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A Note on Type 2 w-Daehee Polynomials

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Abstract: In the paper, by virtue of the p-adic invariant integral on $\mathbb{Z}_p$, the authors consider a type 2 w-Daehee polynomials and present some properties and identities of these polynomials related with well-known special polynomials. In addition, we present some symmetric identities involving the higher order type 2 w-Daehee polynomials. These identities extend and generalize some known results.

Keywords: symmetric identity; type 2 Bernoulli polynomial; type 2 Daehee polynomials; type 2 w-Daehee polynomials; higher order type 2 w-Daehee polynomials

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1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = 1/p$.

It is common knowledge that the usual Bernoulli numbers B_n are given by the generating function to be, for $t \in \mathbb{C}_p$,

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \mathrm{B}_n \frac{t^n}{n!},$$

which can be written symbolically as $e^{\mathrm{B}t} = t/(e^t - 1)$, interpreted to mean that B^n must be replaced by B_n. In addition, usual Bernoulli polynomials $\mathrm{B}_n(x)$ are defined by, for $x \in \mathbb{C}_p$,

$$\mathrm{B}_n(x) = \sum_{l=0}^{n} \binom{n}{l} \mathrm{B}_l x^{n-l}.$$

With the viewpoint of deformed Bernoulli polynomials, the Daehee polynomials $\mathrm{D}_n(x)$ for $n \geq 0$ are defined [1] by the generating function to be , for $t, x \in \mathbb{C}_p$,

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} \mathrm{D}_n(x) \frac{t^n}{n!}.$$

When $x = 0$, we call $\mathrm{D}_n = \mathrm{D}_n(0)$ the Daehee numbers. For more information on the Bernoulli numbers $\mathrm{B}_n = \mathrm{B}_n(0)$, the Bernoulli polynomials $\mathrm{B}_n(x)$, the Daehee numbers D_n and the Daehee polynomials $\mathrm{D}_n(x)$, please refer to [1–3] and the closely related references therein.

We say that f is a uniformly differentiable function, if for a given function $f : \mathbb{Z}_p \to \mathbb{C}_p$, there exists a continuous function $F_f(x, y) \to \mathbb{C}_p$ where for all $x, y \in \mathbb{Z}_p$, $x \neq y$

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}.$$

For a uniformly differentiable function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic integral of f on $\mathbb{Z}_p$ (or the Volkenborn integral of f on $\mathbb{Z}_p$) is defined by the limit, if it exists

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N\to\infty} \frac{1}{p} \sum_{x=0}^{p^N-1} f(x)$$

(see [4–6]). Here, the p-adic Haar distribution μ is given by

$$\mu\left(a + p^N \mathbb{Z}_p\right) = \frac{1}{p^N}.$$

The application of the p-adic integral on $\mathbb{Z}_p$ is an effective way to deduce many important results for p-adic special numbers and polynomials. For more information, please refer to [1–3,7–16]. From the above definition, we can derive

$$I(f_1) = I(f) + f'(0),$$

where $f_1(x) = f(x+1)$ and $f'(0) = \frac{df(x)}{dx}|_{x=0}$.

In the recent year, Kims [11] considered the hyperbolic cosecant numbers by using p-adic integral on $\mathbb{Z}_p$, and investigated many properties on such numbers. The hyperbolic cosecant numbers are presented by p-adic integration on $\mathbb{Z}_p$, for $t, x \in \mathbb{C}_p$ with $|t|_p < p^{-1/(p-1)}$,

$$\begin{aligned} t\,\mathrm{csch}(t) &= \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu(x) \\ &= \frac{2t}{e^{2t}-1} e^t = \frac{2t}{e^t - e^{-t}} = \sum_{n=0}^{\infty} 2^n \mathrm{B}_n\left(\frac{1}{2}\right) \frac{t^n}{n!}. \end{aligned}$$

Motivated by their hyperboric cosecant numbers, they considered the type 2 Daehee polynomials by p-adic integrals on $\mathbb{Z}_p$ as follows, for $t, x, y \in \mathbb{C}_p$ with $|t|_p < p^{-1/(p-1)}$,

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{Z}_p} (1+t)^{x+2y+1} d\mu(y) &= \frac{\log(1+t)}{(1+t)^2 - 1} (1+t)^{x+1} \\ &= \sum_{n=0}^{\infty} d_n(x) \frac{t^n}{n!}, \end{aligned} \tag{1}$$

when $x = 0$, we call $d_n = d_n(0)$ the type 2 Daehee numbers. From Equation (1), we can rewrite the generating function of type 2 Daehee polynomials as follows:

$$\sum_{n=0}^{\infty} d_n(x) \frac{t^n}{n!} = \frac{\log(1+t)}{(1+t) - (1+t)^{-1}} (1+t)^x. \tag{2}$$

In the view of Equation (2), Kim et al. [10,11] considered the type 2 Bernoulli polynomials given by, for $t, x \in \mathbb{C}_p$

$$\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} = \frac{t}{e^t - e^{-t}} e^{xt}. \tag{3}$$

We can easily show that

$$b_n(x) = 2^{n-1} \mathrm{B}_n\left(\frac{x+1}{2}\right), \quad (n \geq 0).$$

The purpose of this paper is to construct a new type of polynomials, the type 2 w-Daehee polynomials, and to investigate some properties and identities of these polynomials. In addition, we will offer some symmetric identities involving the higher order type 2 w-Daehee polynomials. These identities extend and generalize some known results.

2. Some Identities on Type 2 w-Daehee Numbers and Polynomials

The type 2 Daehee polynomials are considered by Kims [11] and various properties on their polynomials are investigated. In Section 3, we want to try to present the symmetric identities of type 2 Daehee polynomials by p-adic integrals on $\mathbb{Z}_p$. On the way to establish such symmetric identities, we need the concept of type 2 w-Daehee polynomials. Thus, in this section, we want to establish some properties on the type 2 w-Daehee polynomials and numbers. Recently, we could see the nice results, which express the central numbers of the second kind in terms of type 2 Bernoulli, type 2 Changhee and type 2 Daehee numbers of negative order [11]. We might express our type 2 w-Daehee polynomials related with new central numbers in the further study (Section 3).

In this section, we assume that $t, x, y \in \mathbb{C}_p$ with $|t|_p < p^{-1/(p-1)}$ and $w \in \mathbb{N}$. In the view of Equations (2) and (3), we define type 2 w-Daehee polynomials by

$$\frac{\log(1+t)}{(1+t)^w - (1+t)^{-w}}(1+t)^x = \sum_{n=0}^{\infty} d_{n,w}(x)\frac{t^n}{n!}. \tag{4}$$

Motivated by Equation (1), we present type 2 w-Daehee polynomials via p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$\frac{1}{2w}\int_{\mathbb{Z}_p}(1+t)^{x+2wy+w}d\mu(y) = \sum_{n=0}^{\infty} d_{n,w}(x)\frac{t^n}{n!}. \tag{5}$$

The following two theorems give us the relation between type 2 Bernoulli polynomials and type 2 w-Daehee polynomials.

Theorem 1. *For $n \geq 0$, we have*

$$d_{n,w}(x) = \sum_{k=0}^{n} w^{k-1} b_k(x) s(n,k),$$

where $s(n,k)$ is the Stirling number of the first kind which is defined as

$$(x)_0 = 1, \ \ (x)_n = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k, \ \ (n \geq 1).$$

Proof. Substituting $w\log(1+t)$ for t in Equation (3) gives

$$\begin{aligned}\frac{w\log(1+t)}{(1+t)^w - (1+t)^{-w}}(1+t)^{wx} &= \sum_{k=0}^{\infty} b_k(x)\frac{(w\log(1+t))^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} w^k b_k(x)s(n,k)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing this with Equation (4) leads to the required identity. □

Especially for the $w = 1$ case, we have

Corollary 1 ([11], Theorem 2.5). *For $n \geq 0$, we have*

$$d_n(x) = \sum_{k=0}^{n} b_k(x)s(n,k).$$

Theorem 2. *For $n \geq 0$, we have*

$$b_n\left(\frac{x}{w}\right) = w^{1-n}\sum_{k=0}^{n} d_{k,w}(x)S(n,k),$$

where $S(n,k)$ for $k \geq 0$, which can be generated by

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!},$$

stands for the Stirling number of the second kind.

Proof. Replacing t by $e^t - 1$ in Equation (4), we obtain

$$\sum_{k=0}^{\infty} d_{k,w}(x)\frac{1}{k!}(e^t-1)^k = \frac{t}{e^{wt}-e^{-wt}}e^{xt}.$$

By Equation (3), it follows that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_{k,w}(x)S(n,k)\right)\frac{t^n}{n!} = \frac{1}{w}\frac{wt}{e^{wt}-e^{-wt}}e^{xt} = \sum_{n=0}^{\infty} b_n\left(\frac{x}{w}\right)\frac{w^{n-1}t^n}{n!}.$$

Equating coeffcients on the very ends of the above identity arrives at the required result. □

For the case of $w = 1$, we have the following corollary.

Corollary 2 ([11], Theorem 2.4). *For $n \geq 0$, we have*

$$b_n = \sum_{k=0}^{n} d_k(x)S(n,k).$$

Recall from [9] that the w-Daehee polynomials $\mathrm{D}_{n,w}(x)$ and the w-Changhee polynomials $\mathrm{Ch}_{n,w}(x)$ are generated by

$$\begin{aligned} \frac{\log(1+t)}{(1+t)^w-1}(1+t)^x &= \sum_{n=0}^{\infty} \mathrm{D}_{n,w}(x)\frac{t^n}{n!}, \\ \frac{\log(1+t)}{(1+t)^w+1}(1+t)^x &= \sum_{n=0}^{\infty} \mathrm{Ch}_{n,w}(x)\frac{t^n}{n!}. \end{aligned} \tag{6}$$

The following theorem gives us the relation between type 2 w-Daehee polynomials, w-Daehee and w-Changhee polynomials.

Theorem 3. *For $n \geq 0$, we have*

$$d_{n,w}(x) = \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\mathrm{D}_{l,w}(x)\mathrm{Ch}_{n-l,w}(w).$$

Proof. From Equations (4) and (6), it can be deduced that

$$\begin{aligned} \sum_{n=0}^{\infty} d_{n,w}\frac{t^n}{n!} &= \frac{\log(1+t)}{(1+t)^{2w}-1}(1+t)^{x+w} \\ &= \frac{\log(1+t)}{(1+t)^w-1}(1+t)^x\frac{1}{2}\frac{\log(1+t)}{(1+t)^w+1}(1+t)^w \\ &= \left(\sum_{l=0}^{\infty} D_{l,w}(x)\frac{t^l}{l!}\right)\left(\frac{1}{2}\sum_{m=0}^{\infty} Ch_{m,w}(w)\frac{t^m}{m!}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\mathrm{D}_{l,w}(x)\mathrm{Ch}_{n-l,w}(w)\frac{t^n}{n!}. \end{aligned}$$

The required result thus follows. □

For the case $w = 1$, we have

Corollary 3. *For $n \geq 0$, we have*

$$\begin{aligned} d_n(x) &= \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\mathrm{D}_l(x)\mathrm{Ch}_{n-l}(1) \\ &= \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\mathrm{D}_l(1)\mathrm{Ch}_{n-l}(x). \end{aligned}$$

If we replace t by $e^t - 1$ in Equation (3),

$$\begin{aligned} \sum_{n=0}^{\infty} d_{n,w}\frac{(e^t-1)^n}{n!} &= \frac{t}{e^{2wt}-1}e^{(x+w)t} \\ &= \frac{t}{e^{wt}-1}e^{xt}\frac{2}{e^{wt}+1}e^{wt} \\ &= \left(\sum_{l=0}^{\infty}\frac{1}{w}\mathrm{B}_{l,w}\left(\frac{x}{w}\right)\frac{t^l}{l!}\right)\left(\frac{1}{2}\sum_{m=0}^{\infty}\mathrm{E}_m(1)\frac{w^m t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\binom{n}{l}\frac{w^{n-l}}{2}\mathrm{B}_{l,w}\left(\frac{x}{w}\right)\mathrm{E}_{n-l}(1)\right]\frac{t^n}{n!}. \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} S(n,k)d_{k,n}(x) = \frac{w^{n-k}}{2}\sum_{k=0}^{n}\binom{n}{k}\mathrm{B}_{k,w}\left(\frac{x}{w}\right)\mathrm{E}_{n-k}(1).$$

For the case $w = 1$, we have the following result.

Corollary 4. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} S(n,k)d_k(x) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathrm{B}_k(x)\mathrm{E}_{n-k}(1).$$

For $g \in \mathbb{N}$, the distribution relation on p-adic integrals on $\mathbb{Z}_p$ is well-known as follows.

Theorem 5. *For $g \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} f(x)d_\mu(x) = \frac{1}{g}\sum_{a=0}^{g-1}\int_{\mathbb{Z}_p} f(a+gx)d_\mu(x).$$

We apply the above theorem to the p-adic representation of type 2 w-Daehee polynomials, we have the following identities.

Theorem 6. *For $g \in \mathbb{N}$, we have*

$$\begin{aligned} &(1) \quad d_{n,w}(x) = \sum_{a=0}^{g-1} d_{n,wg}(x - wg + 2wa + w), \\ &(2) \quad d_{n,w}(x) = \sum_{k=0}^{n}(wg)^{k-1}\sum_{a=0}^{g-1} b_k\left(\frac{2wa+x+w}{wg} - 1\right)s(n,k). \end{aligned}$$

Proof. (1) From Equation (5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,w}(x)\frac{t^n}{n!} &= \frac{1}{2w}\int_{\mathbb{Z}_p}(1+t)^{2wy+x+w}d_\mu(y)\\ &= \frac{1}{2wg}\sum_{a=0}^{g-1}\int_{\mathbb{Z}_p}(1+t)^{2w(a+gy)+x+w}d_\mu(y)\\ &= \sum_{a=0}^{g-1}\frac{1}{2wg}\int_{\mathbb{Z}_p}(1+t)^{2wgy+x+wg+2wa+w-wg}d_\mu(y)\\ &= \sum_{a=0}^{g-1}\sum_{n=0}^{\infty} d_{n,wg}(x+2wa+w-wf)\frac{t^n}{n!}.\end{aligned}$$

The required relation now follows with comparing the coefficients of t^n on both sides.

(2) Similarly, we consider

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,w}(x)\frac{t^n}{n!} &= \frac{1}{2w}\int_{\mathbb{Z}_p}(1+t)^{2wy+x+w}d_\mu(y)\\ &= \frac{1}{2wg}\sum_{a=0}^{g-1}\int_{\mathbb{Z}_p}e^{2wgy\log(1+t)}e^{(2wa+x+w)\log(1+t)}d_\mu(y)\\ &= \frac{1}{2wg}\sum_{a=0}^{g-1}e^{\frac{2wg\log(1+t)}{2wg\log(1+t)-1}}e^{(2wa+x+w)\log(1+t)}\\ &= \frac{1}{wg}\sum_{a=0}^{g-1}e^{\frac{wg\log(1+t)}{wg\log(1+t)-e^{-wg\log(1+t)}}}e^{\left(\frac{2wa+x+w}{wg}-1\right)wg\log(1+t)}\\ &= \frac{1}{wg}\sum_{a=0}^{g-1}\sum_{k=0}^{\infty} b_k\left(\frac{2wa+x+w}{wg}-1\right)\frac{(wg)^k(\log(1+t))^k}{k!}\\ &= \sum_{n=0}^{\infty}\left((wg)^{k-1}\sum_{k=0}^{\infty} b_k\left(\frac{2wa+x+w}{wg}-1\right)s(n,k)\right)\frac{t^n}{n!},\end{aligned}$$

which immediately gives the required result. □

For the case $w=1$ in Theorem 6, we have the following corollary.

Corollary 5. *For $g\in\mathbb{N}$, we have*

$$\begin{aligned}&(1)\quad d_n(x) = \sum_{a=0}^{g-1} d_{n,g}(x-g+2a+1),\\ &(2)\quad d_{n,w}(x) = \sum_{k=0}^{n}\sum_{a=0}^{g-1} g^{k-1}b_k\left(\frac{2a+x+1}{g}-1\right)s(n,k).\end{aligned}$$

3. Symmetric Identities of Higher Order Type 2 w-Daehee Polynomials

In order to study symmetric identities related to type 2 w-Daehee numbers, we need to introduce type 2 w-Daehee polynomials with order $\alpha\in\mathbb{R}$, $d_{n,w}^{(\alpha)}(x)$ as follows, for $t,x\in\mathbb{C}_p$ with $|t|_p<p^{-1/(p-1)}$ and $w\in\mathbb{N}$,

$$\sum_{n=0}^{\infty} d_{n,w}^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{\log(1+t)}{(1+t)^{2w}-1}\right)^{\alpha}(1+t)^{x+w\alpha} = \left(\frac{\log(1+t)}{(1+t)^{w}-(1+t)^{-w}}\right)^{\alpha}(1+t)^{x}. \quad (7)$$

When $x = 0$, $d_{n,w}^{(\alpha)} = d_{n,w}^{(\alpha)}(0)$ are called type 2 w-Daehee numbers of order α. In particular for $w = 1$ and $\alpha = 1$, we have type 2 Daehee numbers in Equation (2) $d_n = d_{n,1}^{(1)}$. For the proof of main theorem, we consider the quotient of p-adic integrals

$$\begin{aligned}
&\frac{n\int_{\mathbb{Z}_p}(1+t)^{2x+1}d\mu(x)}{\int_{\mathbb{Z}_p}(1+t)^{2nx}d\mu(x)} \\
&= \frac{1}{2(1+t)\log(1+t)}\left[\int_{\mathbb{Z}_p}(1+t)^{2x+2n+1}d\mu(x) - \int_{\mathbb{Z}_p}(1+t)^{2x+1}d\mu(x)\right] \\
&= \sum_{l=0}^{n-1}(1+t)^{2l} = \sum_{k=0}^{\infty}\sum_{l=0}^{n-1}(2l)_k\frac{t^k}{k!} = \sum_{k=0}^{\infty}2^k\mathrm{T}_k(n-1 \mid (l)_{k,\frac{1}{2}})\frac{t^k}{k!},
\end{aligned} \tag{8}$$

where the function $\mathrm{T}_k(n \mid (l)_{n,\lambda}) = \sum_{l=0}^{n}(l)_{k,\lambda}$ for each $\lambda \in \mathbb{R}$. In addition, $(l)_{n,\lambda}$ is the well-known λ-falling factorial

$$(l)_{n,\lambda} = \begin{cases} l(l-\lambda)(l-2\lambda)\cdots(l-(n-1)\lambda), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

Now, we start out to state and prove our main results.

Theorem 7. *For $w_1, w_2 \in \mathbb{N}$, $n \geq 0$ and $m \geq 1$, one has*

$$\begin{aligned}
&w_2\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}\binom{j}{k}d_{n-j,w_1}^{(m)}(2w_1w_2x)w_2^k\mathrm{T}_k(w_1-1 \mid (l+1)_{k,\frac{1}{2w}})d_{j-k,w_2}^{(m-1)}(2w_1w_2y) \\
&= w_1\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}\binom{j}{k}d_{n-j,w_2}^{(m)}(2w_1w_2x)w_1^k\mathrm{T}_k(w_2-1 \mid (l+1)_{k,\frac{1}{2w}})d_{j-k,w_1}^{(m-1)}(2w_1w_2y).
\end{aligned}$$

Proof. Consider $\mathrm{T}_m(w_1, w_2)$ as

$$\begin{aligned}
\mathrm{T}_m(w_1,w_2) = &\left(\frac{2w_1\log(1+t)}{(1+t)^{2w_1}-1}\right)^m(1+t)^{2w_1w_2x+mw_1}\frac{(1+t)^{2w_1w_2}-1}{2w_1w_2(1+t)\log(1+t)} \\
&\times\left(\frac{2w_2\log(1+t)}{(1+t)^{2w_2}-1}\right)^m(1+t)^{2w_1w_2y+mw_2}.
\end{aligned} \tag{9}$$

It is clear that $\mathrm{T}_m(w_1, w_2)$ is symmetric in w_1 and w_2, i.e., $\mathrm{T}_m(w_1, w_2) = \mathrm{T}_m(w_2, w_1)$. The above equation can be rewritten as the quotient of p-adic integral form

$$\begin{aligned}
\mathrm{T}_m(w_1,w_2) = &\frac{\int_{\mathbb{Z}_p^m}(1+t)^{2w_1(x_1+x_2+\cdots+x_m+w_2x)+mw_1}d\mu(x_1)d\mu(x_2)\cdots d\mu(x_m)}{\int_{\mathbb{Z}_p}(1+t)^{2w_1w_2x}d\mu(x)} \\
&\times\int_{\mathbb{Z}_p^m}(1+t)^{2w_2(x_1+x_2+\cdots+x_m+w_1y)+mw_2}d\mu(x_1)d\mu(x_2)\cdots d\mu(x_m),
\end{aligned} \tag{10}$$

where $\int_{\mathbb{Z}_p^m}f(x_1+x_2+\cdots+x_m)d\mu(x_1)d\mu(x_2)\cdots d\mu(x_m) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}f(x_1+x_2+\cdots+x_m)d\mu(x_1)d\mu(x_2)\cdots d\mu(x_m)$.

Accordingly, by virtue of Equations (8) and (10), we can represent as follows:

$$\begin{aligned}
&\mathrm{T}_m(w_1,w_2)\\
&=\left(\sum_{l=0}^{\infty}(2w_1)^m d_{l,w_1}^{(m)}(2w_1w_2x)\frac{t^l}{l!}\right)\left(\frac{1}{w_1}\sum_{k=0}^{\infty}(2w_2)^k\mathrm{T}_k(w_1-1\mid(l+1)_{k,\frac{1}{2w_2}})\frac{t^k}{k!}\right)\\
&\quad\times\left(\sum_{i=0}^{\infty}(2w_2)^m d_{i,w_2}^{(m-1)}(2w_1w_2y)\frac{t^i}{i!}\right)\\
&=\left(\sum_{l=0}^{\infty}(2w_1)^m d_{l,w_1}^{(m)}(2w_1w_2x)\frac{t^l}{l!}\right)\\
&\quad\times\left(\frac{1}{w_1}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}(2w_2)^k\mathrm{T}_k(w_1-1\mid(l+1)_{k,\frac{1}{2w_2}})(2w_2)^m d_{j-k,w_2}^{(m-1)}(2w_1w_2y)\right)\frac{t^j}{j!}\right) \qquad (11)\\
&=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\binom{n}{j}(2w_1)^m d_{n-j,w_1}^{(m)}(2w_1w_2x)\right.\\
&\quad\left.\times\frac{1}{w_1}\sum_{k=0}^{j}\binom{j}{k}\mathrm{T}_k(w_1-1)(2w_2)^m d_{j-k,w_2}^{(m-1)}(2w_1w_2y)\right]\frac{t^n}{n!}\\
&=4^m w_1^{m-1}w_2^m\\
&\quad\times\sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}d_{n-j,w_1}^{(m)}(2w_1w_2x)(2w_2)^k\mathrm{T}_k(w_1-1\mid(l+1)_{k,\frac{1}{2w_2}})d_{j-k,w_2}^{(m-1)}(2w_1w_2y)\right]\frac{t^n}{n!}.
\end{aligned}$$

By the symmetry of w_1 and w_2 in $\mathrm{T}_m(w_1,w_2)$, we obtain the following expression:

$$\begin{aligned}
\mathrm{T}_m(w_1,w_2)=4^m w_1^m w_2^{m-1}\sum_{n=0}^{\infty}&\left[\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}d_{n-j,w_2}^{(m)}(2w_1w_2x)\right.\\
&\left.\times(2w_1)^k\mathrm{T}_k(w_2-1\mid(l+1)_{k,\frac{1}{2w_1}})d_{j-k,w_1}^{(m-1)}(2w_1w_2y)\right]\frac{t^n}{n!}.
\end{aligned}$$

Combinig this with Equation (11) yields the required identity. □

Letting $y=0$ and $m=1$ in Theorem 7 results in

Corollary 6. *For $w_1,w_2\in\mathbb{N}$, $n\geq 0$, one has*

$$w_2\sum_{j=0}^{n}\binom{n}{j}d_{n-j,w_1}(2w_1w_2x)\mathrm{T}_j(w_1-1)=w_1\sum_{j=0}^{n}\binom{n}{j}d_{n-j,w_2}(2w_1w_2x)w_1^j\mathrm{T}_j(w_2-1\mid(l+1)_{j,\frac{1}{2w_1}}).$$

Let us take $w_2=1$ in Corollary 6. Then, we have

Corollary 7. *For $w_1\in\mathbb{N}$, $n\geq 0$, one has*

$$d_n(2w_1x)=\sum_{j=0}^{n}\binom{n}{j}d_{n-j,w_1}(2w_1x)\mathrm{T}_j(w_1-1\mid(l+1)_{j,\frac{1}{2}}).$$

Next, we consider the symmetric identities of higher order type 2 w-Daehee polynomials via generating function in the following theorem.

Theorem 8. *For $w_1,w_2\in\mathbb{N}$, $n\geq 0$, $m\geq 1$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{l=0}^{w_1-1}\binom{n}{k}d_{k,w_1}^{(m)}(2w_1w_2x+2w_2l)d_{n-k}^{(m-1)}(2w_1w_2y+w_2)\\
&\qquad=\sum_{k=0}^{n}\sum_{l=0}^{w_2-1}\binom{n}{k}d_{k,w_2}^{(m)}(2w_1w_2x+2w_1l)d_{n-k}^{(m-1)}(2w_1w_2y+w_1).
\end{aligned}$$

Proof. It follows from (9) that

$$
\begin{aligned}
& \mathrm{T}_m(w_1, w_2) \\
&= \left(\frac{2w_1 \log(1+t)}{(1+t)^{2w_1}-1}\right)^m (1+t)^{2w_1w_2x+w_1m} \frac{1}{w_1} \sum_{l=0}^{w_1-1} (1+t)^{2w_2 l} \\
&\times \left(\frac{2w_2 \log(1+t)}{(1+t)^{2w_2}-1}\right)^{m-1} (1+t)^{2w_1w_2y+w_2m} \\
&= \left(\frac{2w_1 \log(1+t)}{(1+t)^{w_1}-(1+t)^{-w_1}}\right)^m \frac{1}{w_1} \sum_{l=0}^{w_1-1} (1+t)^{2w_1w_2x+2w_2l} \\
&\times \left(\frac{2w_2 \log(1+t)}{(1+t)^{w_2}-(1+t)^{-w_2}}\right)^{m-1} (1+t)^{2w_1w_2y+w_2} \\
&= 2^{2m-1}(w_1w_2)^{m-1} \sum_{l=0}^{w_1-1} \sum_{k=0}^{\infty} d_{k,w_1}^{(m)}(2w_1w_2x+2w_2l)\frac{t^k}{k!} \\
&\times \sum_{j=0}^{\infty} d_{n-k}^{(m-1)}(2w_1w_2y+w_2)\frac{t^j}{j!} \\
&= 2^{2m-1}(w_1w_2)^{m-1} \\
&\times \sum_{n=0}^{\infty}\left[\sum_{l=0}^{w_1-1}\sum_{k=0}^{n}\binom{n}{k} d_{k,w_1}^{(m)}(2w_1w_2x+2w_2l)d_{n-k}^{(m-1)}(2w_1w_2y+w_2)\right]\frac{t^n}{n!}.
\end{aligned}
\tag{12}
$$

Furthermore, we observe that

$$
\begin{aligned}
\mathrm{T}_m(w_1, w_2) &= 2^{2m-1}(w_1w_2)^{m-1} \\
&\times \sum_{n=0}^{\infty}\left[\sum_{l=0}^{w_2-1}\sum_{k=0}^{n}\binom{n}{k} d_{k,w_2}^{(m)}(2w_1w_2x+2w_1l)d_{n-k}^{(m-1)}(2w_1w_2y+w_1)\right]\frac{t^n}{n!}.
\end{aligned}
$$

Combination of this identity with Equation (12) leads to the required identity. □

Let $y = 0$ and $m = 1$ in Theorem 8. Then, we have the following symmetric identities for type 2 w-Daehee polynomials.

Corollary 8. *For $w_1, w_2 \in \mathbb{N}$, $n \geq 0$, we have*

$$
\sum_{l=0}^{w_1-1} d_{n,w_1}(2w_1w_2x+2w_2l) = \sum_{l=0}^{w_2-1} d_{n,w_2}(2w_1w_2x+2w_1l).
$$

Finally, taking $w_2 = 1$ in Corollary 8 leads to

Corollary 9. *For $w_1 \in \mathbb{N}$, $n \geq 0$, one has*

$$
d_n(2w_1x) = \sum_{l=0}^{w_1-1} d_{n,w_1}(2w_1x+2l).
$$

Now, we want to provide some other properties of type 2 w-Daehee numbers related with central factorial numbers of the second kind and the type 2 Bernoulli numbers of order α.

For $n \geq 0$, the central factorial is defined as

$$
x^{[0]} = 1,\ \ x^{[n]} = x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-2\right)\cdots\left(x-\frac{n}{2}+1\right),\ \ (n \geq 1).
$$

Then, the central factorial numbers of the second kind are given by, for $n \geq 1$

$$x^n = \sum_{k=0}^{n} \mathrm{T}(n,k)x^{[k]}.$$

Recall from [11] that the type 2 Bernoulli polynomials of order α are generated as follows, for $t, x \in \mathbb{C}_p$

$$\left(\frac{t}{e^t - e^{-t}}\right)^{\alpha} = \sum_{n=0}^{\infty} b_n^{(\alpha)}(x)\frac{t^n}{n!}.$$

Proposition 1 ([11])**.** *For $k \geq 0$, we have*

$$(1) \qquad \frac{1}{k!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k = \sum_{n=k}^{\infty} \mathrm{T}(n,k)\frac{t^n}{n!}$$

$$(2) \qquad 2^{n+k}\mathrm{T}(n+k,k) = \binom{n+k}{k} b_n^{(-k)}.$$

Let us take $\alpha = -k \in \mathbb{Z}$ and $x = 0$ in (7), and replacing t by $e^{t/2w} - 1$ leads to

$$\begin{aligned}\left(\frac{\frac{t}{2w}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{-k} &= \sum_{n=0}^{\infty} d_l^{(-k)} \frac{(e^{\frac{t}{2w}} - 1)^l}{l!} \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{(2w)^n}\sum_{l=0}^{n} d_{l,w}^{(-k)} S_2(n,l)\right]\frac{t^n}{n!}.\end{aligned}$$

On the other hand, the left-hand side of the above equation, by Proposition 1, can be presented by the central factorial numbers of the second kind as follows:

$$\left(\frac{2w}{t}\right)^k \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k = w^k 2^k \sum_{n=0}^{\infty} \mathrm{T}(n+k,k)\frac{1}{\binom{n+k}{k}}\frac{t^n}{n!}.$$

Therefore, we obtain the following property:

$$w^k 2^k \sum_{n=0}^{\infty} \mathrm{T}(n+k,k) = \binom{n+k}{k} \sum_{l=0}^{n} d_{l,w}^{(-k)} S_2(n,l).$$

In addition, the application of Proposition 1 to this gives us

$$2^n w^n b_n^{(-k)} = \sum_{l=0}^{n} d_{l,w}^{(-k)} S_2(n,l).$$

We can summarize these results as follows:

Theorem 9. *For $n, k \geq 0$, we have*

$$(1) \quad (2w)^{n+k}\mathrm{T}(n+k,k) = \binom{n+k}{k}\sum_{l=0}^{n} d_{l,w}^{(-k)} S_2(n,l),$$

$$(2) \quad (2w)^n b_n^{(-k)} = \sum_{l=0}^{n} d_{l,w}^{(-k)} S_2(n,l).$$

4. Discussion

For the case of $w = 1$, $w = \frac{1}{2}$ and $w = \frac{1}{4}$, the symmetry of the type 2 w-Daehee polynomials are related to the works of the type 2 Daehee polynomials, those of well-known Daehee polynomials [1], and we can modify relate to those of the Catalan Daehee polynomials in [9], respectively. Recently, many works are done on some identities of special polynomials in the viewpoint of degenerate sense [12,13]. Our new type 2 w-Daehee polynomials could also be developed in other directions, i.e, on the symmetric identities of the degenerate type 2 w-Daehee polynomials.

Finally, we remark that our work on symmetry of two variables could be extended to the three variable case.

5. Conclusions

In this paper, we have defined the type 2 w-Daehee polynomials and numbers by the generating function, for $t, x \in \mathbb{Z}_p$ and $w \in \mathbb{N}$

$$\frac{\log(1+t)}{(1+t)^w - (1+t)^{-w}}(1+t)^x = \sum_{n=0}^{\infty} d_{n,w}(x)\frac{t^n}{n!}.$$

These are motivated from the pursuit of the symmetric properties of the type 2 Daehee polynomials and numbers, which are defined and investigated by Kims [11]. Our type 2 w-Daehee polynomials are related with λ-Daehee polynomials in [17] and also Catalan Daehee polynomials [9].

We obtained two relations between type 2 Bernoulli polynomials and type 2 w-Daehee polynomials in Theorems 1 and 2. In the Theorem 3, we gave the relationship between type 2 w-Daehee polynomials with w-Daehee and w-Changhee polynomials. After that, we relate type 2 w-Daehee polynomials with w-Bernoulli polynomials and usual Euler polynomials in Theorem 4. In addition, in Theorem 6, we have the distribution relation of type 2 w-Daehee polynomials. In Section 3, we gave symmetric identities involving the type 2 w-Daehee polynomials, which are derived from the p-adic invariant integral on $\mathbb{Z}_p$. In addition, we expressed our type 2 w-Daehee polynomials related with new central numbers.

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