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The Chebyshev Difference Equation

Tom Cuchta [1,*], Michael Pavelites [1] and Randi Tinney [2]

[1] Department of Computer Science and Math, Fairmont State University, Fairmont, WV 26554, USA; mpavelites@gmail.com
[2] TMC^2 Technologies of West Virginia, Fairmont, WV 26554, USA; rtinney93@gmail.com
* Correspondence: tcuchta@fairmontstate.edu

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Abstract: We define and investigate a new class of difference equations related to the classical Chebyshev differential equations of the first and second kind. The resulting "discrete Chebyshev polynomials" of the first and second kind have qualitatively similar properties to their continuous counterparts, including a representation by hypergeometric series, recurrence relations, and derivative relations.

Keywords: discrete analogue; special function; Chebyshev polynomial; difference equation; generalized hypergeometric series

MSC: 39A12; 39A10; 33C20

1. Introduction

The following two differential equations are known as the Chebyshev differential equations:

$$(1-t^2)y'' - ty' + n^2 y = 0, \tag{1}$$

and

$$(1-t^2)y'' - 3ty' + n(n+2)y = 0. \tag{2}$$

For $n \in \{0,1,2,\ldots\}$, the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n = a(a+1)\ldots(a+n-1). \tag{3}$$

The classical generalized hypergeometric series ${}_p\mathcal{F}_q$ is defined by the formula [1]

$${}_p\mathcal{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{t^k}{k!}. \tag{4}$$

The Chebyshev polynomials are sequences of orthogonal polynomials, usually distinguished between Chebyshev polynomials of the first kind, denoted by $\mathcal{T}_n$, which obey

$$\mathcal{T}_n(t) = {}_2\mathcal{F}_1\left(-n, n; \frac{1}{2}; \frac{1-t}{2}\right), \tag{5}$$

and Chebyshev polynomials of the second kind, $\mathcal{U}_n$, given by

$$\mathcal{U}_n(t) = (n+1){}_2\mathcal{F}_1\left(-n, n+2; \frac{3}{2}; \frac{1-t}{2}\right). \tag{6}$$

Equations (5) and (6) are indeed polynomials due to the definition of the Pochhammer symbol, and they also turn out to be solutions to Equations (1) and (2), respectively.

Both sets of Chebyshev polynomials are sequences of orthogonal polynomials: the $\mathcal{T}_n$ are orthogonal with weight function $\dfrac{1}{\sqrt{1-t^2}}$ integrated over the interval $(-1,1)$ and the $\mathcal{U}_n$ are orthogonal with weight function $\sqrt{1-t^2}$ over $(-1,1)$. Both sequences of polynomials obey the same three-term recurrence relation

$$f_{n+1}(t) = 2tf_n(t) - f_{n-1}(t), \tag{7}$$

but with different initial conditions. The following relationship between $\mathcal{T}_n$ and $\mathcal{U}_n$ is known [2] (2.48) for $n \in \{1,2,\ldots\}$:

$$\mathcal{T}_n'(t) = n\mathcal{U}_{n-1}(t). \tag{8}$$

The following formula relates the difference of two Chebyshev polynomials of the second kind to a Chebyshev polynomial of the first kind [3] (p. 9):

$$\mathcal{U}_n(t) - \mathcal{U}_{n-2}(t) = 2\mathcal{T}_n(t). \tag{9}$$

If $f\colon \{0,1,2,\ldots\} \to \mathbb{R}$, then the forward difference of f, written Δf, is defined by the formula $\Delta f(t) = f(t+1) - f(t)$ and we define a "backwards shift operator" by the formula $(\varrho f)(t) = f(t-1)$. In this article, we investigate solutions of the families of second order difference equations with polynomial coefficients

$$t(t-1)\Delta^2 y(t-2) + 2t\Delta y(t-1) + t\Delta y(t-1) + \Delta y(t) - n^2 y(t) = 0, \tag{10}$$

and

$$t(t-1)\Delta^2 y(t-2) + 2t\Delta^2 y(t-1) + 3t\Delta y(t-1) + 3\Delta y(t) - n(n+2)y(t) = 0, \tag{11}$$

where $t, n \in \{0,1,2,\ldots\}$, which we call the Chebyshev difference equations of the first and second kind, respectively.

There has been recent interest in discrete analogues of special functions, by which we mean a function $f\colon \{0,1,2,\ldots\} \to \mathbb{R}$ that obeys some qualitatively similar properties to a related well-known function $\mathcal{F}\colon \mathbb{R} \to \mathbb{R}$. For instance, a Bessel difference equation was investigated in [4], whose solutions were shown to be generalized hypergeometric series with variable parameters. Such "discrete Bessel functions" were applied in [5] to solve discrete wave and diffusion equations. We define the Θ operator by $\Theta = t\varrho\Delta$. In [6], the Bessel difference equation was generalized to the discrete hypergeometric difference equation,

$$\left[\Theta \prod_{j=1}^{q} \left(\Theta + b_j - 1\right) - \xi t\varrho \prod_{i=1}^{p} \left(\Theta + a_i\right)\right] y(t) = 0, \tag{12}$$

where Θ denotes a certain operator containing a forward difference. We shall solve Equation (10) and (11) in terms of solutions of Equation (12), and we will develop some of their properties that justify calling these Chebyshev difference equations.

The phrase "Chebyshev difference equation" sometimes appears in the literature, e.g., in the recent article [7] (40), in reference to the Equation (7) and in [8] (5.2) which refers to a scaled version of Equation (7) for monic Chebyshev polynomials. We do not use the terminology in this way. Instead, we call Equation (7) the "three-term recurrence" for classical Chebyshev polynomials, and we will find a discrete analogue for it in the sequel.

Other similar sounding functions include an existing "Chebyshev polynomial of a discrete variable" which can be found in [9] (p. 33) as a special case of the Hahn polynomials. In Ref. [10,11], the "rth discrete Chebyshev polynomial of order N" is defined. These polynomials are also distinct from the polynomials appearing in this article.

We define the discrete monomials $t^{\underline{k}}$ as "falling factorials", i.e., $t^{\underline{k}} = t(t-1)\ldots(t-k+1)$. Of particular interest is that the falling factorial obeys a "discrete power rule" $\Delta t^{\underline{k}} = kt^{\underline{k-1}}$. We contrast Equation (4) with the discrete hypergeometric series, ${}_pF_q$, defined by

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t,n,\xi) = \sum_{k=0}^{\infty} \frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{\xi^k t^{\underline{nk}}}{k!}, \tag{13} $$

which solves Equation (12). Discrete special functions defined by an instance of Equation (13) import the same parameter set $a_1,\ldots,a_p,b_1,\ldots,b_q$ as its analogous continuous special function defined by Equation (4). Many representations of special functions replace t in Equation (4) with an expression of the form "ξt^n", for some constant ξ. For instance, the sine function is $\sin(t) = t_0\mathcal{F}_1\left(;\frac{3}{2};-\frac{t^2}{4}\right)$ and the discrete sine is given by the formula $\sin_1(t) = t_0F_1\left(;\frac{3}{2};t-1,2,-\frac{1}{4}\right)$ [6] (Proposition 20). Since given an arbitrary $\xi \in \mathbb{R}$ and $n,t \in \{0,1,2,\ldots\}$, $\xi t^n \notin \{0,1,2,\ldots\}$, it is not possible to always naively map what appears in the independent variable arguments of functions defined by Equation (4) to their discrete analogues (13) in general, explaining the extra parameters. The final argument $\frac{1-t}{2}$ appearing in Equations (5) and (6) acts as a barrier to a simple importation of Chebyshev polynomials to the discrete case from the continuous case, but we resolve this dilemma in the sequel.

2. Chebyshev Difference Equation

The natural way suggested by prior work to find the discrete analogue of a polynomial is to replace each monomial t^m in it with $t^{\underline{m}}$. We now demonstrate in the following example that this method fails for the Chebyshev polynomials.

Example 1. *The first few classic Chebyshev polynomials of the first kind (5) appear in the following:*

$n =$	$\mathcal{T}_n(t) =$
0	1
1	t
2	$2t^2 - 1.$

These polynomials obey the recurrence (7). Naively replacing t^2 with $t^{\underline{2}}$, we obtain the following possible discrete analogues:

$n =$	"T_n"(t)
0	1
1	t
2	$2t^{\underline{2}} - 1 = 2t^2 - 2t - 1,$

and we would obtain the an analogue of (7) by replacing all terms of the form $t^m f^{(n)}(t)$ with $t^{\underline{m}}\Delta^n f(t-m)$:

$$ f_{n+1}(t) = 2tf_n(t-1) - f_{n-1}(t). $$

However, this fails even in the case $n = 1$:

$$ \text{“}T_2\text{”}(t) = 2t^2 - 2t - 1 \neq 2t^2 - 1 = 2t\text{“}T_1\text{”}(t) - \text{“}T_0\text{”}(t), $$

and so the well-known method of finding discrete analogues fails in this case.

The problem we have highlighted in Example 1 is caused by the appearance of "$1-t$" in the final argument of Equation (5), and the example demonstrates that the discrete hypergeometric series (13) cannot create a discrete analogue of a function whose classical hypergeometric representation contains horizontal shifts in the independent variable. To fix this problem, we can simply replace t with $t+1$ in Equation (1) to get

$$(t^2+2t)y''+(t+1)y'-n^2y=0, \tag{14}$$

and we do the same in Equation (2) to get

$$(t^2+2t)y''+3(t+1)y'-n(n+2)y=0. \tag{15}$$

Of course, $\mathcal{T}_n(t+1)$ and $\mathcal{U}_n(t+1)$ solve Equations (14) and (15), but they are now in a proper form for obtaining the discrete analogues. We obtain the difference Equations (10) and (11) by replacing all terms of the form $t^m y^{(n)}(t)$ from Equations (14) and (15) with $t^{\underline{m}}\Delta^n y(t-m)$.

The discrete Chebyshev polynomials of the first kind, T_n, are defined by

$$T_n(t) = {}_2F_1\left(-n,n;\frac{1}{2};t,1,-\frac{1}{2}\right) = \sum_{k=0}^{n} \frac{(-1)^k(-n)_k(n)_k}{2^k(\frac{1}{2})_k}\frac{t^{\underline{k}}}{k!}. \tag{16}$$

By applying [6] (Proposition 2), we see that Equation (16) may be written in terms of a classical ${}_p\mathcal{F}_q$ with a variable parameter as

$$T_n(t) = {}_3\mathcal{F}_1\left(-n,n,-t;\frac{1}{2};\frac{1}{2}\right). \tag{17}$$

The discrete Chebyshev polynomials of the second kind, U_n, are defined by

$$U_n(t) = (n+1){}_2F_1\left(-n,n+2;\frac{3}{2};t,1,-\frac{1}{2}\right) = (n+1)\sum_{k=0}^{n} \frac{(-1)^k(-n)_k(n+2)_k}{2^k\left(\frac{3}{2}\right)_k}\frac{t^{\underline{k}}}{k!}, \tag{18}$$

and similarly applying [6] (Proposition 2) here yields

$$U_n(t) = (n+1){}_3\mathcal{F}_1\left(-n,n+2,-t;\frac{3}{2};\frac{1}{2}\right). \tag{19}$$

Both of these functions are finite sums due to Equation (3), since $(-n)_k$ is zero for all $k>n$.

The following lemma will be useful in deriving the difference equations for T_n and U_n.

Lemma 1. *The following formulas hold:*

1. $\Theta y(t) = t\Delta y(t-1)$, *and*
2. $\Theta^2 y(t) = t\Delta y(t-1) + t^{\underline{2}}\Delta^2 y(t-2)$.

Proof. For 1. in Lemma 1, calculate

$$\Theta y(t) = t\varrho\Delta y(t) = t\Delta y(t-1).$$

For 2., use 1. and the discrete product rule to calculate

$$\Theta^2 y(t) = t\varrho\Delta\left[t\Delta y(t-1)\right] = t\Delta y(t-1) + t^{\underline{2}}\Delta y(t-2),$$

completing the proof. □

The following theorem is a discrete analogue of (14).

Theorem 1. *The polynomials* (16) *solve Equation* (10).

Proof. By Equations (12) and (16), we know that $y(t) = T_n(t)$ satisfies

$$\left[\Theta\left(\Theta + \frac{1}{2} - 1\right) - \left(-\frac{1}{2}\right) t\varrho\,(\Theta - n)\,(\Theta + n)\right] y = 0.$$

Thus,

$$\Theta^2 y - \frac{1}{2}\Theta y + \frac{1}{2} t\varrho\left(\Theta^2 - n\Theta + n\Theta - n^2\right) y = 0,$$

and, hence,

$$\left(1 + \frac{1}{2} t\varrho\right)\Theta^2 y(t) - \frac{1}{2}\Theta y(t) - \frac{n^2}{2} t\varrho y(t) = 0.$$

Apply Lemma 1 and multiply by 2 to get

$$\left(2 + t\varrho\right)\left(t\Delta y(t-1) + t^{\underline{2}}\Delta^2 y(t-2)\right) - t\Delta y(t-1) - n^2 t y(t-1) = 0.$$

Expanding yields

$$2t\Delta y(t-1) + 2t^{\underline{2}}\Delta^2 y(t-2) + t^{\underline{2}}\Delta y(t-2) + t^{\underline{3}}\Delta^2 y(t-3) - t\Delta y(t-1) - n^2 t y(t-1) = 0,$$

and, by algebra, we obtain

$$t^{\underline{3}}\Delta^2 y(t-3) + 2t^{\underline{2}}\Delta^2 y(t-2) + t^{\underline{2}}\Delta y(t-2) + t\Delta y(t-1) - n^2 t y(t-1) = 0.$$

Divide by t and then replace t with $t+1$ to arrive at

$$t^{\underline{2}}\Delta^2 y(t-2) + 2t\Delta y(t-1) + t\Delta y(t-1) + \Delta y(t) - n^2 y(t) = 0,$$

completing the proof. □

We now establish the discrete analogue of the three-term-recurrence for the discrete Chebyshev polynomials of the first kind.

Theorem 2. *The polynomials* (16) *obey the recurrence relation*

$$T_{n+1}(t) - 2tT_n(t-1) - 2T_n(t) + T_{n-1}(t) = 0. \tag{20}$$

Proof. Let $\alpha_{k,n} = \dfrac{(-1)^k(-n)_k(n)_k}{2^k\left(\frac{1}{2}\right)_k k!}$. Apply Equation (16) to each term of Equation (20) to get

$$\begin{aligned}
T_{n+1}(t) &= \sum_{k=0}^{n+1} \alpha_{k,n+1} t^{\underline{k}} = 1 + \alpha_{n,n+1} t^{\underline{n}} + \alpha_{n+1,n+1} t^{\underline{n+1}} + \sum_{k=1}^{n-1} \alpha_{k,n+1} t^{\underline{k}}, \\
-2tT_n(t-1) &= -2t \sum_{k=0}^{n} \alpha_{k,n} (t-1)^{\underline{k}} \\
&= -2 \sum_{k=0}^{n} \alpha_{k,n} t^{\underline{k+1}} = -2 \sum_{k=1}^{n+1} \alpha_{k-1,n} t^{\underline{k}} \\
&= -2\alpha_{n-1,n} t^{\underline{n}} - 2\alpha_{n,n} t^{\underline{n+1}} - 2 \sum_{k=1}^{n-1} \alpha_{k-1,n} t^{\underline{k}}, \\
-2T_n(t) &= -2 \sum_{k=0}^{n} \alpha_{k,n} t^{\underline{k}} = -2 - 2\alpha_{n,n} t^{\underline{n}} - 2 \sum_{k=1}^{n-1} \alpha_{k,n} t^{\underline{k}}
\end{aligned}$$

and

$$T_{n-1}(t) \quad = \sum_{k=0}^{n-1} \alpha_{k,n-1} t^{\underline{k}} = 1 + \sum_{k=1}^{n-1} \alpha_{k,n-1} t^{\underline{k}}.$$

Define $\beta_n = \alpha_{n,n+1} - 2\alpha_{n-1,n} - 2\alpha_{n,n}$ and compute

$$\beta_n = \frac{(-1)^{n-1}(-n)_{n-1}(n+1)_{n-2}}{2^{n-1}\left(\frac{1}{2}\right)_{n-1}(n-1)!}\left[\frac{(n+1)(2n-1)(2n)}{(2n-1)n} - 2n - \frac{2n(2n-1)}{(2n-1)n}\right] = 0,$$

define $\gamma_n = \alpha_{n+1,n+1} - 2\alpha_{n,n}$ and compute

$$\gamma_n = \frac{(-1)^n(-n)_n(n+1)_{n-1}}{2^n\left(\frac{1}{2}\right)_n n!}\left[\frac{(n+1)(2n)(2n+1)}{(2n+1)(n+1)} - 2n\right] = 0,$$

and finally define $\delta_{k,n} = \alpha_{k,n+1} - 2\alpha_{k-1,n} - 2\alpha_{k,n} + \alpha_{k,n-1}$ and compute

$$\begin{aligned}\delta_{k,n} \quad &= \frac{(-1)^{k-1}(-n+1)_{k-2}(n+1)_{k-2}}{2^{k-1}\left(\frac{1}{2}\right)_{k-1}(k-1)!}\left[\frac{(n+1)(-n)(n+k-1)(n+k)}{k(2k-1)} + 2n^2\right. \\ &\left. -2\frac{n^2(-n+k-1)(n+k-1)}{k(2k-1)} - \frac{n(-n+k-1)(-n+k)(n-1)}{k(2k-1)}\right] = 0.\end{aligned}$$

Therefore,

$$T_{n+1}(t) - 2tT_n(t-1) - 2T_n(t) + T_{n-1}(t) \quad = \beta_n t^{\underline{n}} + \gamma_n t^{\underline{n+1}} + \sum_{k=1}^{n-1} \delta_{k,n} t^{\underline{k}} = 0,$$

completing the proof. □

In light of Equations (18) and (20), the following classical hypergeometric relation is yielded.

Corollary 1. *The following formula holds for all $t \in \{0,1,2,\ldots\}$ and for all $n \in \{1,2,3,\ldots\}$:*

$$\begin{aligned}&{}_3\mathcal{F}_1\left(-n-1, n+1, -t; \frac{1}{2}; \frac{1}{2}\right) - 2t\,{}_3\mathcal{F}_1\left(-n, n, -t+1; \frac{1}{2}; \frac{1}{2}\right) \\ &\qquad - 2\,{}_3\mathcal{F}_1\left(-n, n, -t; \frac{1}{2}; \frac{1}{2}\right) + {}_3\mathcal{F}_1\left(-n+1, n-1, -t; \frac{1}{2}; \frac{1}{2}\right) = 0.\end{aligned}$$

We now demonstrate the difference equation that the discrete Chebyshev polynomials of the second kind solve.

Theorem 3. *The polynomials (18) solve Equation (11).*

Proof. By Equations (12) and (18), we know that $y(t) = U_n(t)$ satisfies

$$\left[\Theta\left(\Theta + \frac{3}{2} - 1\right) - \left(-\frac{1}{2}\right) t\varrho\left(\Theta - n\right)\left(\Theta + n + 2\right)\right] y = 0,$$

yielding

$$\Theta^2 y(t) + \frac{1}{2}\Theta y(t) + \frac{1}{2} t\varrho\Big(\Theta^2 y(t) + 2\Theta y(t) - n(n+2)y(t)\Big) = 0.$$

Applying Lemma 1, we get

$$\Big[t\Delta y(t-1)+t^{\underline{2}}\Delta^2 y(t-2)\Big]+\frac{1}{2}t\Delta y(t-1)$$
$$+\frac{1}{2}t\varrho\left(\Big(t\Delta y(t-1)+t^{\underline{2}}\Delta^2 y(t-2)\Big)+2t\Delta y(t-1)-n(n+2)y(t)\right)=0.$$

Multiply by $\frac{2}{t}$, replace t with $t+1$, and expand to get

$$t^{\underline{2}}\Delta^2 y(t-2)+2t\Delta^2 y(t-1)+3t\Delta y(t-1)+3\Delta y(t)-n(n+2)y(t)=0,$$

completing the proof. □

We now prove the discrete analogue of the three-term recurrence for the discrete Chebyshev polynomials of the second kind.

Theorem 4. *The polynomials* (18) *obey the recurrence relation*

$$U_{n+1}(t)-2tU_n(t-1)-2U_n(t)+U_{n-1}(t)=0. \tag{21}$$

Proof. Let $\zeta_{k,n}=\dfrac{(n+1)(-1)^k(-n)_k(n+2)_k}{2^k\left(\frac{3}{2}\right)_k k!}$. Apply Equation (18) to each term of Equation (21) to get

$$U_{n+1}(t) = \sum_{k=0}^{n+1}\zeta_{k,n+1}t^{\underline{k}}=1+\zeta_{n+1,n+1}t^{\underline{n+1}}+\zeta_{n,n+1}t^{\underline{n}}+\sum_{k=1}^{n-1}\zeta_{k,n+1}t^{\underline{k}},$$

$$-2tU_n(t-1)=-2t\sum_{k=0}^{n}\zeta_{k,n}(t-1)^{\underline{k}}=-2\sum_{k=1}^{n+1}\zeta_{k-1,n}t^{\underline{k}}=-2\zeta_{n,n}t^{\underline{n+1}}-2\zeta_{n-1,n}t^{\underline{n}}-2\sum_{k=1}^{n-1}\zeta_{k-1,n}t^{\underline{k}},$$

$$-2U_n(t)=-2\sum_{k=0}^{n}\zeta_{k,n}t^{\underline{k}}=-2-2\zeta_{n,n}t^{\underline{n}}-2\sum_{k=1}^{n-1}\zeta_{k,n}t^{\underline{k}},$$

and

$$U_{n-1}(t)=\sum_{k=0}^{n-1}\zeta_{k,n-1}t^{\underline{k}}=1+\sum_{k=1}^{n-1}\zeta_{k,n-1}t^{\underline{k}}.$$

Define $\eta_n=\zeta_{n,n+1}-2\zeta_{n-1,n}-2\zeta_{n,n}$ and compute

$$\eta_n=\frac{(-1)^{n-1}(-n+1)_{n-2}(n+3)_{n-2}}{2^{n-1}(\frac{3}{2})_{n-1}(n-1)!}\Bigg[\frac{(n+2)(n+1)(-n)(2n+1)(2n+2)}{n(2n+1)}$$
$$+2n(n+1)(n+2)+\frac{2n(n+1)(n+2)(2n+1)}{n(2n+1)}\Bigg]=0,$$

define $\theta_n=\zeta_{n+1,n+1}-2\zeta_{n,n}$ and compute

$$\theta_n=\frac{(-1)^n(-n)_n(n+3)_{n-1}}{2^n\left(\frac{3}{2}\right)_n n!}\left[\frac{(n+2)(n+1)(2n+2)(2n+3)}{(2n+3)(n+1)}-2(n+1)(n+2)\right]=0,$$

and finally define $\lambda_{k,n} = \zeta_{k,n+1} - 2\zeta_{k-1,n} - 2\zeta_{k,n} + \zeta_{k,n-1}$ and compute

$$\lambda_{k,n} = \frac{(-1)^{k-1}(-n+1)_{k-2}(n+3)_{k-2}}{2^{k-1}\left(\frac{3}{2}\right)_{k-1}(k-1)!}\left[\frac{(n+2)(n+1)(-n)(n+k+1)(n+k+2)}{k(2k+1)}\right. + 2n(n+1)(n+2) - \frac{2n(n+1)(-n+k-1)(n+2)(n+k+1)}{k(2k+1)} \left. - \frac{n(-n+k-1)(-n+k)(n+1)(n+2)}{k(2k+1)}\right] = 0.$$

Therefore,

$$U_{n+1}(t) - 2tU_n(t-1) - 2U_n(t) + U_{n-1}(t) = \eta_n t^{\underline{n}} + \theta_n t^{\underline{n+1}} + \sum_{k=1}^{n-1} \theta_{k,n} t^{\underline{k}} = 0,$$

completing the proof. □

Using Equation (19), we immediately obtain a corollary that gives us an interesting identity for ${}_3\mathcal{F}_1$.

Corollary 2. *The following formula holds for all $n \in \{1,2,3,\ldots\}$ and for all $t \in \{0,1,2,\ldots\}$:*

$$(n+2)\,{}_3\mathcal{F}_1\left(-n-1, n+3, -t; \frac{3}{2}; \frac{1}{2}\right) - 2t(n+1)\,{}_3\mathcal{F}_1\left(-n, n+2, -t+1; \frac{3}{2}; \frac{1}{2}\right) - 2(n+1)\,{}_3\mathcal{F}_1\left(-n, n+2, -t; \frac{3}{2}; \frac{1}{2}\right) + n\,{}_3\mathcal{F}_1\left(-n+1, n+1, -t; \frac{3}{2}; \frac{1}{2}\right) = 0.$$

The following formula is a discrete analogue of (8).

Theorem 5. *The following difference formula holds for all $n, t \in \{0,1,2,\ldots\}$:*

$$\Delta T_n(t) = nU_{n-1}(t). \tag{22}$$

Proof. Taking the difference of Equation (16) yields

$$\begin{aligned}
\Delta T_n(t) &= \Delta\left[\sum_{k=0}^{n} \frac{(-1)^k(-n)_k(n)_k}{2^k\left(\frac{1}{2}\right)_k}\frac{t^{\underline{k}}}{k!}\right] \\
&= \sum_{k=1}^{n} \frac{(-1)^k(-n)_k(n)_k}{2^k\left(\frac{1}{2}\right)_k}\frac{t^{\underline{k-1}}}{(k-1)!} \\
&= \sum_{k=0}^{n-1} \frac{(-1)^{k+1}(-n)_{k+1}(n)_{k+1}}{2^{k+1}\left(\frac{1}{2}\right)_{k+1}}\frac{t^{\underline{k}}}{k!} \\
&= \frac{-(-n)n}{2\left(\frac{1}{2}\right)}\sum_{k=0}^{n-1} \frac{(-1)^k(-(n-1))_k(n+1)_k}{2^k\left(\frac{3}{2}\right)_k}\frac{t^{\underline{k}}}{k!} \\
&= nU_{n-1}(t),
\end{aligned}$$

completing the proof. □

The following theorem is a discrete analogue of (9).

Theorem 6. *The polynomials (16) and (18) obey the following recurrence relation for all $n, t \in \{0,1,2,\ldots\}$*

$$U_n(t) - U_{n-2}(t) = 2T_n(t).$$

Proof. Take the difference of (20) to obtain

$$\Delta T_{n+1}(t) - [2T_n(t) + 2t\Delta T_n(t-1)] - 2\Delta T_n(t) + \Delta T_{n-1}(t) = 0.$$

By Equation (22), we get

$$(n+1)U_n(t) - 2T_n(t) - 2tnU_{n-1}(t-1) - 2nU_{n-1}(t) + (n-1)U_{n-2}(t) = 0,$$

which simplifies to

$$U_n(t) - U_{n-2}(t) = 2T_n(t) + n\Big[- U_n(t) + 2tU_{n-1}(t-1) + 2U_{n-1}(t) - U_{n-2}(t)\Big],$$

but the second term is identically zero by Equation (21) with n replaced with $n-1$, completing the proof. □

As with many previous results, we obtain a result for ${}_3\mathcal{F}_1$ here as well.

Corollary 3. *The following formula holds for all $n \in \{2, 3, \ldots\}$ and for all $t \in \{0, 1, 2, \ldots\}$:*

$$(n+1){}_3\mathcal{F}_1\left(-n, n+2, -t; \frac{3}{2}; \frac{1}{2}\right) - (n-1){}_3\mathcal{F}_1\left(-n+2, n, -t; \frac{3}{2}; \frac{1}{2}\right) = 2{}_3\mathcal{F}_1\left(-n, n, -t; \frac{1}{2}; \frac{1}{2}\right).$$

Thus far, we have seen multiple properties of the classical Chebyshev polynomials that have direct discrete analogues. We now present an example of a property that is not sustained by the discrete analogue:

$$f_{n+1}(t) = 2tf_n(t) - f_{n-1}(t).$$

Example 2. *Given a sequence of orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$, meaning each is of degree n and there is an inner product $\langle c\cdot, \cdot\rangle$ such that $\langle P_n, P_m\rangle = 0$ whenever $n \neq m$ and $\langle P_n, P_n\rangle \neq 0$, it is known [12] (Theorem 3.2.1) that they obey a three-term-recurrence, i.e., there are constants $A_n, C_n > 0$ and $B_n \in \mathbb{R}$ such that*

$$P_n(t) = (A_n t + B_n)P_{n-1}(t) - C_n P_{n-2}(t). \tag{23}$$

Considering the list of polynomials in Table 1, we will use simple algebra and the contrapositive of [12] (Theorem 3.2.1) to show that the sequence of discrete Chebyshev polynomials (of either kind) does not form a sequence of orthogonal polynomials.

Table 1. The polynomials T_n and U_n for $n \in \{0, 1, 2, 3, 4\}$, fully expanded.

$n =$	$T_n =$	$U_n =$
0	1	1
1	$t+1$	$2t+2$
2	$2t^2+2t+1$	$4t^2+4t+3$
3	$4t^3+5t+1$	$8t^3+12t+4$
4	$8t^4-16t^3+32t^2-8t+1$	$16t^4-32t^3+68t^2-12t+5$

First, suppose that Equation (23) holds for the discrete Chebyshev polynomials of the first kind for $n = 3$. This would mean that there exist constants A_3, B_3, and C_3 such that

$$\begin{aligned} 4t^3 + 5t + 1 &= (A_3 t + B_3)(2t^2 + 2t + 1) - C_3(t+1) \\ &= 2A_3 t^3 + (2A_3 + 2B_3)t^2 + (A_3 + 2B_3 - C_3)t + (B_3 - C_3). \end{aligned} \tag{24}$$

This yields the system of equations

$$\begin{cases} 2A_3 & = 4, \\ 2A_3 + 2B_3 & = 0, \\ A_3 + 2B_3 - C_3 & = 5, \\ B_3 - C_3 & = 1, \end{cases}$$

which has no solution. Therefore, the sequence of discrete Chebyshev polynomials of the first kind does not form a sequence of orthogonal polynomials.

Now suppose that Equation (23) holds for the discrete Chebyshev polynomials of the second kind for $n = 3$. This means there would exist constants A_3, B_3, and C_3 such that

$$\begin{aligned} 8t^3 + 12t + 4 &= (A_3 t + B_3)(4t^2 + 4t + 3) - C_3(2t + 2) \\ &= 4A_3 t^3 + (4A_3 + 4B_3)t^2 + (3A_3 + 4B_3 - 2C_3)t + (3B_3 - 2C_3), \end{aligned} \quad (25)$$

leading to

$$\begin{cases} 4A_3 & = 8, \\ 4A_3 + 4B_3 & = 0, \\ 3A_3 + 4B_3 - 2C_3 & = 12, \\ 3B_3 - 2C_3 & = 4, \end{cases}$$

which similarly has no solution. Hence, the sequence of discrete Chebyshev polynomials of the second kind do not form a sequence of orthogonal polynomials.

3. Conclusions

We have shown that the polynomial solutions to (10) and (11) are a sort of discrete analogue of Chebyshev polynomials for $n \in \{0, 1, 2, \ldots\}$. We have established some of their properties, shown some relationships between them, and demonstrated how these functions yield classic hypergeometric relationships for the ${}_3\mathcal{F}_1$ hypergeometric series. We have shown that this method of finding analogues of special functions does not preserve the orthogonality of the classical Chebyshev polynomials. An immediate question from this lack of orthogonality is whether the polynomials here satisfy some kind of weakening of being orthogonal polynomials, e.g., the notion of "almost orthogonal polynomials" in [13]. Further work can be done with these polynomials, including proving new properties, investigating the second independent solutions to (10) and (11), and investigating the cases of $n \in \mathbb{C} \setminus \{0, 1, 2, \ldots\}$ and $t \in \mathbb{C}$, which may be made well-defined either using the hypergeometric representations (17) and (19) or by using the gamma function to define $t^{\underline{n}}$.

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