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A Deformed Wave Equation and Huygens' Principle

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Abstract: We consider a deformed wave equation where the Laplacian operator has been replaced by a differential-difference operator. We prove that this equation does not satisfy Huygens' principle. Our approach is based on the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

Keywords: generalized Fourier transform; deformed wave equation; Huygens' principle; representation of $\mathfrak{sl}(2,\mathbb{R})$

MSC: 43A30; 22E70

1. Introduction

The wave problem consists of the wave equation and some initial data,

$$\Delta u(x,t) = \partial_{tt}u(x,t), \quad u(x,0) = f(x),\ \partial_t u(x,0) = g(x), \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$

This problem is certainly one of the most interesting problems of mathematical physics. Standard techniques involving the Fourier transform show that there are two distributions P_1 and P_2 on $\mathbb{R}^n \times \mathbb{R}$ such that $u = P_1 * f + P_2 * g$. Here $*$ represents the Euclidean convolution product. The distributions P_1 and P_2 are called propagators.

One of the most celebrated features of the wave equation is Huygens' principle: When the dimension n is odd and starting from 3, the propagators are supported entirely on spherical shells. This is the reason why in our three-dimensional word, transmission of signals is possible and we can hear each other. A two-dimensional world would be drastically different from this point of view.

The problem of classifying all second order differential operators which obey Huygens' principle is known as the Hadamard problem [1]. This problem has received a good deal of attention and the literature is extensive (see, for instance, [2–13]). Nevertheless, this problem is still far from being fully solved.

In this paper we will consider a deformed wave equation where the Laplacian Δ is replaced by a certain differential-difference operator. We will prove the non-existence of Huygens' principle for the deformed wave equation for all $n \geqslant 1$. The main tool is the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

More precisely, we will consider the deformed wave equation $2\|x\|\Delta_k u(x,t) = \partial_{tt}u(x,t)$ with compactly supported initial data (f,g). Here Δ_k is the differential-difference Dunkl Laplacian (see (2)), where k is a multiplicity function for the Dunkl operators. The operator $\|x\|\Delta_k$ appeared in [14] and played a crucial rule in the study of the so-called $(k,1)$-generalized Fourier transform. When $k \equiv 0$, the deformed wave equation becomes $2\|x\|\Delta u(x,t) = \partial_{tt}u(x,t)$ and the $(0,1)$-generalized Fourier transform reduces to a Hankel type transform on $\mathbb{R}^n$. We refer the reader to [14] for a detailed study on the generalized Fourier transform.

We begin with a straightforward treatment of the Cauchy problem for the deformed wave equation by means of the $(k, 1)$-generalized Fourier transform, and derive the existence of propagators $P_{k,1}$ and $P_{k,2}$, in terms of which, the Cauchy problem is solved. Huygens' principle for the deformed wave equation is that $P_{k,1}$ and $P_{k,2}$ are supported entirely on the set $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| - \frac{1}{2}t^2 = 0\}$. It is not a simple task to study the support property from the precise form of the propagators. However, subtler dilatation properties of the propagators allow us to show that Huygens' principle holds true if, and only if, $P_{k,1}$ and $P_{k,2}$ generate a finite dimensional representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. It is here that the construction of a representation of $\mathfrak{sl}(2,\mathbb{R})$ plays a crucial role. This construction was inspired by [14]. A closer investigation shows that $P_{k,1}$ and $P_{k,2}$ cannot generate finite dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$, and therefore, Huygens' principle does not hold for the deformed wave equation for any $n \geqslant 1$ and any multiplicity function k. The strategy uses proof by contradiction. It is noteworthy mentioning that the case $k \equiv 0$ is already new.

It would be interesting to understand the interpretation(s) of the non-existence of Huygens' principle for the deformed wave equation from the physics point of view. It would also be fascinating to ask whether Huygens's principle holds for other seminal Dunkl-type equations such as the Dunkl–Dirac equation (see [15] for more details about the Dunkl–Dirac operator). For the Euclidean Dirac equation, this problem has been investigated in [16].

2. Background

Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean scalar product in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, denote $\|x\| = \langle x, x \rangle^{1/2}$.

For α in $\mathbb{R}^n \backslash \{0\}$, we write r_α for the reflection with respect to the hyperplane $\alpha^\perp$ orthogonal to α defined by

$$r_\alpha(x) := x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha, \quad x \in \mathbb{R}^n.$$

A finite set $\mathcal{R} \subset \mathbb{R}^n \backslash \{0\}$ is called a root system if $r_\alpha(\mathcal{R}) \subset \mathcal{R}$ for every $\alpha \in \mathcal{R}$. The finite group $G \subset O(n)$ generated by the reflections $\{r_\alpha : \alpha \in \mathcal{R}\}$ is called the finite Coxeter group associated with $\mathcal{R}$. A multiplicity function for G is a function $k : \mathcal{R} \to \mathbb{R}_{\geqslant 0}$ which is constant on G-orbits.

For $1 \leqslant j \leqslant n$, the Dunkl operator is defined in [17] by

$$T_j f(x) = \partial_j f(x) + \frac{1}{2}\sum_{\alpha \in \mathcal{R}} k(\alpha) \frac{f(x) - f\big(r_\alpha(x)\big)}{\langle \alpha, x \rangle}\langle \alpha, e_j \rangle, \quad x \in \mathbb{R}^n,$$

where ∂_j is the standard directional derivative and $\{e_1, \ldots, e_n\}$ is the canonical orthonormal basis in $\mathbb{R}^n$. The most important property of these operators is that they commute. The operators T_j and ∂_j are intertwined by the following Laplace type operator

$$V_k f(x) = \int_{\mathbb{R}^n} f(y) d\mu_x^k(y), \tag{1}$$

where μ_x^k is a unique compactly supported probability measure with $\operatorname{supp}(\mu_x^k) \subset \{y \in \mathbb{R}^n : \|y\| \leqslant \|x\|\}$ (see [17,18]).

The Dunkl Laplacian, which is akin to the Euclidean Laplace operator Δ, is defined by $\Delta_k := T_1^2 + \cdots + T_n^2$ and is given explicitly, for suitable function f, by

$$\Delta_k f(x) = \Delta f(x) + \sum_{\alpha \in \mathcal{R}} k(\alpha) \left(\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{\|\alpha\|^2}{2} \frac{f(x) - f\big(r_\alpha(x)\big)}{\langle \alpha, x \rangle^2} \right), \quad x \in \mathbb{R}^n, \tag{2}$$

where ∇ is the gradient. It is worth mentioning that if $k(\alpha) = 0$ for all $\alpha \in \mathcal{R}$, then Δ_k reduces to the Euclidean Laplacian Δ. We refer the reader to [19] for the theory of Dunkl's operators. This theory, which started with the seminal paper [17], was developed extensively afterwards and continues to receive considerable attention (see, e.g., [20–32]).

Next we will introduce some definitions and results for the generalized Fourier transform; for details we refer to [14]. For $a > 0$, let

$$\Delta_{k,a} := \|x\|^{2-a}\Delta_k - \|x\|^a,$$

where $\|x\|^a$ on the right-hand side of the formula stands for the multiplication operator by $\|x\|^a$. The operator $\Delta_{k,a}$ is symmetric on the Hilbert space $L^2(\mathbb{R}^n, \vartheta_{k,a})$ consisting of square integrable functions against the measure $\vartheta_{k,a}(x)dx := \|x\|^{a-2}\prod_{\alpha\in\mathcal{R}}|\langle\alpha, x\rangle|^{k(\alpha)}dx$.

The (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$ was defined in [14] to be

$$\mathcal{F}_{k,a} := e^{i\frac{\pi}{2}\left(\frac{n+2\langle k\rangle+a-2}{a}\right)}\exp\Big(i\frac{\pi}{2a}\Delta_{k,a}\Big),$$

where $\langle k\rangle := \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k(\alpha)$. We pin down that $\mathcal{F}_{k,a}$ is a unitary operator from $L^2(\mathbb{R}^n, \vartheta_{k,a})$ onto itself and the inversion formula is given as

$$\mathcal{F}^{-1}_{k,\frac{1}{r}} = \mathcal{F}_{k,\frac{1}{r}}, \qquad \big(\mathcal{F}^{-1}_{k,\frac{2}{2r+1}}f\big)(x) = \big(\mathcal{F}_{k,\frac{2}{2r+1}}f\big)(-x), \tag{3}$$

where r is any nonnegative integer. The transform $\mathcal{F}_{k,a}$ reduces to the Euclidean Fourier transform if $k \equiv 0$ and $a = 2$; to the Kobayashi–Mano Hankel transform [33] if $k \equiv 0$ and $a = 1$; to the Dunkl transform [34] if $k > 0$ and $a = 2$. In this paper we consider the case $k > 0$ and $a = 1$. For more details, we refer the reader to ([14] Sections 4 and 5) (see also [35–41]).

Let us collect the main properties of the $(k, 1)$-transform $\mathcal{F}_{k,1} := \mathcal{F}_k$. In ([14] Theorem 4.23), the authors proved that for $n + 2\langle k\rangle > 1$, there exists a kernel $B_k(x, y)$ such that for every $f \in L^2(\mathbb{R}^n, \vartheta_{k,1})$,

$$\mathcal{F}_k f(x) = c_k^{-1}\int_{\mathbb{R}^n} f(y)B_k(x, y)\vartheta_{k,1}(y)dy, \quad x \in \mathbb{R}^n,$$

where, for $x = r\theta'$ and $y = t\theta''$, the kernel B_k is given by

$$B_k(x, y) = V_k\Big(\widetilde{J}_{\frac{n-3}{2}+\langle k\rangle}\big(\sqrt{2rt(1 + \langle\theta', \cdot\rangle)}\big)\Big)(\theta'').$$

Here V_k is the Dunkl intertwining operator (1) and $\widetilde{J}_\nu(z)$ is the normalized Bessel function. Above,

$$c_k := \Gamma(n + 2\langle k\rangle - 1)\int_{S^{n-1}}\prod_{\alpha\in\mathcal{R}}|\langle\alpha, \eta\rangle|^{k(\alpha)}d\sigma(\eta).$$

It is noteworthy mentioning that

$$\mathcal{F}_k \circ \|x\| = -\|x\|\Delta_k\mathcal{F}_k, \qquad \mathcal{F}_k \circ (-\|x\|\Delta_k) = -\|x\| \circ \mathcal{F}_k. \tag{4}$$

Recently, in [42] the authors defined a translation operator τ_x, for $x \in \mathbb{R}^n$, on the space $L^1 \cap L^\infty(\mathbb{R}^n, \vartheta_{k,1})$ by

$$\mathcal{F}_k(\tau_x f)(\xi) = B(x, \xi)\mathcal{F}_k(f)(\xi), \qquad \xi \in \mathbb{R}^n.$$

Here are some basic properties of the translation operator:

(i) $\tau_0 = \mathrm{Id}$;
(ii) $\tau_x f(y) = \tau_y f(x)$;
(iii) $\tau_x f_\lambda = (\tau_{\lambda x} f)_\lambda$, where $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$.

By means of the translation operator, a convolution $\circledast$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ was defined by

$$f \circledast g(x) = c_k^{-1} \int_{\mathbb{R}^n} f(y)\tau_x g(y)\vartheta_{k,1}(y)dy, \quad x \in \mathbb{R}^n.$$

In particular, $f \circledast g = g \circledast f$ and $\mathcal{F}_k(f \circledast g) = \mathcal{F}_k f \cdot \mathcal{F}_k g$ (see [42] for more details).

Next we turn our attention to the convolution of distributions (see [42,43]). Denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. If $T \in \mathcal{S}'(\mathbb{R}^n)$, then $\mathcal{F}_k(T)$ is defined by

$$\langle \mathcal{F}_k(T), \varphi \rangle := \langle T, \mathcal{F}_k(\varphi) \rangle, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

It is worth mentioning that $\mathcal{S}(\mathbb{R}^n)$ is stable by $\mathcal{F}_k$ (see [36]). The convolution $T \circledast f$ of $T \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$ is defined in [42] by

$$T \circledast f(x) = \langle T, \tau_x f \rangle.$$

In particular, a result analogous to the Euclidean convolution shows that $T \circledast f \in \mathcal{S}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and $\mathcal{F}_k(T \circledast f) = \mathcal{F}_k T \cdot \mathcal{F}_k f$.

3. The Deformed Wave Equation and Huygens' Principle

For $n + 2\langle k \rangle - 1 > 0$, where $\langle k \rangle = \frac{1}{2} \sum_{\alpha \in \mathcal{R}} k(\alpha)$, we consider the following Cauchy problem for the wave equation

$$\begin{aligned} &2\|x\|\Delta_k^x u_k(x,t) = \partial_{tt} u_k(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ &u_k(x,0) = f(x), \;\; \partial_t u_k(x,0) = g(x), \end{aligned} \tag{5}$$

where the functions f and g belong to the space $\mathcal{D}(\mathbb{R}^n)$ of smooth functions with compact support. Here the superscript in Δ_k^x indicates the relevant variable, while the subscript t indicates differentiation in the t-variable. Next, we will prove the following statements:

($\mathcal{S}$) Let $u_k(x,t)$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, satisfy $2\|x\|\Delta_k^x u_k(x,t) - \partial_{tt} u_k(x,t) = 0$, then u_k does not satisfy Huygen's principle. In other words, the solution u_k is expressed as a sum of $\circledast$-convolution of f and g with distributions that are not supported entirely on the set $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| = \frac{1}{2}t^2\}$.

For $t \in \mathbb{R}$, denote by $P_{k,t}$ the 2×2 matrix of tempered distributions on $\mathbb{R}^n$

$$P_{k,t} = \begin{pmatrix} P_{k,t}^{11} & P_{k,t}^{12} \\ P_{k,t}^{21} & P_{k,t}^{22} \end{pmatrix} := \begin{pmatrix} \mathcal{F}_k\big(\cos(t\sqrt{2\|\cdot\|})\big) & \mathcal{F}_k\big(\sin(t\sqrt{2\|\cdot\|})/\sqrt{2\|\cdot\|}\big) \\ \mathcal{F}_k\big(-\sqrt{2\|\cdot\|}\sin(t\sqrt{2\|\cdot\|})\big) & \mathcal{F}_k\big(\cos(t\sqrt{2\|\cdot\|})\big) \end{pmatrix}. \tag{6}$$

Set $U_k(x,0) := \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$, where the initial data $(f,g) \in \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)$. Thus, we may define the vector column $U_k(x,t)$ by

$$\begin{aligned} U_k(x,t) &:= \{P_{k,t} \circledast U_k(\cdot,0)\}(x) \\ &= \left\{ \begin{pmatrix} P_{k,t}^{11} & P_{k,t}^{12} \\ P_{k,t}^{21} & P_{k,t}^{22} \end{pmatrix} \circledast \begin{pmatrix} f \\ g \end{pmatrix} \right\}(x). \end{aligned} \tag{7}$$

By applying the Fourier transform $\mathcal{F}_k$ to (7), in the x-variable, we get

$$\mathcal{F}_k(U_k(\cdot,t))(\xi) = e^{t\mathbb{A}} \mathcal{F}_k(U_k(\cdot,0))(\xi), \tag{8}$$

where

$$\mathbb{A} := \begin{pmatrix} 0 & 1 \\ -2\|\xi\| & 0 \end{pmatrix}. \tag{9}$$

Above we have used the fact that $\mathcal{F}_k^{-1} = \mathcal{F}_k$ (see (3)). That is $\mathcal{F}_k(U_k(\cdot,t))(\xi)$ is a solution to the following ordinary differential equation

$$\partial_t \mathcal{F}_k(U_k(\cdot,t))(\xi) = \mathbb{A}\mathcal{F}_k(U_k(\cdot,t))(\xi) = \begin{pmatrix} 0 & 1 \\ -2\|\xi\| & 0 \end{pmatrix} \mathcal{F}_k(U_k(\cdot,t))(\xi). \tag{10}$$

Now, recall from (4) that $-\|\xi\|\mathcal{F}_k(f)(\xi) = \mathcal{F}_k(\|x\|\Delta_k f)(\xi)$, and using the injectivity of the Fourier transform $\mathcal{F}_k$, we deduce that

$$\partial_t U_k(x,t) = \begin{pmatrix} 0 & 1 \\ 2\|x\|\Delta_k & 0 \end{pmatrix} U_k(x,t). \tag{11}$$

Hence, if we write $U_k(x,t) = \begin{pmatrix} u_k(x,t) \\ v_k(x,t) \end{pmatrix}$, then $u_k(x,t)$ satisfies the following equation

$$\partial_{tt} u_k(x,t) = 2\|x\|\Delta_k u_k(x,t).$$

Moreover, since $f, g \in \mathcal{D}(\mathbb{R}^n)$, it follows from (7) and the properties of the $\circledast$-convolution that $u_k(\cdot,t) \in C^\infty(\mathbb{R}^n)$ for all $t \in \mathbb{R}$.

Furthermore, $u_k(x,t) \to f(x)$ as $t \to 0$. Indeed, if δ denotes the Dirac functional, then, as $t \to 0$, $\mathcal{F}_k(\cos(t\sqrt{2\|\cdot\|})) \to \delta$ in $\mathcal{S}'(\mathbb{R}^n)$ and thus in $\mathcal{D}'(\mathbb{R}^n)$. On the other hand $\mathcal{F}_k(\sin(t\sqrt{2\|\cdot\|})/\sqrt{2\|\cdot\|}) \to 0$ as $t \to 0$. Using the continuity of the convolution $\circledast$, we deduce that

$$u_k(x,t) \to (\delta \circledast f)(x) = f(x) \qquad \text{as} \quad t \to 0.$$

Similarly, one can prove that $(\partial_t u_k)(x,t) \to g(x)$ as $t \to 0$.

We mention that the solution u_k constructed above is unique. To prove this claim, we need the lemma below. Let

$$\mathcal{E}_k[u_k](t) = \int_{\mathbb{R}^n} \left\{ |\partial_t \mathcal{F}_k(u_k(\cdot,t))(\xi)|^2 + 2\|\xi\| |\mathcal{F}_k(u_k(\cdot,t))(\xi)|^2 \right\} \vartheta_{k,1}(\xi) d\xi. \tag{12}$$

Lemma 1. *Assume that $n + 2\langle k \rangle - 1 > 0$ and that the initial data $f, g \in \mathcal{D}(\mathbb{R}^N)$. Then the total energy $\mathcal{E}_k[u_k]$ is independent of t.*

Proof. Since

$$\mathcal{F}_k(u_k(\cdot,t))(\xi) = \cos(t\sqrt{2\|\xi\|})\mathcal{F}_k f(\xi) + \frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}} \mathcal{F}_k g(\xi), \qquad \text{for all} \ \ t \in \mathbb{R},$$

we deduce that

$$\begin{aligned} |\mathcal{F}_k(u_k(\cdot,t))(\xi)|^2 \quad = \quad & \cos^2(t\sqrt{2\|\xi\|})|\mathcal{F}_k f(\xi)|^2 + \frac{\sin^2(t\sqrt{2\|\xi\|})}{2\|\xi\|}|\mathcal{F}_k g(\xi)|^2 \\ & + \sqrt{2}\frac{\cos(t\sqrt{\|\xi\|})\sin(t\sqrt{\|\xi\|})}{\sqrt{\|\xi\|}} \mathrm{Re}\left(\mathcal{F}_k f(\xi) \overline{\mathcal{F}_k g(\xi)} \right), \end{aligned}$$

and

$$\begin{aligned} |\partial_t \mathcal{F}_k(u_k(\cdot,t))(\xi)|^2 \quad = \quad & \cos^2(t\sqrt{2\|\xi\|})|\mathcal{F}_k g(\xi)|^2 + 2\|\xi\|\sin^2(t\sqrt{2\|\xi\|})|\mathcal{F}_k f(\xi)|^2 \\ & - 2\sqrt{2\|\xi\|}\cos(t\sqrt{2\|\xi\|})\sin(t\sqrt{2\|\xi\|})\mathrm{Re}\left(\mathcal{F}_k f(\xi) \overline{\mathcal{F}_k g(\xi)} \right). \end{aligned}$$

Thus we have

$$\mathcal{E}_k[u_k](t) = \int_{\mathbb{R}^n} \left\{ 2\|\xi\| |\mathcal{F}_k f(\xi)|^2 + |\mathcal{F}_k g(\xi)|^2 \right\} \vartheta_{k,1}(\xi) d\xi. \tag{13}$$

Hence, we established the lemma. □

Now let us go back to the uniqueness of the solution u_k. Assume that $u_k^{(1)}$ and $u_k^{(2)}$ are two solutions of the wave equation with the same initial data, then $u_k^{(1)} - u_k^{(2)}$ is a solution of the wave equation with zero initial data. Therefore, by (13), we have $\mathcal{E}_k[u_k^{(1)} - u_k^{(2)}](t) = 0$. Hence, (12) implies $\partial_t \mathcal{F}_k((u_k^{(1)} - u_k^{(2)})(\cdot, t))(\xi) = 0$ for every $t \in \mathbb{R}$. That is, the function $t \mapsto \mathcal{F}_k((u_k^{(1)} - u_k^{(2)})(\cdot, t))(\xi)$ is a constant, which implies $\mathcal{F}_k((u_k^{(1)} - u_k^{(2)})(\cdot, t))(\xi) = \mathcal{F}_k((u_k^{(1)} - u_k^{(2)})(\cdot, 0))(\xi) = 0$. Using the injectivity of the Fourier transform $\mathcal{F}_k$, we deduce that $(u_k^{(1)} - u_k^{(2)})(x, t) = 0$ for all $x \in \mathbb{R}^n$ and $t > 0$. This proves that the solutions of the wave equation are uniquely determined by the initial Cauchy data.

The following theorem collects all the above facts and discussions.

Theorem 1. *The solution to the Cauchy problem* (5) *is given uniquely by*

$$u_k(x, t) = (P_{k,t}^{11} \circledast f)(x) + (P_{k,t}^{12} \circledast g)(x),$$

where, for a fixed t, $P_{k,t}^{11}$ and $P_{k,t}^{12}$ are the tempered distributions on $\mathbb{R}^n$ given by

$$P_{k,t}^{11} = \mathcal{F}_k\big(\cos(t\sqrt{2\|\cdot\|})\big), \qquad P_{k,t}^{12} = \mathcal{F}_k\big(\sin(t\sqrt{2\|\cdot\|})/\sqrt{2\|\cdot\|}\big).$$

The distributions $P_{k,t}^{ij}$ will be called the propagators.

We shall now prove the statement $(\mathcal{S})$. To do so, we will assume that the propagators $P_{k,t}^{11}$ and $P_{k,t}^{12}$ are supported entirely on the set $\mathcal{C} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| = \frac{1}{2}t^2\}$ and we will show that this assumption cannot hold. Our approach uses the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, following [43,44].

Assume that the propagators $P_{k,t}^{11}$ and $P_{k,t}^{12}$ are supported entirely on the set $\mathcal{C} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| = \frac{1}{2}t^2\}$.

We start by investigating certain properties of the wave equation, which are reflected in properties of the propagators. To see this, we define the 2×2 matrix $P_k = \begin{pmatrix} P_k^{11} & P_k^{12} \\ P_k^{21} & P_k^{22} \end{pmatrix}$ of entrywise distributions on $\mathbb{R}^{n+1}$, where

$$P_k^{ij}(\psi_1 \otimes \psi_2) := \int_{\mathbb{R}} P_{k,t}^{ij}(\psi_1)\psi_2(t)dt, \qquad i, j = 1, 2,$$

for $\psi_1 \in \mathcal{S}(\mathbb{R}^n)$ and $\psi_2 \in \mathcal{S}(\mathbb{R})$. Here we used the fact that $\mathcal{S}(\mathbb{R}^{n+1}) \simeq \mathcal{S}(\mathbb{R}^n)\widehat{\otimes}\mathcal{S}(\mathbb{R})$ is the unique topological tensor product of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R})$ as nuclear spaces. From the constructive proof of Theorem 1, it follows that

$$2\|x\|\Delta_k P_k^{ij} = \partial_{tt} P_k^{ij}, \qquad i, j = 1, 2.$$

Next, we will investigate the dilations of the propagators under a dilation operator. This will inform us on the degree of the "homogeneity" of the distributions P_k^{ij}, with $i, j = 1, 2$. For $\lambda > 0$ and a function ψ on $\mathbb{R}^{n+1}$, let

$$S_\lambda^x \psi(x, t) := \psi(\lambda^2 x, t), \qquad S_\lambda^t \psi(x, t) := \psi(x, \lambda t),$$

where the superscript denotes the relevant variable. Set $S_\lambda := S_\lambda^x \circ S_\lambda^t$. By duality, the operators S_λ^x, S_λ^t, and S_λ act on distributions in the standard way.

We begin by looking to the properties of $P^{ij}_{k,t}$ under the dilation S_λ. Observe that if $u_k(x,t)$ is a solution to (5) with initial data $(f(x), g(x))$, then $S_\lambda u_k(x,t)$ solves the wave equation with initial data $(S^x_\lambda f(x), \lambda S^x_\lambda g(x))$. Thus

$$S_\lambda U_k(x,t) = P_{k,t} \circledast \begin{bmatrix} S^x_\lambda f \\ \lambda S^x_\lambda g \end{bmatrix}. \tag{14}$$

On the other hand

$$\begin{aligned} S_\lambda U_k(x,t) = \begin{bmatrix} S_\lambda u_k(x,t) \\ \partial_t\{S_\lambda u_k(x,t)\} \end{bmatrix} &= \begin{bmatrix} u_k(\lambda^2 x, \lambda t) \\ \lambda\{\partial_t u_k\}(\lambda^2 x, \lambda t) \end{bmatrix} \\ &= \begin{bmatrix} u_k \\ \lambda \partial_t u_k \end{bmatrix}(\lambda^2 x, \lambda t) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} u_k \\ \partial_t u_k \end{bmatrix}(\lambda^2 x, \lambda t) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \left\{ P_{k,\lambda t} \circledast \begin{bmatrix} f \\ g \end{bmatrix} \right\}(\lambda^2 x) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} S^x_\lambda \left\{ P_{k,\lambda t} \circledast \begin{bmatrix} f \\ g \end{bmatrix} \right\}(x). \end{aligned}$$

Using the fact that S^x_λ preserves the convolution of a distribution with a function, a fact that can be easily checked using the properties of the translation operator, we get

$$\begin{aligned} S_\lambda U_k(x,t) &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \left\{ S^x_\lambda P_{k,\lambda t} \circledast \begin{bmatrix} S^x_\lambda f \\ S^x_\lambda g \end{bmatrix} \right\}(x) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \left\{ S^x_\lambda P_{k,\lambda t} \circledast \begin{bmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} S^x_\lambda f \\ \lambda S^x_\lambda g \end{bmatrix} \right\}(x). \end{aligned} \tag{15}$$

Comparing (14) with (15) gives $S^x_\lambda P^{ij}_{k,\lambda t} = \lambda^{j-i} P^{ij}_{k,t}$, for $i, j = 1, 2$. Now we can obtain the dilation properties of P^{ij}_k as follows: For $\psi_1 \in \mathcal{S}(\mathbb{R}^n)$ and $\psi_2 \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned} S_\lambda(P^{ij}_k)(\psi_1 \otimes \psi_2) &= P^{ij}_k(S^x_{\lambda^{-1}}(\psi_1) \otimes S^t_{\lambda^{-1}}(\psi_2)) \\ &= \int_{\mathbb{R}} P^{ij}_{k,t}(S^x_{\lambda^{-1}}(\psi_1)) S^t_{\lambda^{-1}}(\psi_2)(t)dt \\ &= \lambda \int_{\mathbb{R}} P^{ij}_{k,\lambda t}(S^x_{\lambda^{-1}}(\psi_1))\psi_2(t)dt \\ &= \lambda \int_{\mathbb{R}} S^x_\lambda(P^{ij}_{k,\lambda t}(\psi_1))\psi_2(t)dt \\ &= \lambda^{1+j-i} \int_{\mathbb{R}} P^{ij}_{k,t}(\psi_1)\psi_2(t)dt \\ &= \lambda^{1+j-i} P^{ij}_k(\psi_1 \otimes \psi_2). \end{aligned}$$

We summarize the above computations.

Proposition 1. *For $n + 2\langle k \rangle - 1 > 0$, we have*

(1) *The distribution P^{ij}_k satisfies the deformed wave equation, i.e.,*

$$(\|x\|\Delta_k - \frac{1}{2}\partial_{tt})P^{ij}_k = 0, \qquad i, j = 1, 2. \tag{16}$$

(2) *For $\lambda > 0$,*

$$S_\lambda P_k^{ij} = \lambda^{1+j-i} P_k^{ij}, \qquad i,j = 1,2.$$

Next we shall describe the structure of a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{S}(\mathbb{R}^{n+1})$. This structure, together with Proposition 1, will allow us to prove that the Assumption 3 does not hold true.

We take a basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as

$$\mathbf{e} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{f} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{h} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The triple $\{\mathbf{e}, \mathbf{f}, \mathbf{h}\}$ satisfies the commutation relations

$$[\mathbf{e}, \mathbf{f}] = \mathbf{h}, \qquad [\mathbf{h}, \mathbf{e}] = 2\mathbf{e}, \qquad [\mathbf{h}, \mathbf{f}] = -2\mathbf{f},$$

where $[A, B] := AB - BA$.

Choose $x_1, x_2, \ldots, x_n$ as the usual system of coordinates on $\mathbb{R}^n$. Let

$$\mathbb{E}_{n,1} := i(\|x\| - \frac{1}{2}t^2), \quad \mathbb{F}_{n,1} := i(\|x\|\Delta_k - \frac{1}{2}\partial_{tt}), \quad \mathbb{H}_{n,1} := n + 2\langle k\rangle - \frac{1}{2} + 2\sum_{\ell=1}^{n} x_\ell \partial_\ell + t\partial_t.$$

Using ([14] Theorem 3.2), the following commutation relations hold

$$[\mathbb{E}_{n,1}, \mathbb{F}_{n,1}] = \mathbb{H}_{n,1}, \quad [\mathbb{H}_{N,1}, \mathbb{E}_{N,1}] = 2\mathbb{E}_{N,1}, \quad [\mathbb{H}_{n,1}, \mathbb{F}_{n,1}] = -2\mathbb{F}_{n,1}. \tag{17}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (17) gives rise to a representation ω_k of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ on the Schwartz space $\mathcal{S}(\mathbb{R}^{n+1})$ by setting

$$\omega_k(\mathbf{h}) = \mathbb{H}_{n,1}, \qquad \omega_k(\mathbf{e}) = \mathbb{E}_{n,1}, \qquad \omega_{k,a}(\mathbf{f}) = \mathbb{F}_{n,1}. \tag{18}$$

An analogue of the representation ω_k has been intensively studied in [14].

Recall that the Huygens' principle is equivalent to the fact that the propagators P_k^{11} and P_k^{12} are supported on the set $\mathcal{C} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| - \frac{1}{2}t^2 = 0\}$. Since $\mathcal{C}$ is the locus of zeros of $\|x\| - \frac{1}{2}t^2$, then, P_k^{ij} is supported on $\mathcal{C}$ if and only if

$$\mathbb{E}_{n,1}^m \cdot P_k^{ij} = 0 \tag{19}$$

for some positive integer m (see, for instance, ([44] p. 173)). In the light of Proposition 1(1) together with the dilatation property of P_k^{ij}, which implies that P_k^{ij} is an eigen-distribution for $\mathbb{H}_{n,1}$, Equation (19) amounts to saying the distribution P_k^{ij} generates a finite-dimensional representation ω_k^* for $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{S}'(\mathbb{R}^{n+1})$. Thus, the qualitative part of Huygens' principle holds.

Theorem 2. *Huygens' principle holds if and only if P_k^{ij} is supported on the set $\mathcal{C}$, if and only if P_k^{ij} generates a finite-dimensional representation ω_k^* for $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{S}'(\mathbb{R}^{n+1})$.*

Theorem 3. *Huygens' principle cannot hold when*

$$n - 1 + 2\langle k\rangle \notin \mathbb{Z}.$$

Proof. In ([14] Theorem 3.21) the authors proved that the spectrum of the element $\mathbf{k} := i(\mathbf{f} - \mathbf{e})$ acting on $\mathcal{S}'(\mathbb{R}^{n+1})$ via the dual representation ω_k^* is $n - 1 + 2\langle k\rangle + 2\mathbb{Z}$, whereas, it is well known, the spectrum of $\mathbf{k}$ in finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$ is contained in $\mathbb{Z}$. $\square$

The above theorem leaves the possibility that the wave equation may satisfy Huygens' principle when $n-1+2\langle k\rangle \in \mathbb{Z}$.

Now, using Proposition 1(2), we get

$$\Big\{2\sum_{\ell=1}^{n} x_\ell\partial_\ell + t\partial_t\Big\}P_k^{ij} = (1+j-i)P_k^{ij}.$$

Therefore

$$\mathbb{H}_{n,1}P_k^{ij} = -\left(n+2\langle k\rangle - \frac{1}{2} + i - j - 1\right)P_k^{ij}, \qquad i,j=1,2.$$

That is P_k^{ij} is an eigendistributation for $\mathbb{H}_{n,1}$ with eigenvalue $-(n+2\langle k\rangle - \frac{1}{2}+i-j-1)$. Keeping in mind the fact that $\mathbb{F}_{n,1}\cdot P_k^{ij}=0$, and in the light of Theorem 3, clearly each distribution P_k^{ij} cannot generate a finite-dimensional ω_k^* for $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{S}'(\mathbb{R}^{n+1})$; otherwise $n+2\langle k\rangle-\frac{1}{2}+i-j-1\in\mathbb{Z}$ which is impossible in view of Theorem 3. That is our Assumption 3 does not hold true.

Theorem 4. *The solution $u_k(x,t)$ to the Cauchy problem (5) does not satisfy the Huygens' principle.*

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