A_2c_2 - A_3c_3 = I_3, \quad (14)$$

where  $A_i$  ( $i = 1, 2, 3$ ) are given by (9) and

$$\begin{aligned} I_1 &= - \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} \widehat{f}(s) ds, \quad I_2 = - \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} \widehat{f}(s) ds, \\ I_3 &= \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} \widehat{f}(u) du \right) dA(s). \end{aligned} \quad (15)$$

Solving (12) and (13), for  $c_0$  and  $c_3$  in terms of  $c_2$ , we get

$$c_0 = I_1 - \frac{(b-a)}{3} I_2 - \frac{(b-a)^2}{3} c_2, \quad (16)$$

$$c_3 = \frac{1}{3(b-a)^2} I_2 - \frac{2}{3(b-a)} c_2. \quad (17)$$

Substituting (16) and (17) in (14) yields

$$c_2 = \frac{\gamma_2}{\gamma_1} I_2 - \frac{A_1}{\gamma_1} I_1 + \frac{1}{\gamma_1} I_3, \quad \gamma_1 \neq 0, \quad (18)$$

where  $\gamma_1$  and  $\gamma_2$  are defined by (3) and (7) respectively. Using (18) in (16) and (17), we find that

$$c_0 = \lambda_1 I_1 + \lambda_2 I_2 + \lambda_3 I_3,$$

$$c_3 = \lambda_4 I_1 + \lambda_5 I_2 + \lambda_6 I_3.$$

Inserting the values of  $c_0, c_1, c_2$  and  $c_3$  in (10) together with notations (6), we obtain the solution (5). The converse of the lemma can be proved by direct computation.  $\square$

### 3. Existence Results

Let  $\mathcal{E} = C([a, b], \mathbb{R})$  denote the Banach space of all continuous functions from  $[a, b] \rightarrow \mathbb{R}$  equipped with the sup-norm  $\|x\| = \sup\{|x(t)|, t \in [a, b]\}$ . For computational convenience, we introduce

$$\Lambda = \left\{ \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{g}_3 \int_a^b \frac{(s-a)^q}{\Gamma(q+1)} dA(s) \right\}, \quad (19)$$

where  $\bar{g}_1 = \sup_{t \in [a, b]} |g_1(t)|$ ,  $\bar{g}_2 = \sup_{t \in [a, b]} |g_2(t)|$ ,  $\bar{g}_3 = \sup_{t \in [a, b]} |g_3(t)|$ . By Lemma 2, we transform the problems

(1) and (2) into an equivalent fixed-point problem as

$$x = \mathcal{J}x, \quad (20)$$

where  $\mathcal{J} : \mathcal{E} \rightarrow \mathcal{E}$  is defined by

$$\begin{aligned} (\mathcal{J}x)(t) &= \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - g_1(t) \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \\ &\quad - g_2(t) \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} f(s, x(s)) ds + g_3(t) \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, x(u)) du \right) dA(s), \end{aligned} \quad (21)$$

where  $g_1(t), g_2(t)$  and  $g_3(t)$  are given by (6).

Evidently, the existence of fixed points of the operator  $\mathcal{J}$  will imply the existence of solutions for the problems (1) and (2).

Now, the platform is set to present our main results. The following known fixed-point theorem [30] will be used in the proof of our first result.

**Theorem 1.** Let  $X$  be a Banach space. Assume that  $\mathcal{G} : X \rightarrow X$  is a completely continuous operator and the set  $\mathcal{P} = \{x \in X | x = \beta \mathcal{G}x, 0 < \beta < 1\}$  is bounded. Then  $\mathcal{G}$  has a fixed point in  $X$ .

**Theorem 2.** Suppose that there exists  $\varrho \in C([a, b], \mathbb{R}^+)$  such that  $|f(t, x(t))| \leq \varrho(t)$ ,  $\forall t \in [a, b]$ ,  $x \in \mathcal{E}$ , with  $\sup_{t \in [a, b]} |\varrho(t)| = \|\varrho\|$ . Then the problems (1) and (2) has at least one solution on  $[a, b]$ .

**Proof.** Observe that continuity of the operator  $\mathcal{J}$  follows from that of  $f$ . Let  $\Phi \subset \mathcal{E}$  be bounded. Then,  $\forall x \in \Phi$  together with the given assumption  $|f(t, x(t))| \leq \varrho(t)$ , we get

$$\begin{aligned} |(\mathcal{J}x)| &\leq \sup_{t \in [a, b]} \left\{ \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + |g_1(t)| \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \right. \\ &\quad + |g_2(t)| \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds \\ &\quad \left. + |g_3(t)| \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u, x(u))| du \right) dA(s) \right\} \\ &\leq \|\varrho\| \left[ \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{g}_3 \int_a^b \frac{(s-a)^q}{\Gamma(q+1)} dA(s) \right] \\ &= \|\varrho\| \Lambda = M_1, \end{aligned}$$

which shows that  $\mathcal{J}$  is bounded. Next, for  $a < t_1 < t_2 < b$ , we have

$$\begin{aligned} &|(\mathcal{J}x)(t_2) - (\mathcal{J}x)(t_1)| \\ &\leq \int_a^{t_1} \frac{|(t_2-s)^{q-1} - (t_1-s)^{q-1}|}{\Gamma(q)} |f(s, x(s))| ds \\ &\quad + \int_{t_1}^{t_2} \frac{|(t_2-s)^{q-1}|}{\Gamma(q)} |f(s, x(s))| ds + |g_1(t_2) - g_1(t_1)| \int_a^b \frac{|(b-s)^{q-1}|}{\Gamma(q)} |f(s, x(s))| ds \\ &\quad + |g_2(t_2) - g_2(t_1)| \int_a^b \frac{|(b-s)^{q-2}|}{\Gamma(q-1)} |f(s, x(s))| ds \\ &\quad + |g_3(t_2) - g_3(t_1)| \int_a^b \left( \int_a^s \frac{|(s-u)^{q-1}|}{\Gamma(q)} |f(u, x(u))| du \right) dA(s) \\ &\leq \|\varrho\| \left[ \frac{|(t_2-a)^q - (t_1-a)^q| + 2(t_2-t_1)^q}{\Gamma(q+1)} + \frac{|g_1(t_2) - g_1(t_1)| |(b-a)^q|}{\Gamma(q+1)} \right. \\ &\quad \left. + \frac{|g_2(t_2) - g_2(t_1)| |(b-a)^{q-1}|}{\Gamma(q)} + \frac{|g_3(t_2) - g_3(t_1)|}{\Gamma(q+1)} \int_a^b \left( \int_a^s (s-a)^q dA(s) \right) \right], \end{aligned}$$

which tends to zero as  $t_2 \rightarrow t_1$  independent of  $x$ . Thus,  $\mathcal{J}$  is equicontinuous on  $\Phi$ . Hence, by Arzelà-Ascoli theorem,  $\mathcal{J}$  is relatively compact on  $\Phi$ . Therefore,  $\mathcal{J}(\Phi)$  is a relatively compact subset of  $\mathcal{E}$ .

Now we consider a set  $\mathcal{P} = \{x \in \mathcal{E} | x = \beta \mathcal{J}x, 0 < \beta < 1\}$ , and show that the set  $\mathcal{P}$  is bounded. Let  $x \in \mathcal{P}$ , then  $x = \beta \mathcal{J}x$ ,  $0 < \beta < 1$ . For any  $t \in [a, b]$ , we have

$$|x(t)| = \beta |(\mathcal{J}x)(t)|$$

$$\begin{aligned}
&\leq \sup_{t \in [a,b]} \left\{ \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + |g_1(t)| \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \right. \\
&\quad + |g_2(t)| \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds \\
&\quad \left. + |g_3(t)| \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u, x(u))| du \right) dA(s) \right\} \\
&\leq \|q\| \left[ \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{g}_3 \int_a^b \frac{(s-a)^q}{\Gamma(q+1)} dA(s) \right] \\
&= \|q\| \Lambda,
\end{aligned}$$

where  $\Lambda$  given by (19). Thus,  $\|x\| \leq \|q\| \Lambda$  for any  $t \in [a, b]$ . Therefore, the set  $\mathcal{P}$  is bounded. In consequence, the conclusion of Theorem 1 applies and that the operator  $\mathcal{J}$  has at least one fixed point. Thus, there exists at least one solution for the problems (1) and (2) on  $[a, b]$ .  $\square$

**Example 1.** Consider the fractional boundary value problem

$$\begin{cases} {}^c D^{\frac{11}{3}} x(t) = \frac{6e^{-x^2}}{\sqrt{t^4+24}} + \frac{\cos t}{t^2+1} \left( \frac{|x|^3}{1+|x|^3} \right) + t^3 + 6, & t \in [1, 2], \\ x(1) = \sum_{i=1}^4 \alpha_i x(\eta_i) + \int_1^2 x(s) dA(s), & x'(1) = 0, x(2) = 0, x'(2) = 0, \end{cases} \quad (22)$$

where  $q = 11/3$ ,  $a = 1$ ,  $b = 2$ ,  $\alpha_1 = -1/2$ ,  $\alpha_2 = -1/6$ ,  $\alpha_3 = 1/6$ ,  $\alpha_4 = 2$ ,  $\eta_1 = 6/5$ ,  $\eta_2 = 7/5$ ,  $\eta_3 = 8/5$ ,  $\eta_4 = 9/5$  and  $f(t, x) = \frac{6e^{-x^2}}{\sqrt{t^4+24}} + \frac{\cos t}{t^2+1} \left( \frac{|x|^3}{1+|x|^3} \right) + t^3 + 6$ .

Clearly,  $|f(t, x)| \leq \frac{6}{\sqrt{t^4+24}} + \frac{\cos t}{t^2+1} + t^3 + 6 = q(t) > 0$ . Therefore, there exists at least one solution for the problem (22) on  $[1, 2]$  by the conclusion of Theorem 2.

Our next existence result is based on the following fixed-point theorem [30].

**Theorem 3.** Let  $\Omega$  be an open bounded subset of a Banach space  $X$  with  $0 \in \Omega$  and the operator  $\mathcal{F} : \overline{\Omega} \rightarrow X$  is completely continuous satisfying  $\|\mathcal{F}x\| \leq \|x\|$ ,  $\forall x \in \partial\Omega$ . Then the operator  $\mathcal{F}$  has a fixed point in  $\overline{\Omega}$ .

**Theorem 4.** Let  $|f(t, x)| \leq \xi|x|$  for  $0 < |x| < \tau$ , where  $\tau$  and  $\xi$  are positive constants. Then the problems (1) and (2) has at least one solution for small values of  $\xi$ .

**Proof.** Let us choose  $\xi$  such that

$$\Lambda \xi < 1, \quad (23)$$

where  $\Lambda$  given by (19). Define  $B_{r_2} = \{x \in \mathcal{E}; \|x\| \leq r_2\}$  and take  $x \in \mathcal{E}$  such that  $\|x\| = r_2$ , that is,  $x \in \partial B_{r_2}$ . As we argued in Theorem 2, it can be shown that  $\mathcal{J}$  is completely continuous and

$$\begin{aligned}
|(\mathcal{J}x)(t)| &\leq \sup_{t \in [a,b]} \left\{ \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + |g_1(t)| \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \right. \\
&\quad + |g_2(t)| \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds \\
&\quad \left. + |g_3(t)| \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u, x(u))| du \right) dA(s) \right\}
\end{aligned}$$

$$\begin{aligned} &\leq \xi \|x\| \left[ \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{g}_3 \int_a^b \frac{(s-a)^q}{\Gamma(q+1)} dA(s) \right] \\ &= \xi \Lambda \|x\|. \end{aligned}$$

Using (23) and the norm  $\|x\| = \sup_{t \in [a,b]} |x(t)|$ , we get  $\|\mathcal{J}x\| \leq \|x\|$ ,  $x \in \partial B_{r_2}$ . Therefore, conclusion of Theorem 3 applies and hence the problems (1) and (2) has a solution on  $[a, b]$ .  $\square$

**Example 2.** Consider the fractional boundary value problem.

$$\begin{cases} {}^c D^{\frac{27}{7}} x(t) = \frac{2|x|}{\sqrt{100+t^2}} \left( 1 + \frac{|x|}{1+|x|} \right), & t \in [0, 1], \\ x(0) = \sum_{i=1}^4 \alpha_i x(\eta_i) + \int_0^1 x(s) dA(s), & x'(0) = 0, x(1) = 0, x'(1) = 0, \end{cases} \quad (24)$$

where  $q = 27/7$ ,  $a = 0$ ,  $b = 1$ ,  $\alpha_1 = -2$ ,  $\alpha_2 = -1/6$ ,  $\alpha_3 = 1/6$ ,  $\alpha_4 = 15/4$ ,  $\eta_1 = 1/7$ ,  $\eta_2 = 2/7$ ,  $\eta_3 = 3/7$ ,  $\eta_4 = 4/7$ , and  $f(t, x(t)) = \frac{2|x|}{\sqrt{100+t^2}} \left( 1 + \frac{|x|}{1+|x|} \right)$ . Let us take  $A(s) = \frac{s^2}{2}$ . Using the given data, we have that  $A_1 \approx -1.25$ ,  $A_2 \approx 1.45068$ ,  $A_3 \approx 0.903110$ ,  $\gamma_1 \approx -0.43194$ ,  $\gamma_2 \approx -0.115630$ ,  $\lambda_1 \approx 1.96464$ ,  $\lambda_2 \approx -0.422566$ ,  $\lambda_3 \approx 0.771712$ ,  $\lambda_4 \approx 1.92928$ ,  $\lambda_5 \approx 0.154867$ ,  $\lambda_6 \approx 1.54342$ ,  $\bar{g}_1 \approx 1.96464$ ,  $\bar{g}_2 \approx 0.422566$ ,  $\bar{g}_3 \approx 0.771712$ ,  $\Lambda \approx 0.243646$ , where  $\Lambda$  is given by (19). Clearly the hypothesis of Theorem 4 is satisfied with  $\xi = \frac{2}{5}$ . Also  $\xi \Lambda \approx 0.097458 < 1$ . Therefore, the problem (24) has at least one solution on  $[0, 1]$ .

In the next result, we apply a fixed-point theorem due to Krasnoselskii [31] to establish the existence of solutions for the problems (1) and (2).

**Theorem 5.** (Krasnoselskii [31]) Let  $M$  be a closed, convex, bounded and nonempty subset of a Banach space  $X$  and let  $\mathcal{F}_1, \mathcal{F}_2$  be the operators defined from  $M$  to  $X$  such that: (i)  $\mathcal{F}_1 x + \mathcal{F}_2 y \in M$  wherever  $x, y \in M$ ; (ii)  $\mathcal{F}_1$  is compact and continuous; (iii)  $\mathcal{F}_2$  is a contraction. Then there exists  $z \in M$  such that  $z = \mathcal{F}_1 z + \mathcal{F}_2 z$ .

**Theorem 6.** Assume that  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that the following conditions hold:

- (H<sub>1</sub>)  $|f(t, x) - f(t, y)| \leq L|x - y|$ ,  $L > 0$ ,  $\forall t \in [a, b]$ ,  $x, y \in \mathbb{R}$ ;  
 (H<sub>2</sub>)  $|f(t, x)| \leq \mu(t)$ ,  $\forall (t, x) \in [a, b] \times \mathbb{R}$ ,  $\mu \in C([a, b], \mathbb{R}^+)$ .

Then the problems (1) and (2) has at least one solution on  $[a, b]$  if

$$\left( \Lambda - \frac{(b-a)^q}{\Gamma(q+1)} \right) L < 1, \quad (25)$$

where  $\Lambda$  is defined by (19).

**Proof.** Consider a closed ball  $B_r = \{x \in \mathcal{E} : \|x\| \leq r\}$  with  $r \geq \Lambda \|\mu\|$ ,  $\sup_{t \in [a,b]} |\mu(t)| = \|\mu\|$ , and  $\Lambda$  is given by (19). Define operators  $\mathcal{J}_1$  and  $\mathcal{J}_2$  on  $B_r$  as

$$\begin{aligned} (\mathcal{J}_1 x)(t) &= \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds, \\ (\mathcal{J}_2 x)(t) &= -g_1(t) \int_a^b \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - g_2(t) \int_a^b \frac{(t-s)^{q-2}}{\Gamma(q-1)} f(s, x(s)) ds \\ &\quad + g_3(t) \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, x(u)) du \right) dA(s). \end{aligned}$$

Please note that  $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ . For  $x, y \in B_r$ , we have

$$\begin{aligned} \|\mathcal{J}_1 x + \mathcal{J}_2 y\| &\leq \sup_{t \in [a, b]} \left\{ \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + |g_1(t)| \int_a^b \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, y(s))| ds \right. \\ &\quad + |g_2(t)| \int_a^b \frac{(t-s)^{q-2}}{\Gamma(q-1)} |f(s, y(s))| ds \\ &\quad \left. + |g_3(t)| \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u, y(u))| du \right) dA(s) \right\} \\ &\leq \|\mu\| \sup_{t \in [a, b]} \left\{ \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + |g_1(t)| \int_a^b \frac{(t-s)^{q-1}}{\Gamma(q)} ds \right. \\ &\quad \left. + |g_2(t)| \int_a^b \frac{(t-s)^{q-2}}{\Gamma(q-1)} ds + |g_3(t)| \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} du \right) dA(s) \right\} \\ &\leq \|\mu\| \Lambda \leq r, \end{aligned}$$

where we have used (19). Thus,  $\mathcal{J}_1 x + \mathcal{J}_2 y \in B_r$ . Next we show that  $\mathcal{J}_2$  is a contraction. For  $x, y \in B_r$ , we have

$$\begin{aligned} \|\mathcal{J}_2 x - \mathcal{J}_2 y\| &= \sup_{t \in [a, b]} |(\mathcal{J}_2 x)(t) - (\mathcal{J}_2 y)(t)| \\ &\leq \sup_{t \in [a, b]} \left\{ |g_1(t)| \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} |f(s, x(s)) - f(s, y(s))| ds \right. \\ &\quad + |g_2(t)| \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s)) - f(s, y(s))| ds \\ &\quad \left. + |g_3(t)| \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u, x(u)) - f(u, y(u))| du \right) dA(s) \right\} \\ &\leq L \|x - y\| \left[ \bar{g}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{g}_3 \int_a^b \frac{(s-a)^q}{\Gamma(q+1)} dA(s) \right] \\ &= L \left( \Lambda - \frac{(b-a)^q}{\Gamma(q+1)} \right) \|x - y\|, \end{aligned}$$

which shows that  $\mathcal{J}_2$  is a contraction by the condition (25). Continuity of  $f$  implies that the operator  $\mathcal{J}_1$  is continuous. Also,  $\mathcal{J}_1$  is uniformly bounded on  $B_r$  as

$$\|\mathcal{J}_1 x\| \leq \frac{(b-a)^q}{\Gamma(q+1)} \|\mu\|.$$

Next, we establish that the operator  $\mathcal{J}_1$  is compact. Setting  $\mathcal{S} = [a, b] \times B_r$ , we define  $\sup_{(t,x) \in \mathcal{S}} |f(t, x)| = M_r$ . For  $a < t_2 < t_1 < b$ , we get

$$\begin{aligned} |(\mathcal{J}_1 x)(t_1) - (\mathcal{J}_1 x)(t_2)| &= \left| \int_a^{t_2} \frac{(t_1-s)^{q-1} - (t_2-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \right. \\ &\quad \left. + \int_{t_2}^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \right| \\ &\leq \frac{M_r}{\Gamma(q+1)} \left[ |(t_1-a)^q - (t_2-a)^q| + 2(t_1-t_2)^q \right] \\ &\rightarrow 0 \text{ when } t_1 - t_2 \rightarrow 0, \end{aligned}$$



independent of  $x$ . Thus,  $\mathcal{J}_1$  is equicontinuous on  $B_r$ . Hence, by Arzelà-Ascoli theorem,  $\mathcal{J}_1$  is compact on  $B_r$ . Therefore, the conclusion of Theorem 5 applies to the problems (1) and (2).  $\square$

**Remark 1.** By interchanging the role of the operators  $\mathcal{J}_1$  and  $\mathcal{J}_2$  in Theorem 6, the condition (25) becomes:

$$\frac{L(b-a)^q}{\Gamma(q+1)} < 1.$$

**Example 3.** Consider the fractional differential equation:

$${}^c D^{\frac{11}{3}} x(t) = \frac{\delta}{\sqrt{t^2+8}} \tan^{-1} x + e^{2t}, \quad t \in [1, 2]. \quad (26)$$

subject to the boundary conditions of Example 1.

Let us take  $A(s) = \frac{(s-1)^2}{2} + \frac{1}{2}$ . Using the given data (from Example 1), it is found that  $A_1 = -1$ ,  $A_2 \approx 1.543333$ ,  $A_3 \approx 1.245333$ ,  $\gamma_1 \approx -0.379778$ ,  $\gamma_2 \approx 0.081778$ ,  $\lambda_1 \approx 1.877706$ ,  $\lambda_2 \approx -0.261556$ ,  $\lambda_3 \approx 0.877706$ ,  $\lambda_4 \approx 1.755413$ ,  $\lambda_5 \approx 0.476887$ ,  $\lambda_6 \approx 1.755413$ ,  $\bar{g}_1 \approx 1.877706$ ,  $\bar{g}_2 \approx 0.264808$ ,  $\bar{g}_3 \approx 0.877706$ ,  $\Lambda \approx 0.272140$  ( $\Lambda$  is given by (19) and  $\Lambda - \frac{(b-a)^2}{\Gamma(q+1)} \approx 0.204166$ ).

Clearly the hypotheses of Theorem 6 are satisfied with  $L = \delta/3$  and  $\mu(t) = \frac{\delta\pi}{2\sqrt{t^2+8}} + e^{2t}$ . Also  $L\left(\Lambda - \frac{(b-a)^2}{\Gamma(q+1)}\right) < 1$  for  $\delta < 14.693960$ . Therefore, there exists at least one solution for the problem (26) on  $[1, 2]$ .

#### 4. Uniqueness of Solution

Here, we prove the uniqueness of solutions for the problems (1) and (2).

**Theorem 7.** Assume that  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function satisfying the condition  $(H_1)$ . Then the problems (1) and (2) has a unique solution on  $[a, b]$  if

$$L\Lambda < 1, \quad (27)$$

where  $\Lambda$  is given by (19).

**Proof.** Setting  $\sup_{t \in [a, b]} |f(t, 0)| = N < \infty$ , and selecting

$$r_1 \geq N\Lambda(1 - L\Lambda)^{-1},$$

we define  $B_{r_1} = \{x \in \mathcal{E} : \|x\| \leq r_1\}$ , and show that  $\mathcal{J}B_{r_1} \subset B_{r_1}$ , where the operator  $\mathcal{J}$  is defined by (21). For  $x \in B_{r_1}$ ,

$$\begin{aligned} |f(t, x(t))| &= |f(t, x(t)) - f(t, 0) + f(t, 0)| \leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\ &\leq L|x(t)| + N \leq L\|x\| + N \leq Lr_1 + N. \end{aligned}$$

Then,

$$\begin{aligned} \|\mathcal{J}x\| &\leq \sup_{t \in [a, b]} \left\{ \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + |g_1(t)| \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \right. \\ &\quad + |g_2(t)| \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds \\ &\quad \left. + |g_3(t)| \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u, x(u))| du \right) dA(s) \right\} \end{aligned}$$

$$\begin{aligned}
&\leq (L\|x\| + N) \left[ \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{g}_3 \int_a^b \frac{(s-a)^q}{\Gamma(q+1)} dA(s) \right] \\
&\leq (Lr_1 + N)\Lambda \leq r_1.
\end{aligned}$$

This shows that  $\mathcal{J}x \in B_{r_1}$  for any  $x \in B_{r_1}$ . Therefore,  $\mathcal{J}B_{r_1} \subset B_{r_1}$ . Now, we show that  $\mathcal{J}$  is a contraction. For  $x, y \in \mathcal{E}$  and  $t \in [a, b]$ , we obtain

$$\begin{aligned}
\|(\mathcal{J}x) - (\mathcal{J}y)\| &= \sup_{t \in [a, b]} |(\mathcal{J}x)(t) - (\mathcal{J}y)(t)| \\
&\leq \sup_{t \in [a, b]} \left\{ \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s)) - f(s, y(s))| ds \right. \\
&\quad + |g_1(t)| \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} |f(s, x(s)) - f(s, y(s))| ds \\
&\quad + |g_2(t)| \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s)) - f(s, y(s))| ds \\
&\quad \left. + |g_3(t)| \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u, x(u)) - f(u, y(u))| du \right) dA(s) \right\} \\
&\leq L\|x - y\| \left[ \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{g}_3 \int_a^b \frac{(s-a)^q}{\Gamma(q+1)} dA(s) \right] \\
&= L\Lambda\|x - y\|.
\end{aligned}$$

By the condition (27), we deduce from the above inequality that  $\mathcal{J}$  is a contraction. Thus, by the conclusion of Banach fixed-point theorem, the problems (1) and (2) has a unique solution on  $[a, b]$ .  $\square$

**Example 4.** Let us take the problem considered in Example 1, and note that  $L\Lambda < 1$  for  $\delta < 11.023738$ . Clearly the hypothesis of Theorem 7 is satisfied. Hence it follows by the conclusion of Theorem 7 that the problem (22) has a unique solution on  $[1, 2]$ .

## 5. Ulam Stability

In this section, we discuss the Ulam stability for the problems (1) and (2) by means of integral representation of its solution given by

$$\begin{aligned}
y(t) &= \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, y(s)) ds - g_1(t) \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} f(s, y(s)) ds \\
&\quad - g_2(t) \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} f(s, y(s)) ds \\
&\quad + g_3(t) \int_a^b \left( \int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, y(u)) du \right) dA(s).
\end{aligned} \tag{28}$$

Here  $y \in C([a, b], \mathbb{R})$  possesses a fractional derivative of order  $3 < q \leq 4$  and  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function. Then the nonlinear operator  $\mathcal{Q} : C([a, b], \mathbb{R}) \rightarrow C([a, b], \mathbb{R})$  defined by

$$\mathcal{Q}y(t) = {}^c D^q y(t) - f(t, y(t))$$

is continuous.

**Definition 3.** For each  $\epsilon > 0$  and for each solution  $y$  of (1) and (2) such that

$$\|\mathcal{Q}y\| \leq \epsilon, \tag{29}$$

the problems (1) and (2) is said to be Ulam-Hyers stable if we can find a positive real number  $\nu$  and a solution  $x \in C([a, b], \mathbb{R})$  of (1) and (2) satisfying the inequality:

$$\|x - y\| \leq \nu \epsilon_*,$$

where  $\epsilon_*$  is a positive real number depending on  $\epsilon$ .

**Definition 4.** Let there exists  $\kappa \in C(\mathbb{R}^+, \mathbb{R}^+)$  such that for each solution  $y$  of (1) and (2), we can find a solution  $x \in C([a, b], \mathbb{R})$  of (1) and (2) such that

$$|x(t) - y(t)| \leq \kappa(\epsilon), \quad t \in [a, b].$$

Then the problems (1) and (2) is said to be generalized Ulam-Hyers stable

**Definition 5.** For each  $\epsilon > 0$  and for each solution  $y$  of (1) and (2), the problems (1) and (2) is called Ulam-Hyers-Rassias stable with respect to  $\sigma \in C([a, b], \mathbb{R}^+)$  if

$$|Qy(t)| \leq \epsilon \sigma(t), \quad t \in [a, b], \quad (30)$$

and there exist a real number  $\nu > 0$  and a solution  $x \in C([a, b], \mathbb{R})$  of (1) and (2) such that

$$|x(t) - y(t)| \leq \nu \epsilon_* \sigma(t), \quad t \in [a, b],$$

where  $\epsilon_*$  is a positive real number depending on  $\epsilon$ .

**Theorem 8.** Let the assumption  $(H_1)$  hold with  $L\Lambda < 1$ , where  $\Lambda$  is defined by (19). Then the problems (1) and (2) is both Ulam-Hyers and generalized Ulam-Hyers stable.

**Proof.** Let  $x \in C([a, b], \mathbb{R})$  be a solution of (1) and (2) satisfying (21) by Theorem 7. Let  $y$  be any solution satisfying (29). Then by Lemma 2,  $y$  satisfies the integral equation (28). Furthermore, the equivalence in Lemma 2 implies the equivalence between the operators  $Q$  and  $\mathcal{J} - I$  (where  $I$  is identity operator) for every solution  $y \in C([a, b], \mathbb{R})$  of (1) and (2) satisfying (27). Therefore, we deduce by the fixed-point property of the operator  $\mathcal{J}$  (given by (21)) and (29) that

$$\begin{aligned} |y(t) - x(t)| &= |y(t) - \mathcal{J}y(t) + \mathcal{J}y(t) - \mathcal{J}x(t)| \\ &\leq |\mathcal{J}x(t) - \mathcal{J}y(t)| + |\mathcal{J}y(t) - y(t)| \\ &\leq L\Lambda \|x - y\| + \epsilon, \end{aligned}$$

where  $\epsilon > 0$  and  $L\Lambda < 1$ . In consequence, we get

$$\|x - y\| \leq \frac{\epsilon}{1 - L\Lambda}.$$

Fixing  $\epsilon_* = \frac{\epsilon}{1 - L\Lambda}$  and  $\nu = 1$ , we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking  $\kappa(\epsilon) = \frac{\epsilon}{1 - L\Lambda}$ .  $\square$

**Theorem 9.** Assume that  $(H_1)$  holds with  $L < \Lambda^{-1}$  (where  $\Lambda$  is defined by (19)), and there exists a function  $\sigma \in C([a, b], \mathbb{R}^+)$  satisfying the condition (30). Then the problems (1) and (2) is Ulam-Hyers-Rassias stable with respect to  $\sigma$ .

**Proof.** Following the arguments employed in the proof of Theorem 8, we have

$$\|x - y\| \leq \epsilon_* \sigma(t),$$

where  $\epsilon_* = \frac{\epsilon}{1-L\Lambda}$ . This completes the proof.  $\square$

**Example 5.** Consider the following fractional differential equation

$${}^c D^{\frac{27}{7}} x(t) = \frac{3}{t^2 + 16} \left( \cos x + \frac{x}{1+x} \right), \quad t \in [0, 1], \quad (31)$$

subject to the same data and the boundary conditions given in Example 1 with  $f(t, x(t)) = \frac{3}{t^2 + 16} \left( \cos x + \frac{x}{1+x} \right)$ . Obviously  $|f(t, x) - f(t, y)| \leq \frac{3}{8} \|x - y\|$ , so,  $L = 3/8$  and  $L\Lambda \approx 0.102053 < 1$ . Then the problem (31) is Ulam-Hyers stable, and generalized Ulam-Hyers stable. In addition, if there exists a continuous and positive function  $\sigma = e^{1+t^2} + 5$  satisfying the condition (30), then the problem (31) is Ulam-Hyers-Rassias stable with the given value of  $f(t, x)$ .

## 6. Conclusions

We have obtained several existence results for a new class of Caputo-type fractional differential equations of order  $q \in (3, 4]$ , supplemented with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions on an arbitrary domain by imposing different kinds of conditions on the nonlinear function involved in the problem. The existence results, relying on different fixed-point theorems, are presented in Section 3. The uniqueness of solution for the given problem is studied in Section 4 with the aid of Banach fixed-point theorem. Section 5 is concerned with different kinds of Ulam stability for the problem at hand. Some new results follow as special cases of the ones presented in this paper. For example, taking  $A(s) = s$ , our results correspond to the ones for non-conjugate integro-multipoint boundary conditions of the form:  $x(a) = \sum_{i=1}^{n-2} \alpha_i x(\eta_i) + \int_a^b x(s) ds$ ,  $x'(a) = 0$ ,  $x(b) = 0$ ,  $x'(b) = 0$  and the value of  $\Lambda$  given by (19) takes the following form in this situation:

$$\Lambda = \left\{ \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{g}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{g}_3 \frac{(b-a)^{q+1}}{\Gamma(q+2)} \right\}.$$

Letting  $A(s) = 0$  in our results, we get the ones for non-conjugate multipoint boundary conditions of the form:  $x(a) = \sum_{i=1}^{n-2} \alpha_i x(\eta_i)$ ,  $x'(a) = 0$ ,  $x(b) = 0$ ,  $x'(b) = 0$ . In case we take  $\alpha_i = 0$  for all  $i = 1, \dots, n-2$ , our results reduce to the ones with non-conjugate Riemann-Stieltjes boundary conditions. By fixing  $A(s) = 0$  and  $\alpha_i = 0$  for all  $i = 1, \dots, n-2$ , our results correspond to a boundary value problem of fractional order  $q \in (3, 4]$  with conjugate boundary conditions.

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## References

1. Zaslavsky, G.M. *Hamiltonian Chaos and Fractional Dynamics*; Oxford University Press: Oxford, UK, 2005.
2. Zhang, D.; Zhang, Y.; Zhang, Z.; Ahmed, N.; Zhang, Y.; Li, F.; Belic, M.R.; Xiao, M. Unveiling the link between fractional Schrödinger equation and light propagation in honeycomb lattice. *Ann. Phys.* **2017**, *529*, 1700149. [CrossRef]
3. Fallahgoul, H.A.; Focardi, S.M.; Fabozzi, F.J. *Fractional Calculus and Fractional Processes with Applications to Financial Economics. Theory and Application*; Elsevier/Academic Press: London, UK, 2017.
4. Javidi, M.; Ahmad, B. Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system. *Ecol. Model.* **2015**, *318*, 8–18. [CrossRef]
5. Magin, R.L. *Fractional Calculus in Bioengineering*; Begell House Publishers: Danbury, CT, USA, 2006.

6. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
7. Lakshmikantham, V.; Leela, S.; Devi, J.V. *Theory of Fractional Dynamic Systems*; Cambridge Academic Publishers: Cambridge, UK, 2009.
8. Diethelm, K. *The Analysis of Fractional Differential Equations. An Application—Oriented Exposition Using Differential Operators of Caputo Type*; Lecture Notes in Mathematics 2004; Springer: Berlin, Germany, 2010.
9. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Cham, Switzerland, 2017.
10. Ahmad, B.; Nieto, J.J. Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. *Bound. Value Probl.* **2011**, 2011, 36. [[CrossRef](#)]
11. Liang, S.; Zhang, J. Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval. *Math. Comput. Model.* **2011**, 54, 1334–1346. [[CrossRef](#)]
12. Bai, Z.B.; Sun, W. Existence and multiplicity of positive solutions for singular fractional boundary value problems. *Comput. Math. Appl.* **2012**, 63, 1369–1381. [[CrossRef](#)]
13. Agarwal, R.P.; O'Regan, D.; Stanek, S. Positive solutions for mixed problems of singular fractional differential equations. *Math. Nachr.* **2012**, 285, 27–41. [[CrossRef](#)]
14. Ahmad, B.; Ntouyas, S.K.; Alsaedi, A. A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions. *Math. Probl. Eng.* **2013**, 2013, 320415. [[CrossRef](#)]
15. Ahmad, B.; Ntouyas, S.K. Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2013**, 2013, 20. [[CrossRef](#)]
16. O'Regan, D.; Stanek, S. Fractional boundary value problems with singularities in space variables. *Nonlinear Dyn.* **2013**, 71, 641–652. [[CrossRef](#)]
17. Zhai, C.; Xu, L. Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, 19, 2820–2827. [[CrossRef](#)]
18. Graef, J.R.; Kong, L.; Wang, M. Existence and uniqueness of solutions for a fractional boundary value problem on a graph. *Fract. Calc. Appl. Anal.* **2014**, 17, 499–510. [[CrossRef](#)]
19. Wang, G.; Liu, S.; Zhang, L. Eigenvalue problem for nonlinear fractional differential equations with integral boundary conditions. *Abstr. Appl. Anal.* **2014**, 2014, 916260. [[CrossRef](#)]
20. Henderson, J.; Kosmatov, N. Eigenvalue comparison for fractional boundary value problems with the Caputo derivative. *Fract. Calc. Appl. Anal.* **2014**, 17, 872–880. [[CrossRef](#)]
21. Zhang, L.; Ahmad, B.; Wang, G. Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half line. *Bull. Aust. Math. Soc.* **2015**, 91, 116–128. [[CrossRef](#)]
22. Henderson, J.; Luca, R. Nonexistence of positive solutions for a system of coupled fractional boundary value problems. *Bound. Value Probl.* **2015**, 2015, 138. [[CrossRef](#)]
23. Ntouyas, S.K.; Etemad, S. On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions. *Appl. Math. Comput.* **2015**, 266, 235–243. [[CrossRef](#)]
24. Mei, Z.D.; Peng, J.G.; Gao, J.H. Existence and uniqueness of solutions for nonlinear general fractional differential equations in Banach spaces. *Indag. Math.* **2015**, 26, 669–678. [[CrossRef](#)]
25. Ahmad, B.; Ntouyas, S.K. Existence results for fractional differential inclusions with Erdelyi-Kober fractional integral conditions. *Analele Universitatii Ovidius Constanta-Seria Matematica* **2017**, 25, 5–24. [[CrossRef](#)]
26. Srivastava, H.M. Remarks on some families of fractional-order differential equations. *Integr. Transf. Spec. Funct.* **2017**, 28, 560–564. [[CrossRef](#)]
27. Wang, G.; Pei, K.; Agarwal, R.P.; Zhang, L.; Ahmad, B. Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line. *J. Comput. Appl. Math.* **2018**, 343, 230–239. [[CrossRef](#)]
28. Ahmad, B.; Luca, R. Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions. *Appl. Math. Comput.* **2018**, 339, 516–534. [[CrossRef](#)]
29. Zhang, Y.; Wang, R.; Zhong, H.; Zhang, J.; Belic, M.R.; Zhang, Y. Resonant mode conversions and Rabi oscillations in a fractional Schrödinger equation. *Opt. Express* **2017**, 25, 32401. [[CrossRef](#)]

30. Sun, J.X. *Nonlinear Functional Analysis and Its Application*; Science Press: Beijing, China, 2018.
31. Krasnoselskii, M.A. Two remarks on the method of successive approximations. *Uspekhi Matematicheskikh Nauk* **1955**, *10*, 123–127.



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