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Some New Observations on Geraghty and Ćirić Type Results in b-Metric Spaces

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Abstract: We discuss recent fixed point results in b-metric spaces given by Pant and Panicker (2016). Our results are with shorter proofs. In addition, for $\varepsilon \in (1,3]$, our results are genuine generalizations of ones from Pant and Panicker.

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MSC: 47H10; 54H25

1. Introduction

We start with the following.

Definition 1 ([1,2])**.** *Let f be a self-mapping on a metric space (X, d). For $\mu \in X$, take*

$$O(\mu, n) = \{\mu, f\mu, ..., f^n\mu\} \text{ and } O(\mu, \infty) = \{\mu, f\mu, ..., f^n\mu, ...\},$$

where $n \in \mathbb{N}$. The set $O(\mu, \infty)$ is called an orbit of f. Such (X, d) is said to be f-orbitally complete if each Cauchy sequence in $O(\mu, \infty)$ converges in (X, d).

It is well known that every complete metric space is f-orbitally complete for each self-mapping f on X. Its converse does not hold (see [1,2]).

Two very known and important generalizations of the Banach contraction principle [3] obtained by Ćirić [1] and Geraghty [4] as follows:

Theorem 1 ([1])**.** *Let (X, d) be an f-orbitally complete metric space and $f : X \to X$ be a quasi-contraction, i.e., there is $\lambda \in [0, 1)$ so that*

$$d(f\mu, f\tau) \leq \lambda \cdot \max\{d(\mu, \tau), d(\mu, f\mu), d(\tau, f\tau), d(\mu, f\tau), d(f\mu, \tau)\}, \tag{1}$$

for all $\mu, \tau \in X$. Then, f possesses a unique fixed point.

Theorem 2 ([4]). *Let (X, d) be a complete metric space and $f : X \to X$ be such that*

$$d\left(f\mu, f\tau\right) \leq \beta\left(d\left(\mu, \tau\right)\right) d\left(\mu, \tau\right), \tag{2}$$

for all $\mu, \tau \in X$, where $\beta : [0, \infty) \to [0, 1)$ is such that $\beta\left(t_n\right) \to 1$ implies $t_n \to 0$ as $n \to \infty$. Then, f has a unique fixed point.

The concept of quasi-contractions has been generalized by Kumam et al. [2].

Definition 2 ([2]). *A self-mapping f on a metric space (X, d) is called a generalized quasi-contraction if there is $\lambda \in [0, 1)$ so that*

$$d\left(T\mu, T\tau\right) \leq \lambda \cdot M\left(\mu, \tau\right), \tag{3}$$

for all $\mu, \tau \in X$, where

$$M\left(\mu, \tau\right) = \max\left\{d(\mu, \tau), d(\mu, f\mu), d(\tau, f\tau), d(\mu, f\tau), d(f\mu, \tau), d(f^2\mu, \mu), d(f^2\mu, f\mu), d(f^2\mu, \tau), d(f^2\mu, f\tau)\right\}. \tag{4}$$

Theorem 3 ([2]). *Each generalized quasi-contraction self-mapping f on an f-orbitally complete metric space admits a unique fixed point.*

Given $s \geq 1$. A function $d : X \times X \to [0, \infty)$ is called a b-metric on a non-empty set X if for all $\mu, \tau, \xi \in X$, $d\left(\mu, \tau\right) = 0$ iff $\mu = \tau$, $d\left(\mu, \tau\right) = d\left(\tau, \mu\right)$ and $d\left(\mu, \xi\right) \leq s\left[d\left(\mu, \tau\right) + d\left(\tau, \xi\right)\right]$. The concept of b-convergence, b-completeness, b-Cauchyness in b-metric spaces can be found in [5–38].

Definition 3 ([39]). *Given $\Omega : X \times X \to [0, \infty)$ and $f : X \to X$, such f is called Ω-admissible if, for all $\mu, \tau \in X$,*

$$\Omega\left(\mu, \tau\right) \geq 1 \text{ implies } \Omega\left(f\mu, f\tau\right) \geq 1.$$

Definition 4 ([40]). *The mapping $f : X \to X$ is called triangular Ω-admissible if for all $\mu, \tau, \xi \in X$,*
*(**i**) $\Omega\left(\mu, \tau\right) \geq 1$ implies $\Omega\left(f\mu, f\tau\right) \geq 1$;*
*(**ii**) $\Omega\left(\mu, \xi\right) \geq 1$ and $\Omega\left(\xi, \tau\right) \geq 1$ implies $\Omega\left(\mu, \tau\right) \geq 1$.*

Lemma 1 ([40]). *Let f be a triangular Ω-admissible mapping. Suppose there is $\mu_0 \in X$ so that $\Omega\left(\mu_0, f\mu_0\right) \geq 1$. Define $\{\mu_n\}$ by $\mu_n = f^n\mu_0$. Then, $\Omega\left(\mu_m, \mu_n\right) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Very recently, Pant and Panciker [35] initiated the concept of generalized Ω-quasi-contraction in b-metric spaces. Namely, they defined and proved the following:

Definition 5 ([35]). *Let (X, d) be a b-metric space with constant $s \geq 1$. The self-mapping f on X is called a generalized Ω-quasi-contraction if there are $\Omega : X \times X \to [0, \infty)$ and a real number q with $0 < q < \frac{1}{s^2}$ such that*

$$\Omega\left(\mu, \tau\right) d\left(f\mu, f\tau\right) \leq qM\left(\mu, \tau\right), \tag{5}$$

where $M\left(\mu, \tau\right)$ is given by (4).

Lemma 2 ([35]). *Let (X, d) be a b-metric space with $s \geq 1$ and $f : X \to X$ be a generalized $\Omega-$quasi contraction such that*
*(**A**): f is triangular Ω-admissible;*
*(**B**): there is $\mu_0 \in X$ so that $\Omega\left(\mu_0, f\mu_0\right) \geq 1$.*
Then, for all $p, k \in \{1, 2, ..., n\}$ with $(p < k)$, we have

$$d\left(f^p\mu_0, f^k\mu_0\right) \leq q\delta\left[O\left(\mu_0, n\right)\right]. \tag{6}$$

Theorem 4 ([35]). *Let (X,d) be a f-orbitally complete b-metric space with $s \geq 1$ and $f : X \to X$ be a generalized Ω-quasi-contraction so that (**A**) and (**B**) of Lemma 2 hold. Then, f admits a fixed point.*

The proofs of Lemma 2 and Theorem 4 are based on Lemma 1.

Motivated by Ćirić [1], Geraghty [4], Kumam et al. [2], Samet et al. [39] as well as Pant and Panciker [35], we improve some related fixed point theorems in b-metric spaces. Our proofs are much shorter and nicer than the ones in [35].

Corollary 1. *Let (X,d) be a b-complete b-metric space with a constant $s \geq 1$. Given $\Omega : X \times X \to [0,\infty)$ a functional, let $f : X \to X$ be an Ω-quasi-contraction, i.e.,*

$$\Omega(\mu,\tau)\, d(f\mu, f\tau) \leq q \cdot m(\mu,\tau), \tag{7}$$

for all $\mu,\tau \in X$, where $0 \leq q < 1$ and $m(\mu,\tau) = \max\{d(\mu,\tau), d(\mu,f\mu), d(\tau,f\tau), d(\mu,f\tau), d(f\mu,\tau)\}$. Suppose that
*(**i**) f is Ω-admissible;*
*(**ii**) there is $\mu_0 \in X$ so that $\Omega(\mu_0, f\mu_0) \geq 1$.*
If $q < \frac{1}{s^2+s}$, then f admits a fixed point.

Chandok [41] defined the following.

Definition 6 ([41]). *Let (X,d) be a b-metric space with constant $s \geq 1$, $f : X \to X$ and $\Omega, \omega : X \times X \to [0,\infty)$. The mapping f is said to be $(\Omega,\omega)-$admissible if $\Omega(\mu,\tau) \geq 1$ and $\omega(\mu,\tau) \geq 1$ implies $\Omega(f\mu, f\tau) \geq 1$ and $\omega(f\mu, f\tau) \geq 1$ for all $\mu,\tau \in X$.*

Definition 7 ([41]). *Let $\Omega, \omega : X \times X \to [0,\infty)$. A b-metric space (X,d) with a constant $s \geq 1$ is (Ω,ω)-regular if $\{\mu_n\}$ is a sequence in X such that $\mu_n \to x \in X, \Omega(\mu_n,\mu_{n+1}) \geq 1, \omega(\mu_n,\mu_{n+1}) \geq 1$, for all n; then, there is a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\Omega(\mu_{n_k},\mu_{n_k+1}) \geq 1, \omega(\mu_{n_k},\mu_{n_k+1}) \geq 1$, for all $k \in \mathbb{N}$ and $\Omega(\mu, f\mu) \geq 1, \omega(\mu, f\mu) \geq 1$.*

The two following classes of functions are defined in [35].

(**1**) Θ denotes the family of functions $\theta : [0,\infty) \to [0,1)$ so that, for any bounded sequence $\{t_n\}$ of positive reals, $\theta(t_n) \to 1$ implies $t_n \to 0$;

(**2**) Ψ denotes the set of functions $\psi : [0,\infty) \to [0,\infty)$ so that ψ is continuous, strictly increasing and $\psi(0) = 0$.

Definition 8 ([35]). *Let (X,d) be a b-metric space with constant $s \geq 1$. The mapping $f : X \to X$ is called an (Ω,ω)-Geraghty type contraction if there are $\theta \in \Theta$, $\psi \in \Psi$ and $\Omega, \omega : X \times X \to [0,\infty)$ such that*

$$\Omega(\mu, f\mu)\, \omega(\tau, f\tau)\, \psi\left(s^3 d(f\mu, f\tau)\right) \leq \theta(\psi(N(\mu,\tau)))\, \psi(N(\mu,\tau)), \tag{8}$$

for all $\mu,\tau \in X$, where $N(\mu,\tau) = \max\left\{d(\mu,\tau), d(\mu,f\mu), d(\tau,f\tau), \frac{d(\mu,f\tau)+d(\tau,f\mu)}{2s}\right\}$.

Theorem 5 ([35]). *Let (X,d) be a b-complete b-metric space with a constant $s \geq 1$ and $f : X \to X$ be a self-mapping. Suppose that the following assertions hold:*
*(**A**): f is (Ω,ω)-admissible;*
*(**B**): f is an (Ω,ω)-Geraghty type contraction;*
*(**C**): there is $\mu_0 \in X$ so that $\Omega(\mu_0, f\mu_0) \geq 1$ and $\omega(\mu_0, f\mu_0) \geq 1$;*
*(**D**): either f is continuous, or (X,d) is (Ω,ω)-regular.*
Then, f has a unique fixed point.

Corollary 2 ([35]). *Let (X, d) be a b-complete b-metric space with $s \geq 1$ and $f : X \to X$ be a self-mapping. Suppose that the following assertions hold:*

*(**A**): f is Ω-admissible;*

*(**B**): f is an Ω-Geraghty type contraction;*

*(**C**): there is $\mu_0 \in X$ so that $\Omega(\mu_0, f\mu_0) \geq 1$;*

*(**D**): either f is continuous or (X, d) is Ω-regular.*

Then, f admits a unique fixed point.

The proofs of Theorem 5 (and Corollary 2) are based on the following crucial lemma.

Lemma 3 ([8]). *Let (X, d) be a b-metric space with $s \geq 1$. Let $\{\mu_n\}$ and $\{\tau_n\}$ b-converge to $\mu, \tau \in X$, respectively. We have*

$$\frac{1}{s^2} d(\mu, \tau) \leq \liminf_{n\to\infty} d(\mu_n, \tau_n) \leq \limsup_{n\to\infty} d(\mu_n, \tau_n) \leq s^2 d(\mu, \tau). \tag{9}$$

In particular, if $\mu = \tau$, then $\lim_{n\to\infty} d(\mu_n, \tau_n) = 0$. In addition, for any $\xi \in X$,

$$\frac{1}{s} d(\mu, \xi) \leq \liminf_{n\to\infty} d(\mu_n, \xi) \leq \limsup_{n\to\infty} d(\mu_n, \xi) \leq s d(\mu, \xi). \tag{10}$$

2. Main Results

Instead of Lemma 3, we will use in this paper the next result to establish our main results.

Lemma 4. *([28], Lemma 3.1) Let $\{\mu_n\}$ be a sequence in a b-metric space $(X, d, s \geq 1)$ such that*

$$d(\mu_{n+1}, \mu_{n+2}) \leq \lambda d(\mu_n, \mu_{n+1}), \; n \geq 0,$$

for some $\lambda \in [0, \frac{1}{s})$. Then, $\{\mu_n\}$ is a b-Cauchy sequence in X.

By Lemma 4, it would be good to note that each Picard sequence is b-Cauchy, and so the proofs of some results (such as, the ones in the sequel) become shorter.

Remark 1. *In many results based on b-metric spaces with a constant $s \geq 1$, people often suppose that $\lambda \in [0, \frac{1}{s})$ instead of $\lambda \in [0, 1)$, which is clearly a stronger condition due to the fact that $[0, \frac{1}{s}) \subseteq [0, 1)$. To ensure this fact, the following inequality is utilized:*

$$d(\mu_m, \mu_n) \leq s d(\mu_m, \mu_{m+1}) + s^2 d(\mu_{m+1}, \mu_{m+2}) + \cdots + s^{n-m-1} d(\mu_{n-2}, \mu_{n-1}) + s^{n-m-1} d(\mu_{n-1}, \mu_n), \tag{11}$$

for $n, m \in \mathbb{N}$ and $n > m$.

Since there is a doubt in the proof of Theorem 3.5 of [35] (see [35], page 6, line 13: about the inequality: $d(Tu, TT^n\mu_0) \leq \Omega(u, T^n\mu_0) d(Tu, TT^n\mu_0)$), we give the following new result.

Theorem 6. *Let (X, d) be an f-orbitally b-complete b-metric space (with $s > 1$) and let $f : X \to X$ be a generalized Ω-quasi-contraction where (**A**) and (**B**) of Lemma 2 both hold. If either f is continuous, or (X, d) is Ω-regular, then f possesses a fixed point.*

Proof. By assumption, there is $\mu_0 \in X$ so that $\Omega(\mu_0, f\mu_0) \geq 1$. We will show that the sequence $\{\mu_n = f^n \mu_0\}_{n\in\mathbb{N}}$ is b-Cauchy in the b-metric space (X,d). Indeed, let $\mu_n \neq \mu_{n-1}$ for all $n \in \mathbb{N}$. According to (**A**) from Lemma 2, it follows that $\Omega(\mu_n, \mu_{n+1}) \geq 1$ for each $n \geq 0$. Therefore,

$$d(\mu_{n+1}, \mu_{n+2}) \leq \Omega(\mu_n, \mu_{n+1}) d(f\mu_n, f\mu_{n+1}) \leq qM(\mu_n, \mu_{n+1}), \tag{12}$$

where

$$\begin{aligned} M(\mu_n, \mu_{n+1}) &= \max\{d(\mu_n, \mu_{n+1}), d(\mu_{n+1}, \mu_{n+2}), d(\mu_n, \mu_{n+2})\} \\ &\leq \max\{d(\mu_n, \mu_{n+1}), d(\mu_{n+1}, \mu_{n+2}), s(d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_{n+2}))\} \\ &= s(d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_{n+2})). \end{aligned}$$

Hence,

$$d(\mu_{n+1}, \mu_{n+2}) \leq \frac{qs}{1-qs} d(\mu_n, \mu_{n+1}) = \lambda d(\mu_n, \mu_{n+1}), \tag{13}$$

where $\lambda = \frac{qs}{1-qs} < \frac{1}{s}$ because $q < \frac{1}{s^2}$. $\square$

Now, by ([28], Lemma 3.1), the sequence $\{\mu_n = f^n \mu_0\}$ is b-Cauchy. Since (X,d) is f-orbitally b–complete, there is $u \in X$ so that $\lim_{n\to\infty} f^n \mu_0 = u$. If f is continuous, then obviously $fu = u$. When (X,d) is Ω-regular, we have (because $\Omega(u, \mu_n) \geq 1$)

$$\begin{aligned} \frac{1}{s} d(u, fu) &\leq d(u, \mu_{n+1}) + d(f\mu_n, fu) \\ &\leq d(u, \mu_{n+1}) + \Omega(u, \mu_n) d(f\mu_n, fu) \\ &\leq d(u, \mu_{n+1}) + qM(\mu_n, u), \end{aligned}$$

where

$$\begin{aligned} M(\mu_n, u) = \; & \max\{d(\mu_n, u), d(\mu_n, \mu_{n+1}), d(u, fu), d(\mu_n, fu), d(u, \mu_{n+1}), d(\mu_{n+2}, u), \\ & d(\mu_{n+2}, \mu_{n+1}), d(\mu_{n+2}, u), d(\mu_{n+2}, fu)\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} M(\mu_n, u) &\leq \max\{d(\mu_n, u), d(\mu_n, \mu_{n+1}), d(u, fu), s(d(\mu_n, u) + d(u, fu)), d(u, \mu_{n+1}), d(\mu_{n+2}, u), \\ &\qquad d(\mu_{n+2}, \mu_{n+1}), d(\mu_{n+2}, u), s(d(\mu_{n+2}, u) + d(u, fu))\} \\ &\to \max\{0, 0, d(u, fu), sd(u, fu), 0, 0, 0, 0, sd(u, fu)\} \\ &= sd(u, fu), \end{aligned}$$

as $n \to \infty$.

Hence,

$$\frac{1}{s} d(u, fu) \leq sqd(u, fu) \text{ or equivalently } \frac{1}{s^2} d(u, fu) \leq qd(u, fu),$$

which is possible only if $fu = u$.$\square$

Remark 2. *If $q < \frac{1}{s^2}$, then our approach gives a short and nice proof that a generalized Ω-quasi-contraction $f : X \to X$ has a fixed point. However, the proof of the corresponding result in ([35], page 6, line 13+) is not correct without the assumption that the b–metric space (X,d) is Ω-regular. This new compliment of the proof of Theorem 3.5 in [35] is correct if $q < \frac{1}{s^2}$.*

We introduce the following.

Definition 9. *Let $(X, d, s > 1)$ be a b-metric space. The mapping $f : X \to X$ is said to be an (Ω, ω)-type contraction if there are $\Omega, \omega : X \times X \to [0, \infty)$, $\varepsilon > 1$ and $\psi \in \Psi$ such that*

$$\Omega(\mu, f\mu)\,\omega(\tau, f\tau)\,\psi(s^{\varepsilon} d(f\mu, f\tau)) \leq \psi(N(\mu, \tau)), \tag{14}$$

for all $\mu, \tau \in X$, where $N(\mu, \tau) = \max\left\{d(\mu, \tau), d(\mu, f\mu), d(\tau, f\tau), \frac{d(\mu, f\tau) + d(\tau, f\mu)}{2s}\right\}$.

Remark 3. *The contraction (14) generalizes the corresponding ones from ([35], Definition 4.3) in several directions.*

Now, we can prove the next result.

Theorem 7. *Let $(X, d, s > 1)$ be a b-complete b-metric space, $f : X \to X$, and $\Omega, \omega : X \times X \to [0, \infty)$. Suppose that the following assertions hold:*

*(**A**) f is (Ω, ω)-admissible;*
*(**B**) f is an (Ω, ω)-contraction;*
*(**C**) there is $\mu_0 \in X$ so that $\Omega(\mu_0, f\mu_0) \geq 1$ and $\omega(\mu_0, f\mu_0) \geq 1$;*
*(**D**) either f is continuous or $(X, d, s > 1)$ is (Ω, ω)-regular.*

Then, f has a unique fixed point.

Proof. As in [35], page 5748, we get

$$\begin{aligned}
\psi(d(\mu_{n+1}, \mu_{n+2})) &= \psi(d(f\mu_n, f\mu_{n+1})) \\
&\leq \psi(s^{\varepsilon} d(f\mu_n, f\mu_{n+1})) \\
&\leq \Omega(\mu_n, f\mu_n)\,\omega(\mu_{n+1}, f\mu_{n+1})\,\psi(s^{\varepsilon} d(f\mu_n, f\mu_{n+1})) \\
&\leq \psi(N(\mu_n, \mu_{n+1})),
\end{aligned}$$

or equivalently, $s^{\varepsilon} d(f\mu_n, f\mu_{n+1}) \leq N(\mu_n, \mu_{n+1})$, where

$$\begin{aligned}
N(\mu_n, \mu_{n+1}) &= \max\left\{d(\mu_n, \mu_{n+1}), d(\mu_{n+1}, \mu_{n+2}), \frac{d(\mu_n, \mu_{n+2})}{2s}\right\} \\
&\leq \max\left\{d(\mu_n, \mu_{n+1}), d(\mu_{n+1}, \mu_{n+2}), \frac{d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_{n+2})}{2}\right\} \\
&\leq \max\{d(\mu_n, \mu_{n+1}), d(\mu_{n+1}, \mu_{n+2})\} \\
&\leq N(\mu_n, \mu_{n+1}).
\end{aligned}$$

Hence, $s^{\varepsilon} d(f\mu_n, f\mu_{n+1}) \leq \max\{d(\mu_n, \mu_{n+1}), d(\mu_{n+1}, \mu_{n+2})\}$. □

It is not hard to see that $s^{\varepsilon} d(f\mu_n, f\mu_{n+1}) \leq d(\mu_n, \mu_{n+1})$. That is,

$$d(\mu_{n+1}, \mu_{n+2}) \leq \frac{1}{s^{\varepsilon}} d(\mu_n, \mu_{n+1}) = \lambda d(\mu_n, \mu_{n+1}),$$

where $\lambda = \frac{1}{s^{\varepsilon}} < \frac{1}{s}$.

As in the proof of Theorem 6, the sequence $\{\mu_n = f^n \mu_0\}$ is b-Cauchy in b-complete b-metric space, so there is $u \in X$ so that $\mu_n \to u$ as $n \to \infty$. In the case that f is continuous, one writes

$$u = \lim_{n\to\infty} \mu_{n+1} = \lim_{n\to\infty} f\mu_n = f\left(\lim_{n\to\infty} \mu_n\right) = fu.$$

In the case that (X,d) is (Ω,ω)-regular, there is $\{\mu_{n_k}\}$ of $\{\mu_n\}$ so that $\Omega\left(\mu_{n_k+1},\mu_{n_k}\right)\geq 1$ and $\omega\left(\mu_{n_k+1},\mu_{n_k}\right)\geq 1$ for all $k\in\mathbb{N}$ and $\Omega(u,fu)\geq 1$ and $\omega(u,fu)\geq 1$. Using Equation (14) with $\mu=\mu_{n_k}$ and $\tau=u$, we have

$$\begin{aligned}\psi\left(s^{\varepsilon}d\left(f\mu_{n_k},fu\right)\right) &\leq \Omega\left(\mu_{n_k}f\mu_{n_k}\right)\omega(u,fu)\,\psi\left(s^{\varepsilon}d\left(f\mu_{n_k},fu\right)\right)\\ &\leq \psi\left(N\left(\mu_{n_k},u\right)\right).\end{aligned}$$

Consequently, $s^{\varepsilon}d\left(f\mu_{n_k},fu\right)\leq N\left(\mu_{n_k},u\right)$, where

$$\begin{aligned}N\left(\mu_{n_k},u\right) &= \max\left\{d\left(\mu_{n_k},u\right),d\left(\mu_{n_k},f\mu_{n_k}\right),d(u,fu),\frac{d\left(\mu_{n_k},fu\right)+d\left(f\mu_{n_k},u\right)}{2s}\right\}\\ &= \max\left\{d\left(\mu_{n_k},u\right),d\left(\mu_{n_k},\mu_{n_k+1}\right),d(u,fu),\frac{d\left(\mu_{n_k},fu\right)+d\left(\mu_{n_k+1},u\right)}{2s}\right\}.\end{aligned}$$

Since $\frac{1}{s}d\left(\mu_{n_k},fu\right)\leq d\left(\mu_{n_k},u\right)+d(u,fu)$, we get $N\left(\mu_{n_k},u\right)\to d(u,fu)$ as $k\to\infty$

On the other hand,

$$\frac{1}{s}d(u,fu)\leq d\left(u,\mu_{n_k+1}\right)+d\left(f\mu_{n_k},fu\right),$$

that is,

$$\frac{1}{s}d(u,fu)\leq d\left(u,\mu_{n_k+1}\right)+\frac{1}{s^{\varepsilon}}\max\left\{d\left(\mu_{n_k},u\right),d\left(\mu_{n_k},\mu_{n_k+1}\right),d(u,fu),\frac{d\left(\mu_{n_k},fu\right)+d\left(\mu_{n_k+1},u\right)}{2s}\right\},$$

i.e., $\frac{1}{s}d(u,fu)\leq\frac{1}{s^{\varepsilon}}d(u,fu)$. Since $\varepsilon>1$, the last inequality holds unless $u=fu$.

Now, suppose v is so that $fu=u\neq v=fv$. Putting $\mu=u$ and $\tau=v$ in (14),

$$\psi\left(s^{\varepsilon}d(u,v)\right)\leq\Omega(u,fu)\,\omega(v,fv)\,\psi\left(s^{\varepsilon}d(u,v)\right)\leq\psi(N(u,v)),$$

where

$$\begin{aligned}N(u,v) &= \max\left\{d(u,v),d(u,fu),d(v,fv),\frac{d(u,fv)+d(v,fu)}{2s}\right\}\\ &= \max\left\{d(u,v),0,0,\frac{d(u,v)}{s}\right\}\\ &= d(u,v).\end{aligned}$$

Hence, $\psi\left(s^{\varepsilon}d(u,v)\right)\leq\psi(d(u,v))$ is possible only if $u=v$. The proof of the result is finished. □

Remark 4. *It is not hard to check that Example 4.6 from [35] satisfies all conditions of Theorem 7 for* $\varepsilon\in(1,3]$. *Indeed, since for all* $x,y\in X$ *and for all* $\varepsilon\in(1,3]$, *it follows that*

$$\begin{aligned}\alpha(x,y)\,\beta(x,y)\,\psi\left(s^{\varepsilon}d(Tx,Ty)\right) &\leq \alpha(x,y)\,\beta(x,y)\,\psi\left(s^{3}d(Tx,Ty)\right)\\ &\leq \theta(\psi(N(x,y)))\,\psi(N(x,y)).\end{aligned}$$

That is,

$$\alpha(x,y)\,\beta(x,y)\,\psi\left(s^{\varepsilon}d(Tx,Ty)\right)\leq\theta(\psi(N(x,y)))\,\psi(N(x,y)),$$

for all $x,y\in X$ *and for all* $\varepsilon\in(1,3]$. *This means that Example 4.6. from [35] supports Theorem 7. On the other hand, Theorem 7 extends the main result from [35] of* $\{\varepsilon=3\}$ *to* $\varepsilon\in(1,3]$. *Thus, our results are genuine generalizations of ones from [35].*

In [30], the authors introduced so-called Geraghty type functions. Denote by Ψ the set of continuous increasing nonnegative functions ψ defined on $[0,\infty)$ so that $\psi^{-1}(0)=\{0\}$. Take $s\geq 1$. Let $\mathcal{F}$ be the family of all nondecreasing functions $\beta:[0,\infty)\to[0,\frac{1}{s})$ so that

$$\lim_{k\infty}\beta(l_k)=\frac{1}{s}\text{ implies }\lim_{k\to\infty}l_k=0.$$

Definition 10. *Let T be a self-mapping on a b-metric space (M,d). T is a generalized $\Omega-\psi$-Geraghty contractive mapping if there are $\Omega:M\times M\to[0,\infty)$, $\beta\in\mathcal{F}$, $\psi,\phi\in\Psi$ and $L\geq 0$ so that for*

$$E(\mu,\tau)=\max\left\{d(\mu,\tau),d(\mu,T\mu),d(\tau,T\tau),\frac{d(\mu,T\tau)+d(\tau,T\mu)}{2s}\right\}$$

and

$$N(\mu,\tau)=\min\{d(\mu,T\mu),d(\tau,T\tau)\},$$

we have

$$\Omega(\mu,\tau)\psi\left(s^3d(T\mu,T\tau)\right)\leq\beta(E(\mu,\tau))\psi(E(\mu,\tau))+L\phi(N(\mu,\tau)),\tag{15}$$

for all $\mu,\tau\in M$.

Theorem 8. *Let $(M,d,s\geq 1)$ be a b-complete b-metric space and $T:M\to M$ be a generalized $\Omega-\psi$-Geraghty contraction so that*

(i) T is triangular Ω-orbital admissible;
(ii) there is $\mu_0\in M$ so that $\Omega(\mu_0,T\mu_0)\geq 1$;
(iii) T is continuous.

Then, T has a fixed point.

Note that for the proof of the announced result in [30], the authors used Lemma 3. However, our approach does not require this lemma and the proof is much shorter. Namely, we consider the following:

$$\Omega(\mu,\tau)\psi(s^{\varepsilon}d(T\mu,T\tau))\leq\beta(E(\mu,\tau))\psi(E(\mu,\tau))+L\phi(N(\mu,\tau)),$$

where $\varepsilon>1$, instead Equation (15). On the other hand,

$$d(\mu_{n+1},\mu_n)\leq\frac{1}{s^{\varepsilon}}d(\mu_n,\mu_{n-1}),\quad n\geq 1.$$

This further implies that the sequence $\{\mu_n\}$ is b-Cauchy. The proof is now similar to its corresponding one in [30].

Remark 5. *Since $\beta([0,\infty))\subseteq[0,1)$, it is not hard to see that Equation (14) becomes*

$$\Omega(\mu,\tau)\psi(s^3d(T\mu,T\tau))\leq\psi(E(\mu,\tau))+L\phi(N(\mu,\tau),$$

that is, the Geraghty type case in b-metric spaces is superfluous.

It is worth mentioning the following:

Theorem 3 is a consequence of an old theorem of Hegedus [26]. In addition, Ćirić's Definition of quasi-contractions and Definition 2 are special cases of the following Definition of Hegedus [26].

Definition 11 ([26])**.** *A self-mapping T on a metric space X is called a generalized Banach contraction if, for all $x, y \in X, \delta(x,y) < \infty$ and $d(Tx, Ty) \leq \lambda\delta(x,y)$ for some $\lambda < 1$, where $\delta(x,y) = diam[O(x,\infty) \cup O(y,\infty)]$.*

Furthermore, Theorem 1 and Theorem 3 are special cases of the following theorem of Hegedus [26] (by omitting the approximation part of the theorem).

Theorem 9 ([26])**.** *Every generalized Banach contraction on a T-orbitally complete metric space has a unique fixed point.*

3. Conclusions

This paper contains much shorter and more elementary proof of ones given in existing literature for some mappings in the context of b-metric spaces.

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