mathematics

Article

Some Identities of Degenerate Bell Polynomials

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Received: 10 December 2019; Accepted: 26 December 2019; Published: 1 January 2020

Abstract: The new type degenerate of Bell polynomials and numbers were recently introduced, which are a degenerate version of Bell polynomials and numbers and are different from the previously introduced partially degenerate Bell polynomials and numbers. Several expressions and identities on those polynomials and numbers were obtained. In this paper, as a further investigation of the new type degenerate Bell polynomials, we derive several identities involving those degenerate Bell polynomials, Stirling numbers of the second kind and Carlitz's degenerate Bernoulli or degenerate Euler polynomials. In addition, we obtain an identity connecting the degenerate Bell polynomials, Cauchy polynomials, Bernoulli numbers, Stirling numbers of the second kind and degenerate Stirling numbers of the second kind.

Keywords: new type degenerate Bell polynomials; degenerate Bernoulli polynomials; degenerate Euler polynomials; degenerate Cauchy polynomials

MSC: 11B73; 11B83; 05A19

1. Introduction

For any nonzero $\lambda \in \mathbb{R}$, the Carlitz's degenerate Bernoulli polynomials are defined by (see [1])

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

Note that $\lim_{\lambda\to 0} \beta_{n,\lambda}(x) = B_n(x), \quad (n \geq 0)$. Here, $B_n(x)$ are ordinary Bernoulli polynomials which are defined by (see [1–13])

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{2}$$

The degenerate Euler polynomials are given by (see [1])

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \tag{3}$$

When $x = 0$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers.

For any nonzero real number λ, the degenerate exponential functions are defined by (see [8,10])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \tag{4}$$

From (4), we note that (see [8,10])

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \tag{5}$$

where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), (n \geq 1)$.

The Stirling numbers of the first kind are defined by

$$(x)_n = \lim_{\lambda \to 1} (x)_{n,\lambda} = \sum_{k=0}^{n} S_1(n,k) x^k, \quad (n \geq 0), \tag{6}$$

and the Stirling numbers of the second kind are given by (see [12,13])

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{7}$$

As is well known, the Bell polynomials are defined by (see [5,12])

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x) \frac{t^n}{n!}, \tag{8}$$

When $x = 1$, $Bel_n = Bel_n(1), \ (n \geq 0)$, are called the Bell numbers.

We note that the left hand side of (8) is equal to

$$\begin{aligned} \sum_{k=0}^{\infty} x^k \frac{1}{k!}(e^t-1)^k &= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} S_2(n,k) x^k \frac{t^n}{n!}, \end{aligned} \tag{9}$$

and hence we obtain

$$Bel_n(x) = \sum_{k=0}^{n} S_2(n,k) x^k, \quad (n \geq 0). \tag{10}$$

It is known that the Cauchy polynomials are given by the generating function (see [7])

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!}, \tag{11}$$

In view of (11), we may consider the degenerate Cauchy polynomials which are given by

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} C_{n,\lambda}(x) \frac{t^n}{n!}. \tag{12}$$

Here, $\log_\lambda t$ is the compositional inverse function of $e_\lambda(t)$ such that $e_\lambda(\log_\lambda(t)) = \log_\lambda(e_\lambda(t)) = t$.

Recently, the new type degenerate Bell polynomials are introduced by the generating function as (see [10]).

$$e_\lambda^x(e^t-1) = \left(1+\lambda(e^t-1)\right)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x) \frac{t^n}{n!}, \tag{13}$$

When $x = 1$, $Bel_{n,\lambda} = Bel_{n,\lambda}(1)$ are called the degenerate Bell numbers.

Note that $\lim_{\lambda\to 0} Bel_{n,\lambda}(x) = Bel_n(x)$, $(n \geq 0)$. We see that the middle term in (13) is equal to

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\frac{x}{\lambda}\right)_k \lambda^k \frac{1}{k!}(e^t - 1)^k &= \sum_{k=0}^{\infty} (x)_{k,\lambda} \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} S_2(n,k)(x)_{k,\lambda}\frac{t^n}{n!}, \end{aligned} \tag{14}$$

and hence we get

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n} S_2(n,k)(x)_{k,\lambda}. \tag{15}$$

Studying degenerate versions of some special polynomials has been very fruitful and regained lively interest of many mathematicians in recent years. In [1], Carlitz initiated a study of degenerate versions of some special polynomials, namely the degenerate Bernoulli and Euler polynomials and numbers.

In the recent paper [10], the new type degenerate Bell polynomials $Bel_{n,\lambda}(x)$ (see (13)) were introduced and some interesting results about them were obtained, which are different from the previously defined partially degenerate Bell polynomials (see [9]) and a degenerate version of the ordinary Bell polynomials $Bel_n(x)$ (see (8)).

As a further study of the new type degenerate Bell polynomials, we will obtain two expressions involving these degenerate Bell polynomials, Carlitz's degenerate Bernoulli polynomials and the Stirling numbers of the second kind, two identities involving those degenerate Bell polynomials, degenerate Euler polynomials and the Stirling numbers of the second kind. In additon, we will be able to find an identity involving those degenerate Bell polynomials, Cauchy polynomials, Bernoulli numbers, Stirling numbers of the second kind and degenerate Stirling numbers of the second kind.

2. Some Identities of Degenerate Bell Polynomials

The following equation can be easily derived from (13):

$$Bel_{n,\lambda}(x) = \sum_{m=0}^{n} \sum_{l=0}^{m} x^l \lambda^{m-l} S_1(m,l) S_2(n,m), \tag{16}$$

where $n \geq 0$.

Indeed, we see that the middle term in (13) is equal to

$$\begin{aligned} \sum_{m=0}^{\infty} \left(\frac{x}{\lambda}\right)_m \lambda^m \frac{1}{m!}(e^t - 1)^m &= \sum_{m=0}^{\infty} \sum_{l=0}^{m} S_1(m,l) \left(\frac{x}{\lambda}\right)^l \lambda^m \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=0}^{m} x^l \lambda^{m-l} S_1(m,l) S_2(n,m)\frac{t^n}{n!}. \end{aligned} \tag{17}$$

By using (16), we can show that

$$\begin{aligned} &Bel_{0,\lambda}(x) = 1, Bel_{1,\lambda}(x) = x, Bel_{2,\lambda}(x) = -x\lambda + (x^2 + x), \\ &Bel_{3,\lambda}(x) = 2x\lambda^2 + (-3x^2 - 3x)\lambda + (x^3 + 3x^2 + x), \\ &Bel_{4,\lambda}(x) = -6x\lambda^3 + (11x^2 + 12x)\lambda^2 + (-6x^3 - 18x^2 - 7x)\lambda + (x^4 + 6x^3 + 7x^2 + x), \\ &Bel_{5,\lambda}(x) = 24x\lambda^4 + (-50x^2 - 60x)\lambda^3 + (35x^3 + 110x^2 + 50x)\lambda^2 \\ &+(-10x^4 - 60x^3 - 75x^2 - 15x)\lambda + (x^5 + 10x^4 + 25x^3 + 15x^2 + x). \end{aligned}$$

By replacing t by e^t-1 in (1), we get

$$\begin{aligned}\frac{e^t-1}{e_\lambda(e^t-1)-1}e_\lambda^x(e^t-1) &= \sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\frac{1}{m!}(e^t-1)^m \\ &= \sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\beta_{m,\lambda}(x)S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{18}$$

On the other hand,

$$\begin{aligned}\frac{e^t-1}{e_\lambda(e^t-1)-1}e_\lambda^x(e^t-1) &= \sum_{m=0}^{\infty}\beta_{m,\lambda}\frac{1}{m!}(e^t-1)^m\sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\beta_{m,\lambda}S_2(k,m)\frac{t^k}{k!}\sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\beta_{m,\lambda}S_2(k,m)Bel_{n-k,\lambda}(x)\right\}\frac{t^n}{n!}.\end{aligned} \tag{19}$$

Therefore, by comparing the coefficients on both sides of (18) and (19), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\beta_{m,\lambda}(x)S_2(n,m) = \sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\beta_{m,\lambda}S_2(k,m)Bel_{n-k,\lambda}(x).$$

Let us replace t by e^t-1 in (3). Then we get

$$\begin{aligned}\frac{2}{e_\lambda(e^t-1)+1}e_\lambda^x(e^t-1) &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}(x)\frac{1}{m!}(e^t-1)^m \\ &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}(x)\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\mathcal{E}_{m,\lambda}(x)S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{20}$$

The left hand side of (20) is also given by

$$\begin{aligned}\frac{2}{e_\lambda(e^t-1)+1}e_\lambda^x(e^t-1) &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}\frac{(e^t-1)^m}{m!}\sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}\sum_{k=m}^{\infty}S_2(k,m)\frac{t^k}{k!}\sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\mathcal{E}_{m,\lambda}S_2(k,m)\frac{t^k}{k!}\sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\mathcal{E}_{m,\lambda}S_2(k,m)Bel_{n-k,\lambda}(x)\right\}\frac{t^n}{n!}.\end{aligned} \tag{21}$$

Therefore, from (20) and (21), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} \mathcal{E}_{m,\lambda}(x) S_2(n,m) = \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{k} \mathcal{E}_{m,\lambda} S_2(k,m) Bel_{n-k,\lambda}(x).$$

From (1), we have

$$te_\lambda^x(t) = \sum_{l=0}^{\infty} \beta_{l,\lambda}(x) \frac{t^l}{l!} (e_\lambda(t) - 1). \tag{22}$$

Replacing t by $e^t - 1$ in (22), we get

$$(e^t - 1) e_\lambda^x(e^t - 1) = \sum_{l=0}^{\infty} \beta_{l,\lambda}(x) \frac{(e^t-1)^l}{l!} (e_\lambda(e^t - 1) - 1). \tag{23}$$

The right hand side of (23) is equal to

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{l=0}^{m} \beta_{l,\lambda}(x) S_2(m,l) \frac{t^m}{m!} \sum_{k=1}^{\infty} Bel_{k,\lambda} \frac{t^k}{k!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \binom{n}{k} \sum_{l=0}^{n-k} \beta_{l,\lambda}(x) S_2(n-k,l) Bel_{k,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{24}$$

On the other hand, the left hand side of (23) is equal to

$$\sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{m=0}^{\infty} Bel_{m,\lambda}(x) \frac{t^m}{m!} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{n}{k} Bel_{n-k,\lambda}(x) \frac{t^n}{n!}. \tag{25}$$

Therefore, by equating (24) and (25), we obtain the following theorem.

Theorem 3. *For any $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} \binom{n}{k} Bel_{n-k,\lambda}(x) = \sum_{k=1}^{n} \binom{n}{k} \sum_{l=0}^{n-k} \beta_{l,\lambda}(x) S_2(n-k,l) Bel_{k,\lambda}.$$

Setting $x = 0$ in Theorem 3, we get the following corollary.

Corollary 1. *For any $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{l=0}^{n-k} \beta_{l,\lambda} S_2(n-k,l) Bel_{k,\lambda} = 1.$$

From (3), we note that

$$2 = \sum_{l=0}^{\infty} \mathcal{E}_{l,\lambda} \frac{t^l}{l!} (e_\lambda(t) + 1). \tag{26}$$

By (26), we get

$$
\begin{aligned}
2 &= \sum_{l=0}^{\infty} \mathcal{E}_{l,\lambda} \frac{1}{l!}(e^t-1)^l \Big(e_\lambda(e^t-1)+1 \Big) \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \mathcal{E}_{l,\lambda} S_2(k,l) \frac{t^k}{k!} \Big(\sum_{m=0}^{\infty} Bel_{m,\lambda} \frac{t^m}{m!} + 1 \Big) \\
&= \sum_{n=0}^{\infty} \Big\{ \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \mathcal{E}_{l,\lambda} S_2(k,l) Bel_{n-k,\lambda} + \sum_{k=0}^{n} \mathcal{E}_{k,\lambda} S_2(n,k) \Big\} \frac{t^n}{n!}.
\end{aligned}
\tag{27}
$$

By comparing the coefficients on both sides of (27), we get

$$
\sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \mathcal{E}_{l,\lambda} S_2(k,l) Bel_{n-k,\lambda} + \sum_{k=0}^{n} \mathcal{E}_{k,\lambda} S_2(n,k) = \begin{cases} 2, & \text{if } n=0, \\ 0, & \text{if } n \geq 1. \end{cases} \tag{28}
$$

Therefore, by (28), we obtain the following theorem.

Theorem 4. *For $n \in \mathbb{N}$, we have*

$$
\sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \mathcal{E}_{l,\lambda} S_2(k,l) Bel_{n-k,\lambda} = -\sum_{k=0}^{n} \mathcal{E}_{k,\lambda} S_2(n,k).
$$

In other words, for $n \geq 0$, we have

$$
\sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \mathcal{E}_{l,\lambda} S_2(k,l) Bel_{n-k,\lambda} + \sum_{k=0}^{n} \mathcal{E}_{k,\lambda} S_2(n,k) = 2Bel_{n,\lambda}(0).
$$

Let us replace t by $e_\lambda(e^t-1)-1$ in (12). Then we have

$$
\frac{e_\lambda(e^t-1)-1}{e^t-1} e_\lambda^x(e^t-1) = \sum_{m=0}^{\infty} C_{m,\lambda}(x) \frac{1}{m!} (e_\lambda(e^t-1)-1)^m. \tag{29}
$$

As is well known, the degenerate Stirling numbers of second kind are defined by (see [8])

$$
\frac{1}{m!}(e_\lambda(t)-1)^m = \sum_{k=m}^{\infty} S_{2,\lambda}(k,m) \frac{t^k}{k!}, \tag{30}
$$

By (29) and (30), we have

$$
\begin{aligned}
\sum_{m=0}^{\infty} C_{m,\lambda}(x) \frac{1}{m!}(e_\lambda(e^t-1)-1)^m &= \sum_{m=0}^{\infty} C_{m,\lambda}(x) \sum_{k=m}^{\infty} S_{2,\lambda}(k,m) \frac{1}{k!}(e^t-1)^k \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} C_{m,\lambda}(x) S_{2,\lambda}(k,m) \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \sum_{m=0}^{k} C_{m,\lambda}(x) S_{2,\lambda}(k,m) S_2(n,k) \Big) \frac{t^n}{n!}.
\end{aligned}
\tag{31}
$$

Hence, by (29) and (31), we get

$$
\frac{e_\lambda(e^t-1)-1}{e^t-1} e_\lambda^x(e^t-1) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \sum_{m=0}^{k} C_{m,\lambda}(x) S_{2,\lambda}(k,m) S_2(n,k) \Big) \frac{t^n}{n!}. \tag{32}
$$

On the other hand,

$$\begin{aligned}
&\frac{e_\lambda(e^t-1)-1}{e^t-1}e_\lambda^x(e^t-1)=\frac{e_\lambda^{x+1}(e^t-1)-e_\lambda^x(e^t-1)}{e^t-1}\\
&=\left(\frac{t}{e^t-1}\right)\frac{1}{t}\left(e_\lambda^{x+1}(e^t-1)-e_\lambda^x(e^t-1)\right)\\
&=\left(\sum_{k=0}^\infty B_k\frac{t^k}{k!}\right)\frac{1}{t}\sum_{m=0}^\infty\left(Bel_{m,\lambda}(x+1)-Bel_{m,\lambda}(x)\right)\frac{t^m}{m!}\\
&=\sum_{k=0}^\infty B_k\frac{t^k}{k!}\sum_{m=0}^\infty\left(\frac{Bel_{m+1,\lambda}(x+1)-Bel_{m+1,\lambda}(x)}{m+1}\right)\frac{t^m}{m!}\\
&=\sum_{n=0}^\infty\left\{\sum_{m=0}^n\binom{n}{m}\left(\frac{Bel_{m+1,\lambda}(x+1)-Bel_{m+1,\lambda}(x)}{m+1}\right)B_{n-m}\right\}\frac{t^n}{n!},
\end{aligned}\tag{33}$$

where B_n are the ordinary Bernoulli numbers.

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$\begin{aligned}
&\sum_{k=0}^n\sum_{m=0}^k C_{m,\lambda}(x)S_{2,\lambda}(k,m)S_2(n,k)\\
&=\sum_{m=0}^n\binom{n}{m}\left(\frac{Bel_{m+1,\lambda}(x+1)-Bel_{m+1,\lambda}(x)}{m+1}\right)B_{n-m}.
\end{aligned}$$

3. Conclusions

As a further study of the new type degenerate Bell polynomials, we obtained two expressions involving these degenerate Bell polynomials, Carlitz's degenerate Bernoulli polynomials and the Stirling numbers of the second kind, two identities involving those degenerate Bell polynomials, degenerate Euler polynomials and the Stirling numbers of the second kind. In additon, we were able to find an identity involving those degenerate Bell polynomials, Cauchy polynomials, Bernoulli numbers, Stirling numbers of the second kind and degenerate Stirling numbers of the second kind.

In our previous works related to this paper, we studied various degenerate versions of many special polynomials. They have been investigated by using several different means, such as generating functions, combinatorial methods, umbral calculus techniques, probability theory, p-adic analysis, differential equations, and so on. Further, q-analogues of those degenerate versions of special polynomials were also introduced by using bosonic and fermionic p-adic q-integrals, and their number theoretic and combinatorial properties were investigated.

It is one of our research projects to continue this line of study. Namely, we would like to study various degenerate versions of special polynomials and numbers and also their q-analogues, and investigate their possible applications to physics and engineering, as well as to mathematics.

Author Contributions: T.K. and D.S.K. conceived of the framework and structured the whole paper; T.K. wrote the paper; J.K. and H.Y.K. checked the results of the paper; D.S.K. and T.K. completed the revision of the article. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1E1A1A03070882).

Conflicts of Interest: The authors declare no conflict of interest.

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