mathematics

MDPI

Article

Viscosity Methods and Split Common Fixed Point Problems for Demicontractive Mappings

Yaqin Wang [1], Xiaoli Fang [1] and Tae-Hwa Kim [2,*]

[1] Department of Mathematics, Shaoxing University, Shaoxing 312000, China

[2] Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 48513, Korea

* Correspondence: taehwa@pknu.ac.kr

Received: 18 July 2019; Accepted: 7 September 2019; Published: 12 September 2019

Abstract: We, first, propose a new method for solving split common fixed point problems for demicontractive mappings in Hilbert spaces, and then establish the strong convergence of such an algorithm, which extends the Halpern type algorithm studied by Wang and Xu to a viscosity iteration. Above all, the step sizes in this algorithm are chosen without a priori knowledge of the operator norms.

Keywords: split common fixed-point problem; strong convergence; demicontractive mapping; viscosity method

MSC: 47H05; 47H09; 47H20

1. Introduction

In recent years, the split common fixed point problem (SCFPP) has attracted more and more attention [1–7] due to its applications in many areas, such as intensity-modulated radiation therapy, image reconstruction, signal processing, modeling inverse problems, and electron microscopy. The SCFPP is defined as finding a point in one fixed point set, and its image is in another fixed point set under a linear transformation. Specifically, assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces and an operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded and linear. The SCFPP is to find

$$x \in F(U) \ \text{ and } \ Ax \in F(T), \tag{1}$$

where the mappings $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ are nonlinear, and $F(T)$ and $F(U)$ stand for the sets of all fixed points of T and U, respectively. Especially, if U and T are both orthogonal projections, the SCFPP (1) becomes the split feasibility problem (SFP) [8], which can be formulated as:

$$x \in C \ \text{ and } \ Ax \in Q, \tag{2}$$

where the sets $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ are nonempty closed and convex, and A is as (1).

The SCFPP (1) was firstly studied by Censor and Segal [1]. Noting that p is a solution to the SCFPP (1) if the fixed-point equation below holds

$$p = U(p - \tau A^*(I - T)Ap), \ \tau > 0.$$

To solve the SCFPP (1), Censor and Segal [1] introduced the following iterative scheme. For any initial point $x_1 \in \mathcal{H}_1$, define $\{x_n\}$ recursively by

$$x_{n+1} = U(x_n - \gamma A^*(I - T)Ax_n), \tag{3}$$

where U and T are directed operators, $\gamma \in (0, 2/\|A\|^2)$; they show that the sequence generated by (3) is weakly convergent to a solution of (1). Subsequently, this result was extended to the cases of quasi-nonexpansive mappings [3] and demicontractive operators [9], but the sequence $\{x_n\}$ is still weakly convergent to a point of the SCFPP (1).

Though the difficulty occurs when one implements the algorithm (3) because its step size is linked with the computation of the operator norm $\|A\|$, alternative ways of constructing variable step sizes have been considered to surmount the difficulty (see the works by the authors of [5,6,10]). Of these step sizes, Wang and Xu [10] suggested the following one,

$$\tau_n = \frac{\rho_n}{\|x_n - Ux_n + A^*(I-T)Ax_n\|},$$

where $\{\rho_n\} \subset (0, +\infty)$ is a sequence of real numbers satisfying

$$\sum_{k=0}^{\infty} \rho_k = \infty \text{ and } \sum_{k=0}^{\infty} \rho_k^2 < \infty, \tag{4}$$

and they introduced the following iterative Algorithm 1.

Algorithm 1. *Step 1. Choose an anchor, $u \in \mathcal{H}_1$, and initial guess, $x_0 \in \mathcal{H}_1$, arbitrarily.*

Step 2. If

$$\|x_n - Ux_n + A^*(I-T)Ax_n\| = 0,$$

then stop, and x_n is a solution of (1); otherwise, go on to the next step.

Step 3. Update x_{n+1} via the iteration formula:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)[x_n - \tau_n(x_n - Ux_n + A^*(I-T)Ax_n)],$$

where the step size τ_n is chosen as

$$\tau_n = \frac{\rho_n}{\|x_n - Ux_n + A^*(I-T)Ax_n\|}$$

with $\{\rho_n\}$ satisfying (4), and return to Step 2.

Under suitable conditions they obtained a strong convergence result for Algorithm 1.

In addition, following the idea of Attouch [11], in Hilbert spaces, Moudafi [12], first, proposed a viscosity approximation iteration for nonexpansive mappings.

Inspired by the above works, we naturally raise the following question. Can we carry the strong convergence of the SCFPP (1) (Algorithm 1) for nonexpansive mappings due to Wang and Xu [10] over the one of "a viscosity method" for more general "demicontractive mappings"? In this work, we shall give a positive answer for the question.

2. Preliminaries

In this section, assume that $\mathcal{H}$ is a real Hilbert space and $\mathcal{R}$ denotes the set of all real numbers. $\{x_n\} \subset \mathcal{H}$ and $x \in \mathcal{H}$, $x_n \to x$ ($x_n \rightharpoonup x$) denote the strong (weak) convergence of $\{x_n\}$. Let $S : \mathcal{H} \to \mathcal{H}$ be a mapping. The set of all fixed points of S is denote by $F(S)$.

Definition 1. *$S : \mathcal{H} \to \mathcal{H}$ is denoted as*

(i) contractive if there exists $\kappa \in (0, 1)$ such that

$$\|Sy - Sz\| \leq \kappa\|y - z\|, \ \ \forall\, y, z \in \mathcal{H};$$

(ii) *nonexpansive if*

$$\|Sy - Sz\| \leq \|y - z\|, \ \forall\, y, z \in \mathcal{H};$$

(iii) *quasi-nonexpansive if* $F(S) \neq \emptyset$ *and*

$$\|Sy - p\| \leq \|y - p\|, \ \forall\, y \in \mathcal{H}, p \in F(S);$$

(iv) *firmly nonexpansive if*

$$\|Sy - Sz\|^2 \leq \langle y - z, Sy - Sz\rangle, \ \forall\, y, z \in \mathcal{H};$$

(v) *directed if* $F(S) \neq \emptyset$ *and*

$$\|Sy - p\|^2 \leq \|y - p\|^2 - \|y - Sy\|^2, \ \forall\, y \in \mathcal{H}, p \in F(S);$$

(vi) τ*-demicontractive if* $F(S) \neq \emptyset$ *and there exists* $\tau \in (-\infty, 1)$*, such that*

$$\|Sy - p\|^2 \leq \|y - p\|^2 + \tau\|y - Sy\|^2, \ \forall\, y \in \mathcal{H}, p \in F(S),$$

which can also be written as

$$\langle y - Sy, y - p\rangle \geq \frac{1-\tau}{2}\|y - Sy\|^2. \tag{5}$$

Obviously, if S is a quasi-nonexpansive mapping or a directed mapping, then S is demicontractive.

Remark 1. *Note that every 0-demicontractive mapping is quasi-nonexpansive. If* $0 \leq \tau < 1$*, it is also said to be quasi-strictly pseudo-contractive [13]. Moreover, if* $\tau \leq 0$*, every* τ*-demicontractive mapping is quasi-nonexpansive. Thus, we only take* $\tau \in (0, 1)$ *in (vi) of Definition 1. However, from (v) of Definition 1, if* $\tau = -1$*, every directed operator is demicontractive.*

Define an orthogonal projection $P_C : \mathcal{H} \to C$ as follows,

$$P_C u := argmin_{v \in C}\{\|u - v\|\}, \ u \in \mathcal{H}.$$

It is known that P_C is firmly nonexpansive and has the following property [14,15].

$$\langle u - P_C u, v - P_C u\rangle \leq 0, \ \ v \in C. \tag{6}$$

The mapping $I - S$ is said to be demiclosed at 0, if for any $\{y_k\} \subset \mathcal{H}$ and $y^* \in \mathcal{H}$, we obtain

$$\left.\begin{array}{l} y_k \rightharpoonup y^* \\ (I - S)y_k \to 0 \end{array}\right\} \Rightarrow y^* = Sy^*.$$

In uniformly convex Banach spaces, Goebel and Kirk [16] presented a case of the demicloseness principle; especially, if $\mathcal{H}$ is a Hilbert space, $C \subset \mathcal{H}$ is nonempty, closed, and convex, and if $S : C \to \mathcal{H}$ is nonexpansive, then $I - S$ is demiclosed on C. Naturally, we want to know whether $I - S$ is still demiclosed on C if $S : C \to \mathcal{H}$ is quasi-nonexpansive. The following example shows that the conclusion is not true.

Example 1 (see Example 2.11 [17])**.** *The mapping* $S : [0, 1] \to [0, 1]$ *is defined by*

$$Sx = \begin{cases} \frac{x}{5}, \ \ x \in [0, \frac{1}{2}], \\ x \sin \pi x, \ \ x \in (\frac{1}{2}, 1]. \end{cases}$$

Then S is quasi-nonexpansive, but $I - S$ is not demiclosed at 0.

Remark 2. *Note that there exist some demicontractive mappings which are demiclosed at 0, for instant, for $k > 1$, we take $\mathcal{H} = \ell_2$ and $Sx = -kx, \ \forall\, x \in \ell_2$ ([17]; see Example 2.5). Then S is τ-demicontactive but not quasi-nonexpansive, where $\tau := \frac{k-1}{k+1}$. However, $I - S$ is demiclosed at 0. In fact, assume that $\{x_n\}$ is any sequence in ℓ_2 such that $x_n \rightharpoonup x \in \ell_2$ and $\|x_n - Sx_n\| \to 0$, we can get $x = 0 \in F(S)$.*

Next, for making the convergence analysis of our algorithm, we give some lemmas as follows.

Lemma 1 ([18]). *Assume that $\{c_n\}$ is a sequence of non-negative real numbers, such that*

$$c_{n+1} \leq (1 - \gamma_n)c_n + \gamma_n \delta_n + \epsilon_n, \ n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathcal{R}$, such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\sum_{n=0}^{\infty} \epsilon_n < \infty$;
(iii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\Sigma_{n=1}^{\infty} \gamma_n |\delta_n| < \infty$.

Then $\lim_{n\to\infty} c_n = 0$.

Lemma 2 ([13]). *Assume C is a closed convex subset of a Hilbert space $\mathcal{H}$. Let S be a self-mapping of C. If S is τ-demicontractive (which is also said to be τ-quasi-strict pseudo-contractive in the work by the authors of [13]), then $F(S)$ is closed and convex.*

Lemma 3 ([19]). *The demiclosedness principle of nonexpansive mappings. If $V : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping, then $I - V$ is demiclosed at 0.*

3. Main Results

Unless other specified, we always assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces. Let $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be τ-demicontractive and v-demicontractive, respectively. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction with constant $\alpha \in (0, 1/\sqrt{2})$. Let an operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded and linear, and A^* be the adjoint of A.

Let Ω denote the solution set of the SCFPP (1), i.e.,

$$\Omega = \{p : p \in F(U) \text{ and } Ap \in F(T)\} = F(U) \cap A^{-1}(F(T)).$$

Throughout this section, assume $\Omega \neq \emptyset$.

Algorithm 2. *Step 1: Choose an initial guess $x_0 \in \mathcal{H}_1$ arbitrarily.*

Step 2: If

$$\|x_n - Ux_n + A^*(I - T)Ax_n\| = 0,$$

then stop, and x_n is a solution to the problem (1); otherwise, go on to the next step.

Step 3: Update x_{n+1} by the iteration formula

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)[x_n - \tau_n(x_n - Ux_n + A^*(I - T)Ax_n)],$$

where the step size τ_n is chosen as

$$\tau_n = \frac{\rho_n}{\|x_n - Ux_n + A^*(I - T)Ax_n\|},$$

and return to Step 2.

The following lemma of Yao et al. [6] and the proof will be included for the sake of convenience.

Lemma 4. *p solves (1) if and only if $\|p - Up + A^*(I-T)Ap\| = 0$.*

Proof. If p solves (1), then $p = Up$ and $(I-T)Ap = 0$. Obviously $\|p - Up + A^*(I-T)Ap\| = 0$.
Conversely, if $\|p - Up + A^*(I-T)Ap\| = 0$, then for any $z \in \Omega$, we obtain

$$\begin{aligned} 0 &= \|p - Up + A^*(I-T)Ap\|\|p-z\| \\ &\geq \langle p - Up + A^*(I-T)Ap, p - z\rangle \\ &= \langle p - Up, p - z\rangle + \langle A^*(I-T)Ap, p - z\rangle \\ &= \langle p - Up, p - z\rangle + \langle (I-T)Ap, Ap - Az\rangle. \end{aligned} \tag{7}$$

Since U and T are demicontractive, by Equation (5), we obtain

$$\langle p - Up, p - z\rangle \geq \frac{1-\tau}{2}\|p - Up\|^2, \tag{8}$$

$$\langle (I-T)Ap, Ap - Az\rangle \geq \frac{1-v}{2}\|(I-T)Ap\|^2. \tag{9}$$

Then, by Equations (7)–(9), we get

$$\begin{aligned} 0 &\geq \langle p - Up, p - z\rangle + \langle (I-T)Ap, Ap - Az\rangle \\ &\geq \frac{1-\tau}{2}\|p - Up\|^2 + \frac{1-v}{2}\|(I-T)Ap\|^2. \end{aligned} \tag{10}$$

Since $\tau, v \in (0,1)$, we deduce $p \in F(U)$ and $Ap \in F(T)$ by (10). Therefore, p solves the problem (1), completing the proof. □

Lemma 4 implies that Algorithm 2 generally generates an infinite sequence $\{x_n\}$. Otherwise, the algorithm terminates in a finite number of iterations and a solution is found.

Lemma 5. *Suppose $\{x_n\}$ is a bounded sequence, such that*

$$\lim_{n\to\infty} \|x_n - Ux_n + A^*(I-T)x_n\| = 0.$$

Then, $\lim_{n\to\infty}\|x_n - Ux_n\| = 0$ and $\lim_{n\to\infty}\|(I-T)Ax_n\| = 0$.

Proof. Set $y_n = x_n - Ux_n + A^*(I-T)Ax_n$. For any $z \in \Omega$ we get

$$\langle y_n, x_n - z\rangle = \langle x_n - Ux_n, x_n - z\rangle + \langle (I-T)Ax_n, Ax_n - Az\rangle.$$

Since $z \in F(U)$ and $Az \in F(T)$, $\|y_n\| \to 0$ and $\{x_n\}$ is bounded, by Equation (5) we have

$$\frac{1-\tau}{2}\|x_n - Ux_n\|^2 + \frac{1-v}{2}\|(I-T)Ax_n\|^2 \leq \langle y_n, x_n - z\rangle \leq \|y_n\|\|x_n - z\| \to 0.$$

Therefore, by $\tau, v \in (0,1)$, we obtain $\lim_{n\to\infty}\|x_n - Ux_n\| = 0$ and $\lim_{n\to\infty}\|(I-T)Ax_n\| = 0$, completing the proof. □

Theorem 1. *Assume that the sequences $\{\rho_n\} \subseteq (0, +\infty)$, $\{\beta_n\} \subseteq (0,1)$ and the mappings U, T satisfy the following conditions.*

(a) *$I - U$ and $I - T$ are demiclosed at 0.*

(b) $\sum_{n=0}^{\infty} \rho_n^2 < \infty$.
(c) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$.
(d) $\lim_{n\to\infty} \frac{\beta_n}{\rho_n} = 0$.

Then, the sequence $\{x_n\}$ *defined by Algorithm 2 converges strongly to a solution* $p = P_\Omega \circ f(p)$ *to the problem (1).*

Proof. Firstly, Lemma 2 yields that $F(U)$ and $F(T)$ are both closed convex sets. Since A is bounded and linear, $A^{-1}(F(T))$ is also closed convex. Therefore, Ω is closed convex. Thus, $P_\Omega \circ f : \mathcal{H}_1 \to \mathcal{H}_1$ is contractive. By Banach's contraction principle there exists a unique element $p \in \mathcal{H}_1$ such that $p = P_\Omega \circ f(p) \in \Omega$. In particular, by Equation (6), we have

$$\langle f(p) - p, y - p\rangle \leq 0, \ \ y \in \Omega. \tag{11}$$

Secondly we show that $\{x_n\}$ is bounded.

Indeed, set $y_n = x_n - Ux_n + A^*(I-T)Ax_n$, $z_n = x_n - \tau_n y_n$. By Equation (5), we have

$$\begin{aligned}
\langle y_n, x_n - p\rangle &= \langle x_n - Ux_n + A^*(I-T)Ax_n, x_n - p\rangle \\
&= \langle x_n - Ux_n, x_n - p\rangle + \langle (I-T)Ax_n, Ax_n - Ap\rangle \\
&\geq \frac{1-\tau}{2}\|x_n - Ux_n\|^2 + \frac{1-v}{2}\|(I-T)Ax_n\|^2 \\
&\geq \frac{1-\tau}{2}\|x_n - Ux_n\|^2 + \frac{1-v}{2\|A\|^2}\|A^*(I-T)Ax_n\|^2 \\
&\geq \frac{\min\{1-\tau, 1-v\}}{2\max\{1,\|A\|^2\}}(\|x_n - Ux_n\|^2 + \|A^*(I-T)Ax_n\|^2) \\
&\geq \frac{\min\{1-\tau, 1-v\}}{4\max\{1,\|A\|^2\}}\|x_n - Ux_n + A^*(I-T)Ax_n\|^2 \\
&= \tau^*\|y_n\|^2,
\end{aligned} \tag{12}$$

where $\tau^* = \frac{\min\{1-\tau,1-v\}}{4\max\{1,\|A\|^2\}}$. It follows from (12) and $\tau_n\|y_n\| = \rho_n$ that

$$\begin{aligned}
\|z_n - p\|^2 &= \|x_n - p - \tau_n y_n\|^2 \\
&= \|x_n - p\|^2 - 2\tau_n\langle y_n, x_n - p\rangle + \tau_n^2\|y_n\|^2 \\
&\leq \|x_n - p\|^2 - 2\tau^*\tau_n\|y_n\|^2 + \tau_n^2\|y_n\|^2 \\
&\leq \|x_n - p\|^2 - 2\tau^*\rho_n\|y_n\| + \rho_n^2 \qquad (13) \\
&\leq \|x_n - p\|^2 + \rho_n^2. \qquad (14)
\end{aligned}$$

From Algorithm 2, Equation (14), and $\alpha \in (0, 1/\sqrt{2})$ we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
= \ &\|\beta_n(f(x_n) - p) + (1-\beta_n)(z_n - p)\|^2 \\
\leq \ &\beta_n\|f(x_n) - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\
\leq \ &2\beta_n[\alpha^2\|x_n - p\|^2 + \|f(p) - p\|^2] + (1-\beta_n)(\|x_n - p\|^2 + \rho_n^2) \\
\leq \ &[1 - \beta_n(1-2\alpha^2)]\|x_n - p\|^2 + 2\beta_n\|f(p) - p\|^2 + \rho_n^2 \\
= \ &[1 - \beta_n(1-2\alpha^2)]\|x_n - p\|^2 + \beta_n(1-2\alpha^2)\frac{2\|f(p) - p\|^2}{1-2\alpha^2} + \rho_n^2.
\end{aligned}$$

By induction, we can get

$$\|x_{n+1}-p\|^2 \le \max\{\|x_0-p\|^2, \frac{2\|f(p)-p\|^2}{1-2\alpha^2}\} + \sum_{i=0}^{n}\rho_i^2.$$

Hence, $\{x_n\}$ is bounded due to the condition (b). The condition (b) implies that $\rho_n^2 \to 0$. So $\{y_n\}$ and $\{z_n\}$ are bounded by (12) and (14).

It follows from Algorithm 2 and (13) that

$$\begin{aligned}
& \|x_{n+1}-p\|^2 \\
= \; & \|(1-\beta_n)(z_n-p)+\beta_n(f(x_n)-p)\|^2 \\
\le \; & (1-\beta_n)^2\|z_n-p\|^2 + 2\beta_n(1-\beta_n)\langle f(x_n)-p, z_n-p\rangle + \beta_n^2\|f(x_n)-p\|^2 \\
\le \; & (1-\beta_n)^2\|z_n-p\|^2 + \beta_n^2\|f(x_n)-p\|^2 \\
& +2\beta_n(1-\beta_n)(\langle f(x_n)-f(p), z_n-p\rangle + \langle f(p)-p, z_n-p\rangle) \\
\le \; & (1-\beta_n)^2\|z_n-p\|^2 + \beta_n^2\|f(x_n)-p\|^2 \\
& +\beta_n(1-\beta_n)[\|f(x_n)-f(p)\|^2 + \|z_n-p\|^2] + 2\beta_n(1-\beta_n)\langle f(p)-p, z_n-p\rangle \\
\le \; & (1-\beta_n)^2\|z_n-p\|^2 + \beta_n^2\|f(x_n)-p\|^2 \\
& +\beta_n(1-\beta_n)[\alpha^2\|x_n-p\|^2 + \|z_n-p\|^2] + 2\beta_n(1-\beta_n)\langle f(p)-p, z_n-p\rangle \\
= \; & (1-\beta_n)\|z_n-p\|^2 + \beta_n(1-\beta_n)\alpha^2\|x_n-p\|^2 \\
& +2\beta_n(1-\beta_n)\langle f(p)-p, z_n-p\rangle + \beta_n^2\|f(x_n)-p\|^2 \\
\le \; & (1-\beta_n)[\|x_n-p\|^2 - 2\tau^*\rho_n\|y_n\| + \rho_n^2] + \beta_n(1-\beta_n)\alpha^2\|x_n-p\|^2 \\
& +2\beta_n(1-\beta_n)\langle f(p)-p, z_n-p\rangle + \beta_n^2 M \\
\le \; & [1-\beta_n(1-(1-\beta_n)\alpha^2)]\|x_n-p\|^2 + \rho_n^2 \\
& +2\beta_n(1-\beta_n)\langle f(p)-p, z_n-p\rangle - 2\tau^*(1-\beta_n)\rho_n\|y_n\| + \beta_n^2 M \\
= \; & [1-\beta_n(1-(1-\beta_n)\alpha^2)]\|x_n-p\|^2 + \rho_n^2 \\
& +\beta_n(1-(1-\beta_n)\alpha^2)[\frac{2(1-\beta_n)}{1-(1-\beta_n)\alpha^2}\langle f(p)-p, z_n-p\rangle \\
& -\frac{2\tau^*(1-\beta_n)}{1-(1-\beta_n)\alpha^2}\frac{\rho_n}{\beta_n}\|y_n\| + \frac{\beta_n M}{1-(1-\beta_n)\alpha^2}],
\end{aligned} \tag{15}$$

where $M = \sup_{n\ge 0}\{\|f(x_n)-p\|^2\}$. Set

$$b_n = \frac{2(1-\beta_n)}{1-(1-\beta_n)\alpha^2}\langle f(p)-p, z_n-p\rangle - \frac{2\tau^*(1-\beta_n)}{1-(1-\beta_n)\alpha^2}\frac{\rho_n}{\beta_n}\|y_n\| + \frac{\beta_n M}{1-(1-\beta_n)\alpha^2}.$$

Next we show that $\limsup_{n\to\infty} b_n \le 0$.

It is obvious that $\{b_n\}$ is bounded from above, so $\limsup_{n\to\infty} b_n$ is finite, and

$$\|z_n-x_n\| = \|\tau_n y_n\| = \rho_n \to 0.$$

Therefore, we can choose a subsequence $\{n_k\}$ in $\{n\}$ satisfying

$$\begin{aligned}
\limsup_{n\to\infty} b_n &= \lim_{k\to\infty} b_{n_k} \\
&= \lim_{k\to\infty}\Big[\frac{2(1-\beta_{n_k})}{1-(1-\beta_{n_k})\alpha^2}\langle f(p)-p, z_{n_k}-x_{n_k}+x_{n_k}-p\rangle \\
&\quad -\frac{2\tau^*(1-\beta_{n_k})}{1-(1-\beta_{n_k})\alpha^2}\frac{\rho_{n_k}}{\beta_{n_k}}\|y_{n_k}\| + \frac{\beta_{n_k}M}{1-(1-\beta_{n_k})\alpha^2}\Big] \\
&= \lim_{k\to\infty}\Big[\frac{2(1-\beta_{n_k})}{1-(1-\beta_{n_k})\alpha^2}\langle f(p)-p, x_{n_k}-p\rangle \\
&\quad -\frac{2\tau^*(1-\beta_{n_k})}{1-(1-\beta_{n_k})\alpha^2}\frac{\rho_{n_k}}{\beta_{n_k}}\|y_{n_k}\|\Big]. \qquad (16)
\end{aligned}$$

Due to the boundedness of $\{x_n\}$, there exists a weakly convergent subsequence, and we suppose that $\{x_{n_k}\}$ converges weakly to some point x^*, such that

$$\lim_{k\to\infty}\langle f(p)-p, x_{n_k}-p\rangle = \langle f(p)-p, x^*-p\rangle. \qquad (17)$$

It follows from (16) and (17) and $\beta_{n_k}\to 0$ that

$$\lim_{k\to\infty}\frac{\rho_{n_k}}{\beta_{n_k}}\|y_{n_k}\|$$

exists. Therefore, from condition (d), we have

$$\|y_{n_k}\| = \frac{\beta_{n_k}}{\rho_{n_k}}\frac{\rho_{n_k}}{\beta_{n_k}}\|y_{n_k}\| \to 0,$$

that is,

$$\lim_{k\to\infty}\|y_{n_k}\| = \lim_{k\to\infty}\|x_{n_k}-Ux_{n_k}+A^*(I-T)Ax_{n_k}\| = 0,$$

which together with Lemma 5 implies that

$$\lim_{k\to\infty}\|x_{n_k}-Ux_{n_k}\| = \lim_{k\to\infty}\|(I-T)Ax_{n_k}\| = 0.$$

Therefore, by the condition (a) we have $x^*\in F(U)$ and $Ax^*\in F(T)$, i.e., $x^*\in\Omega$. So from (11), (16), and (17) we get

$$\limsup_{n\to\infty} b_n \le \frac{2}{1-\alpha^2}\lim_{k\to\infty}\langle f(p)-p, x_{n_k}-p\rangle = \frac{2}{1-\alpha^2}\langle f(p)-p, x^*-p\rangle \le 0.$$

Finally, we prove that $\{x_n\}$ is strongly convergent to $p = P_\Omega\circ f(p)$.

From the condition $(b),(c)$, applying Lemma 1 to Equation (15) we get $\|x_n-p\|\to 0$, that is, the sequence $\{x_n\}$ converges strongly to $p = P_\Omega\circ f(p)$, completing the proof. □

Remark 3. *Choose* $\rho_n = \frac{1}{n^r}, \beta_n = \frac{1}{n^s}, \frac{1}{2} < r < s \le 1$. *The sequences* $\{\rho_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions (b)–(d) in Theorem 1.*

If U and T are nonexpansive with $F(U) \neq \emptyset$ and $F(T) \neq \emptyset$, then U and T are demicontractive; by Lemma 3, the condition (a) in Theorem 1 is satisfied. Hence, by Theorem 1, we get the result below.

Corollary 1. *Assume that U and T are two nonexpansive mappings and $\Omega \neq \emptyset$. "Assume that two control sequences $\{\beta_n\}$ and $\{\rho_n\}$ satisfy the conditions the conditions (b)–(d) in Theorem 1. Then, the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to a solution $p = P_\Omega \circ f(p)$ to the problem (1).*

Author Contributions: All authors read and approved the final manuscript. Conceptualization, Y.W.; Validation, Y.W. and T.-H.K.; writing-original draft preparation, Y.W., X.F. and T.-H.K.; writing-review and editing, Y.W., X.F. and T.-H.K.

Funding: The work was supported by the Natural Science Fundation of China (No. 11975156, 11671365), and Zhejiang Provincial Natural Science Foundation of China (No. LQ13A010007, LY14A010006).

Acknowledgments: The authors would like to thank the referee for valuable suggestions to improve the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Censor, Y.; Segal, A. The split common fixed point problem for directed operators. *J. Convex Anal.* **2009**, *16*, 587–600.
2. Jailoka, P.; Suantai, S. Split common fixed point and null point problems for demicontractive operators in Hilbert spaces. *Optim. Methods Softw.* **2019**, *34*, 248–263. [CrossRef]
3. Moudafi, A. A note on the split common fixed point problem for quasinonexpansive operators. *Nonlinear Anal.* **2011**, *74*, 4083–4087. [CrossRef]
4. Wang, Y.Q.; Kim, T.H.; Fang, X.L.; He, H.M. The split common fixed-point problem for demicontractive mappings and quasi-nonexpansive mappings. *J. Nonlinear Sci. Appl.* **2017**, *10*, 2976–2985. [CrossRef]
5. Yao, Y.H.; Liou, Y.C.; Postolache, M. Self-adaptive algorithms for the split problem of the demicontractive operators. *Optimization* **2018**, *67*, 1309–1319. [CrossRef]
6. Yao, Y.H.; Yao, J.C.; Liou, Y.C.; Postolache, M. Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms. *Carpathian J. Math.* **2018**, *34*, 451–458.
7. Yao, Y.; Leng, L.; Postolache, M.; Zheng, X. Mann-type iteration method for solving the split common fixed point problem. *J. Nonlinear Convex Anal.* **2017**, *18*, 875–882.
8. Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms* **1994**, *8*, 221–239. [CrossRef]
9. Moudafi, A. The split common fixed point problem for demicontractive mappings. *Inverse Probl.* **2010**, *26*, 055007. [CrossRef] [PubMed]
10. Wang, F.; Xu, H.K. Weak and strong convergence of two algorithms for the split fixed point problem. *Numer. Math. Theory Method Appl.* **2018**, *11*, 770–781.
11. Attouch, H. Viscosity solutions of minimization problems. *SIAM J. Optim.* **1996**, *6*, 769–806. [CrossRef]
12. Moudafi, A. Viscosity approximation methods for fixed points problems. *J. Math. Anal. Appl.* **2000**, *241*, 46–55. [CrossRef]
13. Marino, G.; Xu, H.K. Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. *J. Math. Anal. Appl.* **2007**, *329*, 336–346. [CrossRef]
14. Geobel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Marcel Dekker, Inc.: New York, NY, USA; Basel, Switzerland, 1984.
15. Takahashi, W. *Nonlinear Functional Analysis*; Yokohama Publishers: Yokohama, Japan, 2000.
16. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Volume 28 of Cambridge Studies in Advanced Mathematics; Cambridge University Press: Cambridge, UK, 1990.
17. Wang, Y.Q.; Kim, T.H. Simultaneous iterative algorithm for the split equality fixed-point problem of demicontractive mappings. *J. Nonlinear Sci. Appl.* **2017**, *10*, 154–165. [CrossRef]

18. Xu, H.K. Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **2002**, *66*, 240–256. [CrossRef]
19. Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–597. [CrossRef]