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Existence of Solutions for Fractional Multi-Point Boundary Value Problems on an Infinite Interval at Resonance

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Abstract: This paper aims to investigate a class of fractional multi-point boundary value problems at resonance on an infinite interval. New existence results are obtained for the given problem using Mawhin's coincidence degree theory. Moreover, two examples are given to illustrate the main results.

Keywords: fractional differential equation; multi-point boundary value problem; resonance; infinite interval; coincidence degree theory

MSC: 34A08; 34B15

1. Introduction

Fractional calculus is a generalization of classical integer-order calculus and has been studied for more than 300 years. Unlike integer-order derivatives, the fractional derivative is a non-local operator, which implies that the future states depend on the current state as well as the history of all previous states. From this point of view, fractional differential equations provide a powerful tool for mathematical modeling of complex phenomena in science and engineering practice (see [1–7]). For example, an epidemic model of non-fatal disease in a population over a lengthy time interval can be described by fractional differential equations:

$$\begin{cases} D_0^\alpha x(t) = -\beta x(t)y(t), \\ D_0^\alpha y(t) = \beta x(t)y(t) - \gamma y(t), \\ D_0^\alpha z(t) = \gamma y(t), \end{cases}$$

where $0 < \alpha \leq 1$, D_0^α is the Caputo fractional derivative of order α, $x(t)$ represents the number of susceptible individuals, $y(t)$ expresses the number of infected individuals that can spread the disease to susceptible individuals through contact, and $z(t)$ is the number of isolated individuals who cannot contract or transmit the disease for various reasons (see [1]). In [2], Ateş and Zegeling investigated the following fractional-order advection–diffusion–reaction boundary value problem (BVP):

$$\begin{cases} \varepsilon {}^C D^\alpha x + \gamma x' + f(x) = S(t), \quad t \in [0,1], \\ x(0) = x_L, \quad x(1) = x_R, \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \varepsilon \leq 1$, $\gamma \in \mathbb{R}$, ${}^C D^\alpha$ is the Caputo fractional derivative of order α and $S(t)$ is a spatially dependent source term.

In recent years, the discussion of fractional initial value problems (IVPs) and BVPs have attracted the attention of many scholars and valuable results have been obtained (see [8–33]). Various methods have been utilized to study fractional IVPs and BVPs such as the Banach contraction map principle (see [8–11]), fixed point theorems (see [12–18]), monotone iterative method (see [19–21]), variational method (see [22–24]), fixed point index theory (see [17–25]), coincidence degree theory (see [26–29]), and numerical methods [30,31]. For instance, Jiang (see [26]) studied the existence of solutions using coincidence degree theory for the following fractional BVP:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)), \quad a.e.\ t \in [0,1]. \\ u(0) = 0, \ \ D_{0+}^{\alpha-1}u(0) = \sum_{i=1}^{m} a_i D_{0+}^{\alpha-1}u(\xi_i), \\ D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m} b_j D_{0+}^{\alpha-2}u(\eta_j), \end{cases}$$

where $2 < \alpha < 3$, D_{0+}^{α} is the Riemann–Liouville fractional derivative of order α.

BVPs on an infinite interval arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena such as plasmas, unsteady flow of gas through a semi-infinite porous medium, and electric potential of an isolated atom (see [34]). Numerous papers discuss BVPs of integer-order differential equations on infinite intervals (see [35–38]). Naturally, BVPs of fractional differential equations on infinite intervals have received some attention (see [8,12,14–16,18–20,27,29,32]). For example, Wang et al. [8] considered the following fractional BVPs on an infinite interval:

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t)) = 0, \ 2 < \alpha \leq 3, \ t \in [0, +\infty) \\ u(0) = u'(0) = 0, \ D^{\alpha-1}u(\infty) = \xi I^{\beta}u(\eta), \ \beta > 0. \end{cases}$$

where D^{α} is the Riemann–Liouville fractional derivative of order α, I^{β} is the Riemann–Liouville fractional integral of order β, $f \in C([0, +\infty) \times \mathbb{R}, \mathbb{R})$, $\xi \in \mathbb{R}$ and $\eta \in [0, +\infty)$. Then, employing the Banach contraction mapping principle, the author established the existence results.

Motivated by the aforementioned work, this paper uses coincidence degree theory to investigate the existence of solutions for the following fractional BVP:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-1}u(t)), \ \ 0 < t < +\infty, \\ u(0) = 0, \ D_{0+}^{\alpha-2}u(0) = \sum_{i=1}^{m} \alpha_i D_{0+}^{\alpha-2}u(\xi_i), \\ D_{0+}^{\alpha-1}u(+\infty) = \sum_{j=1}^{n} \beta_j D_{0+}^{\alpha-1}u(\eta_j), \end{cases} \tag{1}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, $2 < \alpha \leq 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < +\infty$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < +\infty$, $\alpha_i, \beta_j \in \mathbb{R}$, $f : \ [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ Carathéodory's criterion, i.e., $f(t, u, v, w)$ is Lebesgue measurable in t for all $(u, v, w) \in \mathbb{R}^3$, and continuous in (u, v, w) for a.e. $t \in [0, +\infty)$.

Throughout this paper, we assume the following conditions hold:

(H_1) $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = 1, \ \sum_{i=1}^{m} \alpha_i \xi_i = 0.$

(H_2) There exist nonnegative functions $\delta(t), \beta(t), \eta(t), \gamma(t) \in L^1[0, +\infty)$ such that $\forall t \in [0, +\infty)$ and $(u, v, w) \in \mathbb{R}^3$,

$$|f(t, u, v, w)| \leq \delta(t)\frac{|u|}{1 + t^{\alpha-1}} + \beta(t)\frac{|v|}{1 + t} + \eta(t)\,|w| + \gamma(t),$$

where we let $\Sigma := ||\delta||_{L^1} + ||\beta||_{L^1} + ||\eta||_{L^1}$, $||\kappa||_{L^1} = \int_0^{+\infty} |\kappa(t)|dt$, $\kappa = \delta, \beta, \eta$.

(H_3) $\Delta := a_{11}a_{22} - a_{12}a_{21} \neq 0$, where

$$a_{11} = -1 + \sum_{i=1}^{m} \alpha_i e^{-\xi_i}, \quad a_{12} = \sum_{j=1}^{n} \beta_j e^{-\eta_j},$$
$$a_{21} = -2 + \sum_{i=1}^{m} \alpha_i (2 + \xi_i) e^{-\xi_i}, \quad a_{22} = \sum_{j=1}^{n} \beta_j (1 + \eta_j) e^{-\eta_j}.$$

A BVP is called a resonance problem if the corresponding homogeneous BVP has nontrivial solution. According to (H_1), we will consider the following homogeneous BVP of fractional BVP (1):

$$\begin{cases} D_{0+}^{\alpha}u(t)=0, \ \ 0<t<+\infty, \\ u(0)=0, \ D_{0+}^{\alpha-2}u(0)=\sum_{i=1}^{m}\alpha_i D_{0+}^{\alpha-2}u(\xi_i), \\ D_{0+}^{\alpha-1}u(+\infty)=\sum_{j=1}^{n}\beta_j D_{0+}^{\alpha-1}u(\eta_j). \end{cases} \tag{2}$$

By Lemma 2 (see Section 2), BVP (2) has nontrivial solution $u(t)=at^{\alpha-1}+bt^{\alpha-2}$, $a,b\in\mathbb{R}$, which implies that BVP (1) is a resonance problem and the kernel space of linear operator $Lu=D_{0+}^{\alpha}u$ is two-dimensional, i.e., $\dim\mathrm{Ker}L=2$ (see Section 3, Lemma 7).

In this paper we aim to show the existence of solutions for BVP (1). To the authors' knowledge, the existence of solutions for fractional BVPs at resonance with $\dim\mathrm{Ker}L=2$ on an infinite interval has not been reported. Thus, this article provides new insights. Firstly, our paper extends results from $\dim\mathrm{Ker}L=1$ to $\dim\mathrm{Ker}L=2$ [27,29] and from finite interval to infinite interval [26]. Secondly, we generalize the results of [37,38] to fractional-order cases. Meanwhile, in the previously literature [37,38] authors established the existence results are based on similar conditions to (H_4) and (H_5) (see Section 3, Theorem 1). In the present paper we also show that existence results can be obtained by imposing sign conditions (see Section 3, Theorem 2).

The main difficulties in solving the present BVP are: Constructing suitable Banach spaces for BVP (1); Since $[0,+\infty)$ is noncompact, it is difficult to prove that operator N is L-compact; The theory of Mawhin's continuation theorem is characterized by higher dimensions of the kernel space on resonance BVPs, therefore, constructing projections P and Q is difficult; Estimating a priori bounds of the resonance problem on an infinite interval with $\dim\mathrm{Ker}L=2$ (see Section 3, Lemmas 11–16).

The rest of this paper is organized as follows. Section 2, we recall some preliminary definitions and lemmas; Section 3, existence results are established for BVP (1) using Mawhin's continuation theorem; Section 4 provides two examples to illustrate our main results; Finally, conclusions of this work are outlined in Section 5.

2. Preliminaries

In this section, we recall some definitions and lemmas which are used throughout this paper.

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two real Banach spaces. Suppose $L:\mathrm{dom}L\subset X\to Y$ is a Fredholm operator with index zero then there exist two continuous projectors $P:X\to X$ and $Q:Y\to Y$ such that

$$\mathrm{Im}\,P=\mathrm{Ker}L,\ \mathrm{Im}\,L=\mathrm{Ker}Q,\ X=\mathrm{Ker}L\oplus\mathrm{Ker}P,\ Y=\mathrm{Im}\,L\oplus\mathrm{Im}\,Q,$$

and the mapping $L|_{\mathrm{dom}L\cap\mathrm{Ker}P}:\mathrm{dom}L\to\mathrm{Im}\,L$ is invertible. We denote $K_p=(L|_{\mathrm{dom}L\cap\mathrm{Ker}P})^{-1}$. Let Ω be an open bounded subset of X and $\mathrm{dom}L\cap\bar{\Omega}\neq\emptyset$. The map $N:X\to Y$ is called L-compact on $\bar{\Omega}$, if $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N(\bar{\Omega})=K_p(I-Q)N:\bar{\Omega}\to X$ is compact (see [39,40]).

Lemma 1. *(see [39,40]). Let $L:\mathrm{dom}L\subset X\to Y$ be a Fredholm operator of index zero and $N:X\to Y$ is L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lu\neq\lambda Nu$ *for any* $u\in(\mathrm{dom}L\backslash\mathrm{Ker}L)\cap\partial\Omega$, $\lambda\in(0,1)$;
(ii) $Nu\notin\mathrm{Im}\,L$ *for any* $u\in\mathrm{Ker}L\cap\partial\Omega$;
(iii) $\deg\{QN|_{\mathrm{Ker}L},\Omega\cap\mathrm{Ker}L,0\}\neq 0$.

Then the equation $Lu=Nu$ *has at least one solution in* $\mathrm{dom}L\cap\bar{\Omega}$.

Definition 1. *(see [4,5]). The Rieman-Liouville fractional integral of order $\alpha > 0$ for a function $u : (0, +\infty) \to \mathbb{R}$ is defined as*

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Definition 2. *(see [4,5]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function $u : (0, +\infty) \to \mathbb{R}$ is defined as*

$$D_{0+}^{\alpha} u(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Lemma 2. *(see [18]). Let $\alpha > 0$. Assume that $u \in C[0, +\infty) \cap L^1(0, +\infty)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0,$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, c_i \in \mathbb{R}, i = 1, 2, \ldots, n, n = [\alpha] + 1$, as the unique solution.

Lemma 3. *(see [4,5]) Assume that $\alpha > 0,\ \lambda > -1,\ t > 0$, then*

$$I_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} t^{\alpha+\lambda},\ \ D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} t^{\lambda-\alpha},$$

in particular $D_{0+}^{\alpha} t^{\alpha-m} = 0,\ m = 1, 2, \cdots, n$, where $n = [\alpha] + 1$.

Lemma 4. *(see [4,5]) Let $\alpha > \beta > 0$. Assume that $f(t) \in L^1(\mathbb{R}^+)$, then the following formulas hold:*

$$D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t),\ \ D_{0+}^{\beta} I_{0+}^{\alpha} f(t) = I_{0+}^{\alpha-\beta} f(t).$$

Lemma 5. *(see [4,5]) Let $\alpha > 0,\ m \in \mathbb{N}$ and $D = d/dx$. If the fractional derivatives $(D_{0+}^{\alpha} u)(t)$ and $(D_{0+}^{\alpha+m} u)(t)$ exist, then*

$$(D^m D_{0+}^{\alpha} u)(t) = (D_{0+}^{\alpha+m} u)(t).$$

3. Main Result

Let

$$\begin{aligned} X = \Bigg\{ u \,\Big|\, & u, D_{0+}^{\alpha-2} u, D_{0+}^{\alpha-1} u \in C[0, +\infty),\ \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\alpha-1}} < +\infty, \\ & \sup_{t \geq 0} \frac{|D_{0+}^{\alpha-2} u(t)|}{1+t} < +\infty,\ \sup_{t \geq 0} \left| D_{0+}^{\alpha-1} u(t) \right| < +\infty \Bigg\}, \\ Y = {} & L^1[0, +\infty), \end{aligned}$$

with norms

$$\|u\|_X = \max \left\{ \|u\|_0, \left\| D_{0+}^{\alpha-2} u \right\|_1, \left\| D_{0+}^{\alpha-1} u \right\|_{\infty} \right\}, \|y\|_Y = ||y||_{L^1},$$

respectively, where

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)|\, dt, \quad \left\|D_{0+}^{\alpha-1}u\right\|_\infty = \sup_{t\geq 0}\left|D_{0+}^{\alpha-1}u(t)\right|,$$
$$\|u\|_0 = \sup_{t\geq 0}\frac{|u(t)|}{1+t^{\alpha-1}}, \quad \left\|D_{0+}^{\alpha-2}u\right\|_1 = \sup_{t\geq 0}\frac{\left|D_{0+}^{\alpha-2}u(t)\right|}{1+t}.$$

It is easy to check that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two Banach spaces.

Define the linear operator $L : \mathrm{dom}L \subset X \to Y$ and the nonlinear operator $N : X \to Y$ as follows:

$$Lu = D_{0+}^{\alpha}u, \ u \in \mathrm{dom}L, \quad Nu = f(t, u, D_{0+}^{\alpha-2}u, D_{0+}^{\alpha-1}u), \ u \in X,$$

where

$$\mathrm{dom}\, L = \{u \in X | D_{0+}^{\alpha}u(t) \in Y, \ u \text{ satisfies boundary value conditions of } (1)\}.$$

Then BVP (1) is equivalent to $Lu = Nu$.

Lemma 6. *(see [34]). Let $M \subset X$ be a bounded set. Then M is relatively compact if the following conditions hold:*

(i) *the functions from M are equicontinuous on any compact interval of* $[0, +\infty)$;
(ii) *the functions from M are equiconvergent at infinity.*

Lemma 7. *Assume that* (H_1) *and* (H_3) *hold. Then we have*

$$KerL = \left\{u(t) \in domL : u(t) = at^{\alpha-1} + bt^{\alpha-2}, \forall t \in [0, +\infty), \ a, b \in \mathbb{R}\right\},$$
$$\mathrm{Im}\, L = \{y \in Y : Q_1 y = Q_2 y = 0\},$$

where

$$Q_1 y = \textstyle\sum_{i=1}^m \alpha_i \displaystyle\int_0^{\xi_i} (\xi_i - s)y(s)ds, \ Q_2 y = \textstyle\sum_{j=1}^n \beta_j \displaystyle\int_{\eta_j}^{+\infty} y(s)ds.$$

Proof. By Lemmas 2 and 3 and boundary conditions, we obtain

$$\mathrm{Ker}L = \left\{u(t) \in \mathrm{dom}L : u(t) = at^{\alpha-1} + bt^{\alpha-2}, \forall t \in [0, +\infty), \ a, b \in \mathbb{R}\right\} \cong \mathbb{R}^2.$$

Now, we prove that $\mathrm{Im}\, L = \{y \in Y : Q_1 y = Q_2 y = 0\}$. In fact, if $y \in \mathrm{Im}\, L$, then there exists a function $u \in \mathrm{dom}L$, such that $y(t) = D_{0+}^{\alpha}u(t)$. By Lemma 2, we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds.$$

Using Lemmas 3 and 4 and boundary condition $u(0) = 0$, we have $c_3 = 0$,

$$D_{0+}^{\alpha-1}u(t) = c_1\Gamma(\alpha) + \int_0^t y(s)ds$$

and

$$D_{0+}^{\alpha-2}u(t) = c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) + \int_0^t (t-s)y(s)ds.$$

Since $D_{0+}^{\alpha-2}u(0) = \sum_{i=1}^m \alpha_i D_{0+}^{\alpha-2}u(\xi_i)$ and $D_{0+}^{\alpha-1}u(+\infty) = \sum_{j=1}^n \beta_j D_{0+}^{\alpha-1}u(\eta_j)$, we obtain

$$\begin{aligned} D_{0+}^{\alpha-2}u(0) =& c_2\Gamma(\alpha-1) = \sum_{i=1}^m \alpha_i D_{0+}^{\alpha-2}u(\xi_i) \\ =& \sum_{i=1}^m \alpha_i \left[c_1\Gamma(\alpha)\xi_i + c_2\Gamma(\alpha-1) + \int_0^{\xi_i} (\xi_i - s)y(s)ds\right] \\ =& c_2\Gamma(\alpha-1) + \sum_{i=1}^m \alpha_i \int_0^{\xi_i} (\xi_i - s)y(s)ds \end{aligned}$$

and

$$\begin{aligned} D_{0+}^{\alpha-1}u(+\infty) =& c_1\Gamma(\alpha) + \int_0^{+\infty} y(s)ds = \sum_{j=1}^n \beta_j D_{0+}^{\alpha-1}u(\eta_j) \\ =& \sum_{j=1}^n \beta_j \left[c_1\Gamma(\alpha) + \int_0^{\eta_j} y(s)ds\right] \\ =& c_1\Gamma(\alpha) + \sum_{j=1}^n \beta_j \int_0^{\eta_j} y(s)ds. \end{aligned}$$

Thus,

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i} (\xi_i - s)y(s)ds = 0, \quad \sum_{j=1}^n \beta_j \int_{\eta_j}^{+\infty} y(s)ds = 0. \tag{3}$$

On the other hand, for any $y \in Y$ satisfying (3), take $u(t) = I_{0+}^{\alpha}y(t)$, then $u \in \mathrm{dom}L$ and $D_{0+}^{\alpha}u(t) = y \in \operatorname{Im} L$. Thus we have derived that $\operatorname{Im} L = \{y \in Y : Q_1y = Q_2y = 0\}$. □

Define the linear operators $T_1, T_2 : Y \to Y$ by

$$T_1y = \frac{1}{\Delta}(a_{22}Q_1y - a_{21}Q_2y)e^{-t}, \quad T_2y = \frac{1}{\Delta}(-a_{12}Q_1y + a_{11}Q_2y)e^{-t},$$

where $\Delta, a_{ij}(i,j = 1,2)$ are the constants which have been given in (H_3).

Lemma 8. *Define the operators* $P : X \to X_1, \quad Q : Y \to Y_1$ *by*

$$Pu = \frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}u(0)t^{\alpha-2}, \quad Qy = T_1y + (T_2y)t,$$

where $X_1 := \mathrm{Ker}L,\ Y_1 := \operatorname{Im} Q$. *Then* L *is a Fredholm operator with index zero.*

Proof. Obviously, P is a projection operator and $\operatorname{Im} P = \mathrm{Ker}L$. For $u \in X$, we have $u = (u - Pu) + Pu$, that is, $X = \mathrm{Ker}P + \mathrm{Ker}L$. It is easy to show that $\mathrm{Ker}L \cap \mathrm{Ker}P = \{0\}$. So, $X = \mathrm{Ker}L \oplus \mathrm{Ker}P$. Noting that the definitions of the operators T_1 and T_2, we see Q is a linear operator. On the other hand, for $y \in Y$, a routine computation gives

$$T_1(T_1y) = T_1y,\ T_1((T_2y)t) = 0,\ T_2(T_1y) = 0,\ T_2((T_2y)t) = T_2y.$$

It follows that $Q^2y = Q(Qy) = Qy$. Thus, Q is a projection operator. Let $y = (y - Qy) + Qy$, then $Qy \in \operatorname{Im} Q$ and $Q(y - Qy) = 0$, which together with (H_3), yields that

$$Q_1(y - Qy) = Q_2(y - Qy) = 0, i.e., \ (y - Qy) \in \operatorname{Im} L.$$

Hence, $Y = \operatorname{Im} L + \operatorname{Im} Q$. If $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, then $y = Qy = 0$. Therefore, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$ and dim Ker$L$=codim Im L=2. Consequently, we infer that L is a Fredholm operator with index zero. □

Lemma 9. *Define operator $K_p : \operatorname{Im} L \to domL \cap KerP$ by*

$$K_p y = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds, \ \ y \in \operatorname{Im} L.$$

Then K_p is the inverse operator of $L\,|_{domL \cap KerP}$ and $\left\|K_p y\right\|_X \le \|y\|_{L^1}$.

Proof. For any $y \in \operatorname{Im} L \subset Y$, then $Q_1 y = Q_2 y = 0$ and $K_p y = I_{0+}^{\alpha} y$. By Lemma 4 and condition (H_1), it is not difficult to verify that $K_p y \in \mathrm{dom}L \cap \mathrm{Ker}P$. Hence, K_p is well defined. We now prove that $K_p = (L\,|_{\mathrm{dom}L \cap \mathrm{Ker}P})^{-1}$. In fact, for $u \in \mathrm{dom}L \cap \mathrm{Ker}P$, by Lemma 3, we have

$$K_p L u = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_{0+}^{\alpha} u(s) ds = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

Since $K_p L u \in \mathrm{dom}L \cap \mathrm{Ker}P$, then $K_p L u(0) = 0$ and $P(K_p L u) = 0$, which yields that $c_1 = c_2 = c_3 = 0$. Therefore, $K_p L u = u$, for any $u \in \mathrm{dom}L \cap \mathrm{Ker}P$. In view of Lemma 4, it is straightforward to show that $L K_p y = y$ for any $y \in \operatorname{Im} L$. Then

$$K_p = (L\,|_{\mathrm{dom}L \cap \mathrm{Ker}P})^{-1}.$$

It remains to show that $\left\|K_p y\right\|_X \le \|y\|_{L^1}$. Indeed,

$$\begin{aligned}\left\|K_p y\right\|_0 =& \sup_{t \ge 0} \frac{\left|K_p y\right|}{1+t^{\alpha-1}} = \sup_{t \ge 0} \frac{1}{\Gamma(\alpha)} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} y(s) ds \right| \\ \le& \frac{1}{\Gamma(\alpha)} \|y\|_{L^1} \le \|y\|_{L^1},\end{aligned}$$

$$\left\|D_{0+}^{\alpha-2} K_p y\right\|_1 = \sup_{t \ge 0} \frac{\left|D_{0+}^{\alpha-2} K_p y\right|}{1+t} = \sup_{t \ge 0} \left| \int_0^t \frac{t-s}{1+t} y(s) ds \right| \le \|y\|_{L^1}$$

and

$$\left\|D_{0+}^{\alpha-1} K_p y\right\|_\infty = \sup_{t \ge 0} \left| \int_0^t y(s) ds \right| \le \|y\|_{L^1}.$$

Thus we arrive at the conclusion that $\left\|K_p y\right\|_X \le \|y\|_{L^1}$ for any $y \in \operatorname{Im} L$. □

Lemma 10. *Suppose that (H_2) holds and Ω is an open bounded subset of X such that $domL \cap \bar{\Omega} \neq \emptyset$, then N is L-compact on $\bar{\Omega}$.*

Proof. Since Ω is bounded in X, there exists a constant $r > 0$ such that $\|u\|_X \leq r$ for any $u \in \bar{\Omega}$. Then, by (H_2), we have

$$\begin{aligned}
|Q_1 Nu| &= \left|\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} (\xi_i - s) f(s, u(s), D_{0+}^{\alpha-2}u(s), D_{0+}^{\alpha-1}u(s)) ds\right| \\
&= \left|\sum_{i=1}^{m} \alpha_i \xi_i \int_0^{\xi_i} \frac{\xi_i - s}{\xi_i} f(s, u(s), D_{0+}^{\alpha-2}u(s), D_{0+}^{\alpha-1}u(s)) ds\right| \\
&\leq \sum_{i=1}^{m} |\alpha_i \xi_i| \int_0^{\xi_i} \left| f(s, u(s), D_{0+}^{\alpha-2}u(s), D_{0+}^{\alpha-1}u(s)) \right| ds \\
&\leq \sum_{i=1}^{m} |\alpha_i \xi_i| \int_0^{+\infty} \left| f(s, u(s), D_{0+}^{\alpha-2}u(s), D_{0+}^{\alpha-1}u(s)) \right| ds \\
&\leq \sum_{i=1}^{m} |\alpha_i \xi_i| \left(\Sigma \|u\|_X + \|\gamma\|_{L^1}\right) := m_1
\end{aligned}$$

and

$$\begin{aligned}
|Q_2 Nu| &= \left|\sum_{j=1}^{n} \beta_j \int_{\eta_j}^{+\infty} f(s, u(s), D_{0+}^{\alpha-2}u(s), D_{0+}^{\alpha-1}u(s)) ds\right| \\
&\leq \sum_{j=1}^{n} |\beta_j| \int_{\eta_j}^{+\infty} \left| f(s, u(s), D_{0+}^{\alpha-2}u(s), D_{0+}^{\alpha-1}u(s)) \right| ds \\
&\leq \sum_{j=1}^{n} |\beta_j| \left(\Sigma \|u\|_X + \|\gamma\|_{L^1}\right) := m_2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|QNu\|_{L^1} &= \int_0^{+\infty} |QNu(s)| ds \leq \int_0^{+\infty} |T_1 Nu(s)| ds + \int_0^{+\infty} |T_2 Nu(s)| sds \\
&= \int_0^{+\infty} \left| \frac{1}{\Delta} \left(a_{22} Q_1 Nu(s) - a_{21} Q_2 Nu(s)\right) e^{-s} \right| ds \\
&\quad + \int_0^{+\infty} \left| \frac{1}{\Delta} \left(-a_{12} Q_1 Nu(s) + a_{11} Q_2 Nu(s)\right) se^{-s} \right| ds \\
&\leq \frac{1}{|\Delta|} \int_0^{+\infty} \left(|a_{22}| |Q_1 Nu(s)| + |a_{21}| |Q_2 Nu(s)|\right) e^{-s} ds \\
&\quad + \frac{1}{|\Delta|} \int_0^{+\infty} \left(|a_{12}| |Q_1 Nu(s)| + |a_{11}| |Q_2 Nu(s)|\right) se^{-s} ds \\
&\leq \frac{1}{|\Delta|} \left(|a_{22}| m_1 + |a_{21}| m_2\right) + \frac{1}{|\Delta|} \left(|a_{12}| m_1 + |a_{11}| m_2\right) \\
&= \frac{1}{|\Delta|} \left[\left(|a_{12}| + |a_{22}|\right) m_1 + \left(|a_{11}| + |a_{21}|\right) m_2\right] := m.
\end{aligned}$$

This means that $QN(\bar{\Omega})$ is bounded. Next, we show that $K_{P,Q}N(\bar{\Omega})$ on $[0, +\infty)$ is compact. To this end, we divide our proof in three steps. First, we need to prove that $K_{P,Q}N : \bar{\Omega} \to Y$ is bounded. In fact, for any $u \in \bar{\Omega}$, we have

$$\begin{aligned}
\|Nu\|_{L^1} &= \left| \int_0^{+\infty} f(s, u(s), D_{0+}^{\alpha-2}u(s), D_{0+}^{\alpha-1}u(s)) ds \right| \\
&\leq \int_0^{+\infty} \left| f(s, u(s), D_{0+}^{\alpha-2}u(s), D_{0+}^{\alpha-1}u(s)) \right| ds \\
&\leq \Sigma \|u\|_X + \|\gamma\|_{L^1} := m_3.
\end{aligned}$$

Then

$$\begin{aligned}\frac{\left|K_{P,Q}Nu(t)\right|}{1+t^{\alpha-1}} &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\,(I-Q)\,Nu(s)ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{+\infty}(|Nu(s)|+|QNu(s)|)ds \\ &= \frac{1}{\Gamma(\alpha)}\left(\|Nu\|_{L^1}+\|QNu\|_{L^1}\right) \leq \frac{m+m_3}{\Gamma(\alpha)},\end{aligned}$$

$$\begin{aligned}\frac{\left|D_{0+}^{\alpha-2}K_{P,Q}Nu(t)\right|}{1+t} &= \left|\int_0^t \frac{t-s}{1+t}\,(I-Q)\,Nu(s)ds\right| \\ &\leq \int_0^{+\infty}(|Nu(s)|+|QNu(s)|)ds \\ &= \left(\|Nu\|_{L^1}+\|QNu\|_{L^1}\right) \leq m+m_3\end{aligned}$$

and

$$\begin{aligned}\left|D_{0+}^{\alpha-1}K_{P,Q}Nu(t)\right| &= \left|\int_0^t (I-Q)\,Nu(s)ds\right| \\ &\leq \int_0^{+\infty}(|Nu(s)|+|QNu(s)|)ds \\ &= \left(\|Nu\|_{L^1}+\|QNu\|_{L^1}\right) \leq m+m_3.\end{aligned}$$

Thus we conclude that $K_{P,Q}N\,(\bar{\Omega})$ is bounded. The next thing to do in the proof is that $K_{P,Q}N\,(\bar{\Omega})$ is equicontinuous on any subcompact interval of $[0,+\infty)$. Indeed, for $u\in\bar{\Omega}$, by (H_2), we have

$$|Nu(s)| \leq \alpha(s)\frac{|u(s)|}{1+s^{\alpha-1}}+\beta(s)\frac{\left|D_{0+}^{\alpha-2}u(s)\right|}{1+s}+\eta(s)\left|D_{0+}^{\alpha-1}u(s)\right|+\gamma(s)$$

and

$$\begin{aligned}&|QNu(s)| = |T_1Nu+(T_2Nu)s| \\ &\leq \frac{1}{|\Delta|}|(a_{22}Q_1Nu-a_{21}Q_2Nu)e^{-s}|+\frac{1}{|\Delta|}|(-a_{12}Q_1Nu+a_{11}Q_2Nu)se^{-s}| \\ &\leq \frac{1}{|\Delta|}\left[(|a_{22}||Q_1Nu|+|a_{21}||Q_2Nu|)+(|a_{12}||Q_1Nu|+|a_{11}||Q_2Nu|)s\right]e^{-s} \\ &\leq \frac{1}{|\Delta|}\left[(|a_{22}|m_1+|a_{21}|m_2)+(|a_{12}|m_1+|a_{11}|m_2)s\right]e^{-s}.\end{aligned}$$

Let κ be any finite positive constant on $[0,+\infty)$, then for any $t_1,t_2\in[0,\kappa]$ (without loss of generality we assume that $t_1<t_2$), we obtain

$$
\begin{aligned}
&\left|\frac{K_{P,Q}Nu(t_1)}{1+t_1^{\alpha-1}}-\frac{K_{P,Q}Nu(t_2)}{1+t_2^{\alpha-1}}\right| \\
=&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}(I-Q)\,Nu(s)ds-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}(I-Q)\,Nu(s)ds\right| \\
\le&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}(I-Q)\,Nu(s)ds-\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}(I-Q)\,Nu(s)ds\right| \\
&+\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}(I-Q)\,Nu(s)ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}(I-Q)\,Nu(s)ds\right| \\
\le&\frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}(I-Q)\,Nu(s)ds\right| \\
&+\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left(\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right)(I-Q)\,Nu(s)ds\right| \\
\le&\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}|(I-Q)\,Nu(s)|\,ds+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right||(I-Q)\,Nu(s)|\,ds \\
&\to 0,\quad \text{as } t_1\to t_2.
\end{aligned}
$$

Proceeding as in the proof of above, we can obtain

$$
\left|\frac{D_{0+}^{\alpha-2}K_{P,Q}Nu(t_1)}{1+t_1}-\frac{D_{0+}^{\alpha-2}K_{P,Q}Nu(t_2)}{1+t_2}\right|\to 0,\quad \text{as } t_1\to t_2,
$$

and

$$
\begin{aligned}
&\left|D_{0+}^{\alpha-1}K_{P,Q}Nu(t_1)-D_{0+}^{\alpha-1}K_{P,Q}Nu(t_2)\right| \\
=&\left|\int_{t_1}^{t_2}(I-Q)\,Nu(s)ds\right|\le\int_{t_1}^{t_2}|(I-Q)\,Nu(s)|\,ds\to 0,\quad \text{as } t_1\to t_2.
\end{aligned}
$$

Consequently, we infer that $K_{P,Q}N\,(\bar{\Omega})$ is equicontinuous on $[0,\kappa]$. Finally, we have to show that $K_{P,Q}N\,(\bar{\Omega})$ is equiconvergent at infinity. As a matter of fact, for any $u\in\bar{\Omega}$, we have

$$
\int_0^{+\infty}|(I-Q)\,Nu(t)|\,dt\le\|Nu\|_{L^1}+\|QNu\|_{L^1}\le m_3+m.
$$

Hence, for given $\varepsilon>0$, there exists a positive constant L such that

$$
\int_L^{+\infty}|(I-Q)\,Nu(t)|\,dt<\varepsilon. \tag{4}
$$

On the other hand, since $\lim\limits_{t\to+\infty}\dfrac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}}=1$ and $\lim\limits_{t\to+\infty}\dfrac{t-L}{1+t}=1$, then for above $\varepsilon>0$ there exists a constant $T>L>0$ such that for any $t_1,t_2\ge T$ and $0\le s\le L$, we have

$$
\begin{aligned}
&\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|=\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-1+1-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right| \\
\le&\left(1-\frac{(t_1-L)^{\alpha-1}}{1+t_1^{\alpha-1}}\right)+\left(1-\frac{(t_2-L)^{\alpha-1}}{1+t_2^{\alpha-1}}\right)<\varepsilon
\end{aligned} \tag{5}
$$

and

$$
\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right|\le\left(1-\frac{t_1-L}{1+t_1}\right)+\left(1-\frac{t_2-L}{1+t_2}\right)<\varepsilon. \tag{6}
$$

Thus, for any $t_1, t_2 \geq T > L > 0$, by (4)–(6), we get

$$\begin{aligned}
&\left|\frac{K_{P,Q}Nu(t_1)}{1+t_1^{\alpha-1}} - \frac{K_{P,Q}Nu(t_2)}{1+t_2^{\alpha-1}}\right| \\
=&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\left(I-Q\right)Nu(s)ds - \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\left(I-Q\right)Nu(s)ds\right| \\
\leq&\frac{1}{\Gamma(\alpha)}\int_0^L\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\left|\left(I-Q\right)Nu(s)\right|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_L^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\left|\left(I-Q\right)Nu(s)\right|ds + \frac{1}{\Gamma(\alpha)}\int_L^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\left|\left(I-Q\right)Nu(s)\right|ds \\
\leq&\frac{\varepsilon}{\Gamma(\alpha)}\int_0^L\left|\left(I-Q\right)Nu(s)\right|ds + \frac{2}{\Gamma(\alpha)}\int_L^{+\infty}\left|\left(I-Q\right)Nu(s)\right|ds \\
<&\frac{(m+m_3+2)\varepsilon}{\Gamma(\alpha)}.
\end{aligned}$$

Using the similar argument as in the proof of above, we can show that

$$\left|\frac{D_{0+}^{\alpha-2}K_{P,Q}Nu(t_1)}{1+t_1} - \frac{D_{0+}^{\alpha-2}K_{P,Q}Nu(t_2)}{1+t_2}\right| < (m+m_3+2)\varepsilon,$$

and

$$\begin{aligned}
&\left|D_{0+}^{\alpha-1}K_{P,Q}Nu(t_1) - D_{0+}^{\alpha-1}K_{P,Q}Nu(t_2)\right| \\
=&\left|\int_{t_1}^{t_2}\left(I-Q\right)Nu(s)ds\right| \leq \int_L^{+\infty}\left|\left(I-Q\right)Nu(t)\right|dt < \varepsilon.
\end{aligned}$$

Thus we arrive at the conclusion that $K_{P,Q}N\left(\bar{\Omega}\right)$ is equiconvergent at infinity. According to Lemma 6, it follows that $K_{P,Q}N\left(\bar{\Omega}\right)$ is relatively compact. Therefore, N is L-compact on $\bar{\Omega}$. □

Theorem 1. *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *and the following conditions hold:*

(H_4) *There exist positive constants A and B such that, for all $u(t) \in \mathrm{dom}L\backslash\mathrm{Ker}L$, if one of the following conditions is satisfied:*

$$(i)\ |D_{0+}^{\alpha-2}u(t)| > A \text{ for any } t \in [0,B]; \quad (ii)\ |D_{0+}^{\alpha-1}u(t)| > A \text{ for any } t \in [0,+\infty),$$

then either $Q_1Nu \neq 0$ or $Q_2Nu \neq 0$.

(H_5) *There exists a positive constant C such that, for every $a, b \in \mathbb{R}$ satisfying $|a| > C$ or $|b| > C$, then either*

$$aQ_1N(at^{\alpha-1}+bt^{\alpha-2}) + bQ_2N(at^{\alpha-1}+bt^{\alpha-2}) < 0, \tag{7}$$

or

$$aQ_1N(at^{\alpha-1}+bt^{\alpha-2}) + bQ_2N(at^{\alpha-1}+bt^{\alpha-2}) > 0. \tag{8}$$

Then boundary value problem (1) has at least one solution in X provided that

$$[(3+B)\Gamma(\alpha) + (\alpha-1)B + 1]\Sigma < \Gamma(\alpha).$$

To prove the Theorem 1, we need several lemmas.

Lemma 11. *Assume that* (H_2) *and* (H_4) *hold, then the set*

$$\Omega_1 = \{u \in domL \backslash KerL : Lu = \lambda Nu,\ \lambda \in (0,1)\}$$

is bounded in X.

Proof. For $u \in \Omega_1$, then $Nu \in \operatorname{Im} L$, this implies

$$Q_1 Nu = Q_2 Nu = 0.$$

Thus, it follows from assumption (H_4) that there exist constants $t_0 \in [0, B]$ and $t_1 \in [0, +\infty)$ such that $\left|D_{0+}^{\alpha-2}u(t_0)\right| \le A$ and $\left|D_{0+}^{\alpha-1}u(t_1)\right| \le A$. These, combined with the Lemma 5, we obtain

$$\begin{aligned}
\left|D_{0+}^{\alpha-1}u(t)\right| &= \left|D_{0+}^{\alpha-1}u(t_1) + \int_{t_1}^{t} D_{0+}^{\alpha}u(s)ds\right| \\
&\le \left|D_{0+}^{\alpha-1}u(t_1)\right| + \int_{t_1}^{t} \left|D_{0+}^{\alpha}u(s)\right| ds \le A + \|Nu\|_{L^1}
\end{aligned}$$

and

$$\begin{aligned}
\left|D_{0+}^{\alpha-2}u(0)\right| &= \left|D_{0+}^{\alpha-2}u(t_0) - \int_0^{t_0} D_{0+}^{\alpha-1}u(s)ds\right| \le A + \left|\int_0^{t_0} D_{0+}^{\alpha-1}u(s)ds\right| \\
&\le A + \left\|D_{0+}^{\alpha-1}u(t)\right\|_{\infty} B \le A(1+B) + B\|Nu\|_{L^1}.
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}
\|Pu\|_0 &= \sup_{t\ge 0} \frac{|Pu|}{1+t^{\alpha-1}} \\
&= \sup_{t\ge 0} \frac{1}{1+t^{\alpha-1}} \left|\frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2}u(0)t^{\alpha-2}\right| \\
&\le \frac{1}{\Gamma(\alpha)} \left|D_{0+}^{\alpha-1}u(0)\right| \sup_{t\ge 0} \frac{t^{\alpha-1}}{1+t^{\alpha-1}} + \frac{1}{\Gamma(\alpha-1)} \left|D_{0+}^{\alpha-2}u(0)\right| \sup_{t\ge 0} \frac{t^{\alpha-2}}{1+t^{\alpha-1}} \\
&\le \frac{1}{\Gamma(\alpha)} \left(A + \|Nu\|_{L^1}\right) + \frac{1}{\Gamma(\alpha-1)} \left[A(1+B) + B\|Nu\|_{L^1}\right]
\end{aligned}$$

and

$$\begin{aligned}
\left\|D_{0+}^{\alpha-1}Pu\right\|_{\infty} &= \left|D_{0+}^{\alpha-1}u(0)\right| \le A + \|Nu\|_{L^1}, \\
\left\|D_{0+}^{\alpha-2}Pu\right\|_{1} &= \sup_{t\ge 0} \frac{\left|D_{0+}^{\alpha-1}u(0)t + D_{0+}^{\alpha-2}u(0)\right|}{1+t} \\
&\le \left(A + \|Nu\|_{L^1}\right) + A(1+B) + B\|Nu\|_{L^1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|Pu\|_X &= \max\left\{\|Pu\|_0, \left\|D_{0+}^{\alpha-2}Pu\right\|_0, \left\|D_{0+}^{\alpha-1}Pu\right\|_{\infty}\right\} \\
&\le \|Pu\|_0 + \left\|D_{0+}^{\alpha-2}Pu\right\|_0 + \left\|D_{0+}^{\alpha-1}Pu\right\|_{\infty} \\
&\le \frac{2\Gamma(\alpha)+1}{\Gamma(\alpha)} \left(A + \|Nu\|_{L^1}\right) + \frac{\Gamma(\alpha-1)+1}{\Gamma(\alpha-1)} \left[A(1+B) + B\|Nu\|_{L^1}\right].
\end{aligned} \tag{9}$$

Noting that $(I-P)\,u \in \mathrm{dom}L \cap \mathrm{Ker}P$ and $LPu = 0$, by Lemma 9, we have

$$\|(I-P)\,u\|_X = \left\|K_p L\,(I-P)\,u\right\|_X \le \|L\,(I-P)\,u\|_{L^1} = \|Lu\|_{L^1} \le \|Nu\|_{L^1}. \tag{10}$$

Combining Formulas (9) and (10), we obtain

$$\begin{aligned}
&\|u\|_X = \|Pu + (I-P)\,u\|_X \le \|Pu\|_X + \|(I-P)\,u\|_X \\
&\le \frac{2\Gamma(\alpha)+1}{\Gamma(\alpha)}\left(A + \|Nu\|_{L^1}\right) + \frac{\Gamma(\alpha-1)+1}{\Gamma(\alpha-1)}\left[A\,(1+B) + B\,\|Nu\|_{L^1}\right] + \|Nu\|_{L^1} \\
&= \Xi A + \Theta||Nu||_{L^1} \le \Xi A + \Theta(\Sigma||u||_X + ||\gamma||_{L^1}),
\end{aligned}$$

where

$$\Xi = 3 + B + \frac{1}{\Gamma(\alpha)} + \frac{1+B}{\Gamma(\alpha-1)},\;\; \Theta = 3 + B + \frac{1}{\Gamma(\alpha)} + \frac{B}{\Gamma(\alpha-1)}.$$

Solving the above inequality gives

$$\|u\|_X \le \frac{\Xi A + \Theta\,\|\gamma\|_{L^1}}{1 - \Theta\Sigma}.$$

Thus we have derived that Ω_1 is bounded. □

Lemma 12. *Assume that* (H_5) *holds, then the set*

$$\Omega_2 = \{u \in KerL : Nu \in \mathrm{Im}\,L\}$$

is bounded in X.

Proof. Let $u \in \Omega_2$, then u can be written as $u = at^{\alpha-1} + bt^{\alpha-2},\ a, b \in \mathbb{R}$ and $Q_1Nu = Q_2Nu = 0$. According to the assumption (H_5), it follows that $|a| \le C$ and $|b| \le C$. Hence, we have

$$\left\|D_{0+}^{\alpha-1}u\right\|_\infty = |a\Gamma(\alpha)| \le C\Gamma(\alpha)$$

and

$$\begin{aligned}
&\sup_{t\ge 0}\frac{|u|}{1+t^{\alpha-1}} = \sup_{t\ge 0}\frac{\left|at^{\alpha-1} + bt^{\alpha-2}\right|}{1+t^{\alpha-1}} \le |a| + |b| \le 2C, \\
&\sup_{t\ge 0}\frac{\left|D_{0+}^{\alpha-2}u\right|}{1+t} = \sup_{t\ge 0}\frac{|a\Gamma(\alpha)t + b\Gamma(\alpha-1)|}{1+t} \\
&\qquad \le |a|\,\Gamma(\alpha) + |b|\,\Gamma(\alpha-1) \le \left(\Gamma(\alpha) + \Gamma(\alpha-1)\right)C.
\end{aligned}$$

Thus we conclude that Ω_2 is bounded. □

Lemma 13. *Assume that* (H_5) *holds, then the set*

$$\Omega_3 = \{u \in KerL : \vartheta\lambda Ju + (1-\lambda)QNu = 0,\ \lambda \in [0,1]\}$$

is bounded in X, where

$$\vartheta = \begin{cases} -1, & \textit{if (7) holds,} \\ 1, & \textit{if (8) holds,} \end{cases}$$

$J : KerL \to \mathrm{Im}\, Q$ *is the linear isomorphism operator defined by*

$$J(at^{\alpha-1} + bt^{\alpha-2}) = \frac{1}{\Delta}(a_{22}a - a_{21}b)e^{-t} + \frac{1}{\Delta}(-a_{12}a + a_{11}b)te^{-t}\ a, b \in \mathbb{R}.$$

Proof. Without loss of generality, we may assume hypothesis (7) holds. For $u \in \Omega_3$, we can write u in the form $u = at^{\alpha-1} + bt^{\alpha-2}, a, b \in \mathbb{R}$ and $\lambda Ju = (1-\lambda)QNu, \lambda \in [0,1]$. Using the same argument as in the proof of Lemma 12, we need only show that $|a| \le C$ and $|b| \le C$. In fact, if $\lambda = 0$, then $QNu = 0$, that is,

$$\frac{1}{\Delta}(a_{22}Q_1Nu - a_{21}Q_2Nu)e^{-t} + \frac{1}{\Delta}(-a_{12}Q_1Nu + a_{11}Q_2Nu)te^{-t} = 0.$$

Thus,

$$\begin{cases} a_{22}Q_1Nu - a_{21}Q_2Nu = 0, \\ -a_{12}Q_1Nu + a_{11}Q_2Nu = 0. \end{cases}$$

It follows from $\Delta \neq 0$ that $Q_1Nu = Q_2Nu = 0$. By (H_5), we obtain $|a| \le C,\ |b| \le C$. If $\lambda = 1$, then $Ju = 0$, that is,

$$\frac{1}{\Delta}(a_{22}a - a_{21}b)e^{-t} + \frac{1}{\Delta}(-a_{12}a + a_{11}b)te^{-t} = 0.$$

From this it follows that

$$\begin{cases} a_{22}a - a_{21}b = 0, \\ -a_{12}a + a_{11}b = 0. \end{cases}$$

Since $\Delta \neq 0$, we obtain $a = b = 0$. For $\lambda \in (0,1)$, by $\lambda Ju = (1-\lambda)QNu$, we have

$$\begin{aligned} &\lambda \left[\frac{1}{\Delta}(a_{22}a - a_{21}b)\, e^{-t} + \frac{1}{\Delta}(-a_{12}a + a_{11}b)\, te^{-t}\right] \\ &= (1-\lambda)\left[\frac{1}{\Delta}(a_{22}Q_1Nu - a_{21}Q_2Nu)\, e^{-t} + \frac{1}{\Delta}(-a_{12}Q_1Nu + a_{11}Q_2Nu)\, te^{-t}\right], \end{aligned}$$

from which we deduce that

$$\begin{cases} \lambda a_{22}a - \lambda a_{21}b = (1-\lambda)a_{22}Q_1Nu - (1-\lambda)a_{21}Q_2Nu, \\ \lambda a_{11}b - \lambda a_{12}a = (1-\lambda)a_{11}Q_2Nu - (1-\lambda)a_{12}Q_1Nu. \end{cases}$$

In view of $\Delta \neq 0$, we get

$$\begin{cases} \lambda a = (1-\lambda)Q_1Nu, \\ \lambda b = (1-\lambda)Q_2Nu. \end{cases}$$

We are now in a position to claim that $|a| \le C$ and $|b| \le C$. If the assertion would not hold, then by (7), we obtain

$$\lambda(a^2 + b^2) = (1-\lambda)(aQ_1Nu + bQ_2Nu) < 0.$$

This leads to a contradiction. Consequently, we infer that Ω_3 is bounded. □

We now turn to the proof of Theorem 1.

Proof. Let $\Omega \subset X$ be a bounded open set such that $\cup_{i=1}^{3}\bar{\Omega}_i \subset \Omega$. It follows from Lemma 10 that N is L-compact on $\bar{\Omega}$. Applying Lemmas 11 and 12, we obtain

(i) $Lu \neq \lambda Nu$ for any $u \in (\mathrm{dom}L \backslash \mathrm{Ker}L) \cap \partial\Omega,\ \lambda \in (0,1)$;
(ii) $Nu \notin \mathrm{Im}\, L$ for any $u \in \mathrm{Ker}L \cap \partial\Omega$.

We finally remark that $\deg\{QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$. To show this, we define

$$H(u,\lambda) = \vartheta\lambda Ju + (1-\lambda)QNu.$$

From Lemma 13 we conclude that $H(u,\lambda) \neq 0$ for any $u \in \mathrm{Ker}L \cap \partial\Omega,\ \lambda \in [0,1]$.

Hence, by the homotopy of degree, we have

$$\begin{aligned}\deg\{QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0\} &= \deg\{H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0\}\\ &= \deg\{H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0\}\\ &= \deg\{\vartheta J, \Omega \cap \mathrm{Ker}L, 0\} \neq 0.\end{aligned}$$

According to Lemma 1, it follows that $Lu = Nu$ has at least one solution in $\mathrm{dom}L \cap \bar{\Omega}$, that is, (1) has at least one solution in X. □

Theorem 2. *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *and the following conditions hold:*

(H_6) *There exists a positive constant M such that, for each $u(t) \in domL$ satisfying $|D_{0+}^{\alpha-1}u(t)| > M$ for all $t \in [0,+\infty)$, we have either*

$$\mathrm{sgn}\{D_{0+}^{\alpha-1}u(t)\}Q_2Nu(t) > 0, \quad \forall t \in [0,+\infty) \tag{11}$$

or

$$\mathrm{sgn}\{D_{0+}^{\alpha-1}u(t)\}Q_2Nu(t) < 0, \quad \forall t \in [0,+\infty); \tag{12}$$

(H_7) *There exist positive constants G and $\mathcal{J}$ such that, for every $u(t) \in domL$ satisfying $|D_{0+}^{\alpha-2}u(t)| > G$ for all $t \in [0,\mathcal{J}]$, we have either*

$$\mathrm{sgn}\{D_{0+}^{\alpha-2}u(t)\}Q_1Nu(t) > 0, \quad \forall t \in [0,\mathcal{J}] \tag{13}$$

or

$$\mathrm{sgn}\{D_{0+}^{\alpha-2}u(t)\}Q_1Nu(t) < 0, \quad \forall t \in [0,\mathcal{J}]. \tag{14}$$

Then boundary value problem (1) has at least one solution in X provided that

$$[3 + 2(\alpha-1)\mathcal{J}]\Sigma < \Gamma(\alpha).$$

We shall adopt the same procedure as in the proof of Theorem 1.

Lemma 14. *Assume that* (H_2), (H_6) *and* (H_7) *hold, then Ω_1 (same define as Lemma 11) is bounded in X.*

Proof. For $u \in \Omega_1$, we get $Nu \in \mathrm{Im}\,L = \mathrm{Ker}Q$. By (H_6) and (H_7), there exist constants $t_1 \in [0,+\infty)$, $t_2 \in [0,\mathcal{J}]$ such that $|D_{0+}^{\alpha-1}u(t_1)| \leq M,\ |D_{0+}^{\alpha-2}u(t_2)| \leq G$. This together with the Lemma 5 implies that

$$\begin{aligned}D_{0+}^{\alpha-1}u(t) =& D_{0+}^{\alpha-1}u(t_1) + \int_{t_1}^{t} D_{0+}^{\alpha}u(s)ds,\\ D_{0+}^{\alpha-2}u(t) =& D_{0+}^{\alpha-2}u(t_2) + \int_{t_2}^{t} D_{0+}^{\alpha-1}u(s)ds\\ =& D_{0+}^{\alpha-2}u(t_2) + (t-t_2)D_{0+}^{\alpha-1}u(t_1) + \int_{t_2}^{t}\int_{t_1}^{s} D_{0+}^{\alpha}u(\tau)d\tau ds.\end{aligned}$$

Then, we obtain

$$||D_{0+}^{\alpha-1}u||_\infty \leq M + ||D_{0+}^{\alpha}u||_{L^1}, \tag{15}$$

$$||D_{0+}^{\alpha-2}u||_1 \leq G + M + ||D_{0+}^{\alpha}u||_{L^1}. \tag{16}$$

On the other hand, by Lemma 2, for $u \in \Omega_1 \subset \mathrm{dom}L$, we have

$$u(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \ \ c_1, c_2 \in \mathbb{R},$$

it follows that

$$\frac{u(t)}{1+t^{\alpha-1}} = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} D_{0+}^{\alpha} u(s) ds + \frac{c_1 t^{\alpha-1}}{1+t^{\alpha-1}} + \frac{c_2 t^{\alpha-2}}{1+t^{\alpha-1}}, \tag{17}$$

$$\begin{aligned} D_{0+}^{\alpha-1} u(t) &= \int_0^t D_{0+}^{\alpha} u(s) ds + c_1 \Gamma(\alpha), \\ D_{0+}^{\alpha-2} u(t) &= \int_0^t (t-s) D_{0+}^{\alpha} u(s) ds + c_1 \Gamma(\alpha) t + c_2 \Gamma(\alpha-1) \\ &= -\int_0^t s D_{0+}^{\alpha} u(s) ds + t D_{0+}^{\alpha-1} u(t) + c_2 \Gamma(\alpha-1). \end{aligned}$$

By solving the above equations, we obtain

$$\begin{aligned} c_1 =& \frac{1}{\Gamma(\alpha)} \left(D_{0+}^{\alpha-1} u(t) - \int_0^t D_{0+}^{\alpha} u(s) ds \right), \\ c_2 =& \frac{1}{\Gamma(\alpha-1)} \left[D_{0+}^{\alpha-2} u(t_2) - t_2 D_{0+}^{\alpha-1} u(t_2) + \int_0^{t_2} s D_{0+}^{\alpha} u(s) ds \right]. \end{aligned}$$

These together with the inequalities (15) and (16), we find

$$\begin{aligned} |c_1| \leq& \frac{1}{\Gamma(\alpha)} (||D_{0+}^{\alpha-1} u||_{\infty} + ||D_{0+}^{\alpha} u||_{L^1}) \leq \frac{1}{\Gamma(\alpha)} (M + 2||D_{0+}^{\alpha} u||_{L^1}), \\ |c_2| \leq& \frac{1}{\Gamma(\alpha-1)} (G + \mathcal{J}||D_{0+}^{\alpha-1} u||_{\infty} + \mathcal{J}||D_{0+}^{\alpha} u||_{L^1}) \\ \leq& \frac{1}{\Gamma(\alpha-1)} (G + \mathcal{J}M + 2\mathcal{J}||D_{0+}^{\alpha} u||_{L^1}). \end{aligned} \tag{18}$$

Substituting (18) into (17), one has

$$\begin{aligned} \left| \frac{u(t)}{1+t^{\alpha-1}} \right| \leq& \frac{1}{\Gamma(\alpha)} ||D_{0+}^{\alpha} u||_{L^1} + |c_1| + |c_2| \\ \leq& \frac{1}{\Gamma(\alpha)} [3 + 2(\alpha-1)\mathcal{J}] ||D_{0+}^{\alpha} u||_{L^1} + \frac{M}{\Gamma(\alpha)} + \frac{G + \mathcal{J}M}{\Gamma(\alpha-1)}, \ \forall t \in [0, +\infty). \end{aligned}$$

From this it follows that

$$||u||_0 \leq \frac{1}{\Gamma(\alpha)} [3 + 2(\alpha-1)\mathcal{J}] ||D_{0+}^{\alpha} u||_{L^1} + \frac{M}{\Gamma(\alpha)} + \frac{G + \mathcal{J}M}{\Gamma(\alpha-1)}. \tag{19}$$

Combining formulas (15), (16) and (19) gives

$$\begin{aligned} ||u||_X =& \max\{||u||_0, ||D_{0+}^{\alpha-2} u||_1, ||D_{0+}^{\alpha-1} u||_{\infty}\} \\ \leq& \frac{1}{\Gamma(\alpha)} [3 + 2(\alpha-1)\mathcal{J}] ||D_{0+}^{\alpha} u||_{L^1} + M + \frac{G + \mathcal{J}M}{\Gamma(\alpha-1)}. \end{aligned} \tag{20}$$

Noting that $Lu = \lambda Nu$, by (H_2), we have

$$||D_{0+}^{\alpha} u||_{L^1} \leq ||Nu||_{L^1} \leq \Sigma ||u||_X + ||\gamma||_{L^1}. \tag{21}$$

It follows from (20) and (21) that

$$||u||_X \le \frac{[3+2(\alpha-1)\mathcal{J}]||\gamma||_{L^1} + M\Gamma(\alpha) + (\alpha-1)(G+\mathcal{J}M)}{\Gamma(\alpha) - [3+2(\alpha-1)\mathcal{J}]\Sigma}.$$

Thus we arrive at the conclusion that Ω_1 is bounded. □

Lemma 15. *Assume that* (H_6), (H_7) *hold, then* Ω_2 *(same define as Lemma 12) is bounded in X.*

Proof. For any $u \in \Omega_2$, then u can be expressed as $u(t) = at^{\alpha-1} + bt^{\alpha-2},\ a, b \in \mathbb{R}$, $t \in [0, +\infty)$ and $Q_1Nu = Q_2Nu = 0$. Using the same argument as in the proof of Lemma 12, to get the desired result, we just need to show that $|a|$ and $|b|$ are bounded. By (H_6) and (H_7), there exist constants $t_3 \in [0, +\infty)$ and $t_4 \in [0, \mathcal{J}]$ such that $|D^{\alpha-1}_{0+}u(t_3)| \le M$, $|D^{\alpha-2}_{0+}u(t_4)| \le G$, i.e.,

$$|D^{\alpha-1}_{0+}u(t_3)| = |a\Gamma(\alpha)| \le M,\ |D^{\alpha-2}_{0+}u(t_4)| = |a\Gamma(\alpha)t_4 + b\Gamma(\alpha-1)| \le G.$$

Then, we obtain

$$|a| \le \frac{M}{\Gamma(\alpha)},\ \ |b| \le \frac{G+\mathcal{J}M}{\Gamma(\alpha-1)}.$$

The proof is completed. □

Lemma 16. *Assume that* (H_6) *and* (H_7) *hold, then the set*

$$\Omega_4 = \left\{u \in KerL : \vartheta\mu\tilde{J}u + (1-\mu)QNu = 0,\ \mu \in [0,1]\right\}.$$

is bounded in X, where

$$\vartheta = \begin{cases} \vartheta_1 = \begin{cases} 1, & \text{if } (3.9) \text{ and } (3.11) \text{ hold}, \\ -1, & \text{if } (3.10) \text{ and } (3.12) \text{ hold}, \end{cases} \\ \vartheta_2 = \begin{cases} 1, & \text{if } (3.10) \text{ and } (3.11) \text{ hold}, \\ -1, & \text{if } (3.9) \text{ and } (3.12) \text{ hold}, \end{cases} \end{cases}$$

$\tilde{J} : KerL \to \operatorname{Im} Q$ *is the linear isomorphism operator defined by*

$$\tilde{J}(at^{\alpha-1} + bt^{\alpha-2}) = \begin{cases} \frac{1}{\Delta}(a_{22}b - a_{21}a)e^{-t} + \frac{1}{\Delta}(a_{11}a - a_{12}b)te^{-t}, \text{ if } \vartheta = \vartheta_1, \\ \frac{1}{\Delta}(a_{22}b + a_{21}a)e^{-t} + \frac{1}{\Delta}(-a_{11}a - a_{12}b)te^{-t}, \text{ if } \vartheta = \vartheta_2, \end{cases} \quad a, b \in \mathbb{R}.$$

Proof. Without loss of generality, we may prove the lemma in the case that (12) and (14) hold. Indeed, for $u \in \Omega_4$, we can express u as $u = at^{\alpha-1} + bt^{\alpha-2},\ a, b \in \mathbb{R}$ and $\mu\tilde{J}u = (1-\mu)QNu,\ \mu \in [0,1]$. Similar proof as Lemma 13, we can show that $|a|$ and $|b|$ are bounded when $\mu = 0$ or $\mu = 1$. Now we prove that $|a|$ and $|b|$ are also bounded for $\mu \in (0,1)$. In fact, by $\mu\tilde{J}u = (1-\mu)QNu$, we have

$$\begin{cases} \mu(a_{22}b - a_{21}a) = (1-\mu)(a_{22}Q_1Nu - a_{21}Q_2Nu), \\ \mu(a_{11}a - a_{12}b) = (1-\mu)(a_{11}Q_2Nu - a_{12}Q_1Nu). \end{cases}$$

Since $\Delta \ne 0$, we obtain

$$\mu a = (1-\mu)Q_2Nu, \tag{22}$$
$$\mu b = (1-\mu)Q_1Nu. \tag{23}$$

From (12) and (22), we can get $|a|\Gamma(\alpha) \le M$; otherwise, by (12) and (22), we have

$$0 \le \mu a \operatorname{sgn}\{a\} = \mu a \operatorname{sgn}\{D^{\alpha-1}_{0+}u(t)\} = (1-\mu)\operatorname{sgn}\{D^{\alpha-1}_{0+}u(t)\}Q_2Nu(t) < 0.$$

It is a contradiction. Similarly, from (14) and (23), we can derive $|b|\Gamma(\alpha-1) \leq G + M\mathcal{J}$; otherwise, by (14) and (23), a contradiction will be obtained:

$$0 \leq \mu b \,\mathrm{sgn}\{b\} = \mu b \,\mathrm{sgn}\{D_{0+}^{\alpha-2}u(t)\} = (1-\mu)\,\mathrm{sgn}\{D_{0+}^{\alpha-2}u(t)\}Q_1 Nu(t) < 0.$$

Consequently, we infer that Ω_4 is bounded. $\square$

With the help of the preceding three lemmas we can now prove the Theorem 2.

Proof. Set $\Omega' \subset X$ be a bounded open set such that $\cup_{i=1}^{2}\bar{\Omega}_i \cup \bar{\Omega}_4 \subset \Omega'$. Using Lemma 10, N is L-compact on $\bar{\Omega}'$. It follows from Lemma 14 and Lemma 15 that conditions (i) and (ii) of Lemma 1 hold. In what follows, we prove that condition (iii) is satisfied. To this end, we set

$$H(u,\mu) = \vartheta\mu\tilde{J}u + (1-\mu)QNu.$$

By Lemma 16, we obtain $H(u,\mu) \neq 0$ for any $u \in \mathrm{Ker}L \cap \partial\Omega',\ \mu \in [0,1]$. Based on the homotopy of degree, we have

$$\begin{aligned}\deg\left\{QN|_{\mathrm{Ker}L}, \Omega' \cap \mathrm{Ker}L, 0\right\} &= \deg\left\{H(\cdot,0), \Omega' \cap \mathrm{Ker}L, 0\right\} \\ &= \deg\left\{H(\cdot,1), \Omega' \cap \mathrm{Ker}L, 0\right\} \\ &= \deg\left\{\vartheta\tilde{J}, \Omega' \cap \mathrm{Ker}L, 0\right\} \neq 0.\end{aligned}$$

According to Lemma 1, the equation $Lu = Nu$ has at least one solution in $\mathrm{dom}L \cap \bar{\Omega}'$, which means (1) has at least one solution in X. $\square$

4. Example

Example 1. *Consider the following boundary value problem:*

$$\begin{cases} D_{0+}^{2.5}u(t) = f(t, u(t), D_{0+}^{0.5}u(t), D_{0+}^{1.5}u(t)), \ \ t \in (0,+\infty), \\ u(0) = 0, \ \ D_{0+}^{0.5}u(0) = 2D_{0+}^{0.5}u(1/2) - D_{0+}^{0.5}u(1), \\ D_{0+}^{1.5}u(+\infty) = D_{0+}^{1.5}u(1). \end{cases} \tag{24}$$

Corresponding to problem (1), here

$$m = 2,\ n = 1,\ \alpha_1 = 2,\ \alpha_2 = -1,\ \xi_1 = \frac{1}{2},\ \xi_2 = 1,\ \ \beta_1 = \eta_1 = 1,$$

$$\begin{aligned}&f(t, u(t), D_{0+}^{0.5}u(t), D_{0+}^{1.5}u(t)) \\ &= \begin{cases} e^{-10t}D_{0+}^{\alpha-2}u(t),\ t \in [0,1], \\ 0.4(-0.1e^{-5t} + 0.1e^{-10t} + 0.01e^{1-t})D_{0+}^{\alpha-1}u(t),\ t \in (1,+\infty). \end{cases}\end{aligned}$$

Let

$$\delta(t) = \gamma(t) = 0,\ \ \beta(t) = \begin{cases} (1+t)e^{-10t}, & t \in [0,1], \\ 0, & t \in (1,+\infty), \end{cases}$$

$$\eta(t) = \begin{cases} 0, & t \in [0,1], \\ \dfrac{1}{25}e^{-5t} + \dfrac{1}{25}e^{-10t} + \dfrac{1}{250}e^{1-t}, & t \in (1,+\infty). \end{cases}$$

We can easily check $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold and*

$$||\beta||_1 = \frac{11}{100} - \frac{21}{100}e^{-10},\ \ ||\eta||_1 = \frac{1}{250} + \frac{1}{125}e^{-5} + \frac{1}{250}e^{-10}.$$

Take $A = 100,\ B = 1$, *we can check that for any* $t \in [0,1]$ *if* $|D^{0.5}_{0+}u(t)| > A$, *we have* $Q_1Nu \neq 0$ *and for any* $t \in [0,+\infty)$ *if* $|D^{1.5}_{0+}u(t)| > A$, *we get* $Q_2Nu \neq 0$. *Moreover, for every* $C > 0$, *if* $|a| > C$, *then we have*

$$\begin{aligned}
&aQ_1\left(N\left(at^{\alpha-1}+bt^{\alpha-2}\right)\right)+bQ_2\left(N\left(at^{\alpha-1}+bt^{\alpha-2}\right)\right)\\
=&a^2\Gamma(\alpha)\left(-\frac{3}{1000}+\frac{7}{500}e^{-5}-\frac{11}{1000}e^{-10}\right)<0.
\end{aligned}$$

By Theorem 1, BVP (24) has at least one solution.

Example 2. *Consider the following fractional boundary value problem:*

$$\begin{cases}
D^{2.5}_{0+}u(t) = f(t,u(t),D^{0.5}_{0+}u(t),D^{1.5}_{0+}u(t)),\ \ 0<t<+\infty,\\
u0 = 0,\ D^{0.5}_{0+}u(0) = 2D^{0.5}_{0+}u(1) - D^{0.5}_{0+}u(2),\\
D^{1.5}_{0+}u(+\infty) = 0.5D^{1.5}_{0+}u(2) + 0.5D^{1.5}_{0+}u(3).
\end{cases} \tag{25}$$

Corresponding to problem (1), here

$$\alpha{=}2.5,\ m{=}n{=}2,\ \alpha_1{=}2,\ \alpha_2{=}-1,\ \xi_1{=}1,\ \xi_2{=}2,\ \beta_1{=}\beta_2{=}0.5,\ \eta_1{=}2,\ \eta_2{=}3,$$

$$\begin{aligned}
&f(t,u(t),D^{0.5}_{0+}u(t),D^{1.5}_{0+}u(t))\\
=&\frac{1}{20}e^{-3t}\sin\left(\frac{u(t)}{1+t^{1.5}}\right)+\frac{1}{15}g_1(t)e^{-2t}D^{0.5}_{0+}u(t)+\frac{1}{15}g_2(t)e^{-2t}D^{1.5}_{0+}u(t)+\frac{1}{10}e^{-t},
\end{aligned}$$

where

$$g_1(t)=\begin{cases}1,\ \ t\in(1,2),\\ 0,\ t\in[0,1]\cup[2,+\infty),\end{cases}\quad g_2(t)=\begin{cases}0,\ \ t\in[0,2],\\ 1,\ t\in(2,+\infty).\end{cases}$$

Let

$$\delta(t)=\frac{1}{20}e^{-3t},\ \beta(t)=\frac{1}{15}(1+t)e^{-2t},\ \eta(t)=\frac{1}{15}e^{-2t},\ \gamma(t)=\frac{1}{10}e^{-t},\ \mathcal{J}=2.$$

We can easily check that $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold and*

$$[3+2(\alpha-1)\mathcal{J}]\Sigma=\frac{9}{10}<\frac{3}{4}\sqrt{\pi}=\Gamma(\alpha).$$

To verify the conditions (H_6) *and* (H_7), *we let*

$$\Phi(t)=\frac{1}{20}e^{-3t}\sin\left(\frac{u(t)}{1+t^{1.5}}\right)+\frac{1}{10}e^{-t}.$$

Then, we have

$$\begin{aligned}
&\frac{1}{2}\int_2^{+\infty}\Phi(t)dt+\frac{1}{2}\int_3^{+\infty}\Phi(t)dt\\
\le&\frac{1}{2}\int_2^{+\infty}\left(\frac{1}{20}e^{-3t}+\frac{1}{10}e^{-t}\right)dt+\frac{1}{2}\int_3^{+\infty}\left(\frac{1}{20}e^{-3t}+\frac{1}{10}e^{-t}\right)dt\\
=&\frac{1}{120}e^{-6}+\frac{1}{20}e^{-2}+\frac{1}{120}e^{-9}+\frac{1}{20}e^{-3}<\frac{1}{10}(1+e^{-2}),
\end{aligned}$$

and

$$
\begin{aligned}
&2\int_0^1 (1-t)\Phi(t)dt - \int_0^2 (2-t)\Phi(t)dt \\
&\leq \int_0^1 (1-t)\left(\frac{1}{10}e^{-3t} + \frac{1}{5}e^{-t}\right)dt + \int_0^2 (2-t)\left(\frac{1}{20}e^{-3t} + \frac{1}{10}e^{-t}\right)dt \\
&= \frac{1}{5}e^{-1} + \frac{1}{10}e^{-2} + \frac{1}{90}e^{-3} + \frac{1}{180}e^{-6} + \frac{3}{20} < \frac{1}{5}(e+e^{-1}).
\end{aligned}
$$

Choosing $M = 6e^4,\ G = 12e^3$, *we conclude that*

(i) *for* $|D_{0+}^{1.5}u(t)| > M,\ t \in [0,+\infty)$, *one has*

$$
\begin{aligned}
&\operatorname{sgn}\{D_{0+}^{\alpha-1}u(t)\}Q_2Nu(t) \\
&= \operatorname{sgn}\{D_{0+}^{1.5}u(t)\}\left[\frac{1}{2}\int_2^{+\infty}\Phi(t)dt + \frac{1}{30}\int_2^{+\infty} e^{-2t}D_{0+}^{1.5}u(t)dt \right.\\
&\left. + \frac{1}{2}\int_3^{+\infty}\Phi(t)dt + \frac{1}{30}\int_3^{+\infty} e^{-2t}D_{0+}^{1.5}u(t)dt\right] > 0,
\end{aligned}
$$

(ii) *for* $|D_{0+}^{0.5}u(t)| > G,\ t \in [0,2]$, *one gets*

$$
\begin{aligned}
&\operatorname{sgn}\{D_{0+}^{\alpha-2}u(t)\}Q_1Nu(t) \\
&= \operatorname{sgn}\{D_{0+}^{0.5}u(t)\}\left[2\int_0^1 (1-t)\Phi(t)dt - \int_0^2 (2-t)\Phi(t)dt \right.\\
&\left. - \frac{1}{15}\int_1^2 (2-t)e^{-2t}D_{0+}^{0.5}u(t)dt\right] < 0.
\end{aligned}
$$

Therefore, (H_6) *and* (H_7) *hold. By Theorem 2, BVP (25) has at least one solution.*

5. Conclusions

In the present work, we considered a class of fractional differential equations with multi-point boundary conditions at resonance on an infinite interval. With the aid of Mawhin's continuation theorem, we obtained existence results for solutions of BVP (1). Two practical examples were presented to illustrate the main results. BVPs of fractional differential equations on an infinite interval have been widely discussed in recent years. However, there is still more work to be done in the future on this interesting problem. For example, establishing the existence of solutions for fractional differential equations with infinite-point boundary conditions, as well as the existence of non-negative solutions for fractional BVPs, at resonance on an infinite interval in the case of $\dim \mathrm{Ker} L = 2$.

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