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Quaternionic Product of Equilateral Hyperbolas and Some Extensions

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Abstract: This note concerns a product of equilateral hyperbolas induced by the quaternionic product considered in a projective manner. Several properties of this composition law are derived and, in this way, we arrive at some special numbers as roots or powers of unit. Using the algebra of octonions, we extend this product to oriented equilateral hyperbolas and to pairs of equilateral hyperbolas. Using an inversion we extend this product to Bernoulli lemniscates and q-lemniscates. Finally, we extend this product to a set of conics. Three applications of the given products are proposed.

Keywords: equilateral hyperbola; quaternion; product; projective geometry; octonion

MSC: 51N20; 51N14; 11R52; 11R06

1. Introduction

The aim of this paper is to introduce some products of the set of equilateral hyperbolas and give some extensions of them. For our hyperbolas considered in a projective way we use the well-known product of quaternions to define a first product, denoted $\odot_c$. Since the c-square of the unit hyperbola $H(1) : x^2 - y^2 - 1 = 0$ is the degenerate hyperbola $H(0) : x^2 - y^2 = 0$ we introduce a second product, denoted $\odot_{pc}$. A detailed study of both of these products is the content of Section 2. By looking at examples, as well as to roots/powers of the unit $1 \in \mathbb{R}$, we obtain some remarkable numbers, some of them algebraic but other of difficult nature.

Starting from the above results, some interesting extensions are obtained. In Section 3, using an inversion, we extend the above products from the set of equilateral hyperbolas to the sets of Bernoulli lemniscates and q-lemniscates. A strong motivation for this extension is that equilateral hyperbolas share with Bernoulli lemniscates the property of having rational chord-length parametrization, as in pointed out in Reference [1] (p. 210).

In Section 4 we give another extension of the products of equilateral hyperbolas to a larger set of conics $\mathcal{Q}_{\Gamma_0}$ and we prove that $\odot_c$ is a commutative and associative law and has a neutral element, thus the triple $(\mathcal{Q}_{\Gamma_0}, \oplus, \odot_c)$ is a field isomorphic to the field of complex numbers. For some particular values of the parameters, we obtain the product of equilateral hyperbolas considered in the first section.

Using also an inversion, in Section 5 we extend the product on conics from $\mathcal{Q}_{\Gamma_0}$ to other curves.

Inspired by the expression of an octonion as a pair of quaternions, we introduce in Section 6 an octonionic product of pairs of equilateral hyperbolas. For this new composition law we compute the square of a fixed pair and several products involving the unit hyperbola $H(1)$. We note that the products of the first section are commutative while the considered product of pairs of equilateral hyperbolas is not.

In the last section, we propose three applications of the given products. The first two of them are regarding hyperbolic objects, namely the reduced equilateral hyperbola $H_e : xy = 1$ and hyperbolic matrices, but, in general, concerns with multi-valued maps. The last possible application returns to the Euclidean plane geometry and defines a chain of labels for a given polygon. This application can be put in correspondence with the recent studies on the moduli space of polygons, studies based on Reference [2].

We note that the present study is the hyperbolic counter-part of a similar work concerning circles in Reference [3] while a more general Clifford product for EPH-cycles is introduced in Reference [4]. In fact, the present paper is a natural continuation of Reference [3] due to the Lambert's and Riccati's analogies between the circle and the equilateral hyperbola as are exposed in Reference [5], also published as Reference [6].

2. Quaternionic Product of Hyperbolas and Quaternionic Product of Oriented Hyperbolas

The starting point of this paper is the identification of a given equilateral hyperbola H in the Euclidean plane with coordinates (x, y):

$$H : x^2 - y^2 + ax + by + c = 0 \tag{1}$$

with a quaternion:

$$q(H) = c + ai + bj + k = (c, a, b, 1) \in \mathbb{R}^4. \tag{2}$$

The quaternion $q(H)$ is pure imaginary if and only if the origin $O(0, 0)$ belongs to H. Let us point out that the given hyperbola is expressed in a *projective* manner since the coefficient of the quadratic part is chosen as being 1. Hence the set of equilateral hyperbolas is a 3-dimensional projective subspace of the 5-dimensional projective space of conics. Our study will be a mix of elements from Euclidean and projective geometry.

From the real algebra structure of the quaternions it follows a product of equilateral hyperbolas:

$$H_1 \odot_c H_2 := q^{-1}(q(H_1) \cdot q(H_2)), \tag{3}$$

where the dot of the right-hand side denotes the product of quaternions. For H_i, $i = 1, 2$ given by (a_i, b_i, c_i) we derive immediately:

$$\begin{aligned} q(H_1 \odot_c H_2) = {} & (a_1 b_2 - a_2 b_1 + c_1 + c_2)k + (b_1 - b_2 + a_1 c_2 + a_2 c_1)i + (a_2 - a_1 + b_1 c_2 + b_2 c_1)j + \\ & + (c_1 c_2 - 1 - a_1 a_2 - b_1 b_2), \end{aligned} \tag{4}$$

which gives non-commutative expressions for the coefficients of i, j and k and commutative expression for the free term.

Due to the chosen projective setting we restrict our study to equilateral hyperbolas $H(r)$ already centered in O; hence their set is a 1-dimensional projective subspace of the projective spaces considered above. For such a hyperbola we have:

$$H(r) : x^2 - y^2 - r = 0, \quad (a, b, c) = (0, 0, -r) \tag{5}$$

and hence the Equation (4) yields:

$$q(H(r_1) \odot_c H(r_2)) = (c_1 + c_2)k + (c_1 c_2 - 1) = -(r_1 + r_2)k + (r_1 r_2 - 1). \tag{6}$$

From the properties of quaternionic product we have that the above product can be also expressed in matrix product manner:

$$(-r_2, 0, 0, 1) \cdot \begin{pmatrix} -r_1 & 0 & 0 & 1 \\ 0 & -r_1 & 1 & 0 \\ 0 & -1 & -r_1 & 0 \\ -1 & 0 & 0 & -r_1 \end{pmatrix} = (r_1 r_2 - 1, 0, 0, -(r_1 + r_2)). \tag{7}$$

We derive the product law:

$$H(r_1) \odot_c H(r_2) = H(R), \quad R := \frac{r_1 r_2 - 1}{r_1 + r_2}. \tag{8}$$

In conclusion, on the set $M = (0, +\infty)$ we define a *non-internal* law of composition:

$$r_1 \odot_c r_2 := \frac{r_1 r_2 - 1}{r_1 + r_2} < \min\{r_1, r_2\} \tag{9}$$

and the rest of this section concerns with several of its properties.

Remark 1.1 We have:

$$H(r_1) \odot_c H(r_2) = H(r_1 \odot_c r_2). \tag{10}$$

Property 1.1 The product $\odot_c$ is commutative and associative but does not have a neutral element:

$$r_1 \odot_c r_2 \odot_c r_3 = \frac{r_1 r_2 r_3 - (r_1 + r_2 + r_3)}{r_1 r_2 + r_2 r_3 + r_3 r_1 - 1}, \quad r^3_{\odot_c} = \frac{r^3 - 3r}{3r^2 - 1}, \quad r > \frac{1}{\sqrt{3}}. \tag{11}$$

Property 1.2 With $r_i = \tan \varphi_i$ we get:

$$\tan \varphi_1 \odot_c \tan \varphi_2 := -\cot(\varphi_1 + \varphi_2). \tag{12}$$

Property 1.3 Concerning the unit hyperbola $H(1) : x^2 - y^2 = 1$ we have:

$$r \odot_c 1 = \frac{r - 1}{r + 1} < \min\{1, r\}, \quad \lim_{r \to +\infty} (r \odot_c 1) = 1. \tag{13}$$

In particular, the unit hyperbola is the square root of the degenerate hyperbola, $H(1) \odot_c H(1) = H(0) : x^2 - y^2 = 0$; in fact $[q(H(1))]^2 = (k-1)^2 = -2k$. With a rational $r = \frac{x}{y}$:

$$\frac{x}{y} \odot_c 1 = \frac{x - y}{x + y}, \quad \frac{1}{2} \odot_c 1 = -\frac{1}{3}. \tag{14}$$

For example, two remarkable positive numbers are provided by the radius involved in the well-known Hopf fibration as the Riemannian submersion $S^3(1) \to S^2(\frac{1}{2})$ and hence we compute: $1 \odot_c 2 = \frac{1}{3}$. Also, the eccentricity of an equilateral hyperbola is $\sqrt{2}$ and $1 \odot_c \sqrt{2} = 3 - 2\sqrt{2}$, $\sqrt{2}^2_{\odot_c} = \frac{1}{2\sqrt{2}}$.

Property 1.4 Concerning the squares we have:

$$r^2_{\odot_c} = \frac{r^2 - 1}{2r} < r, \quad (\tan \varphi)^2_{\odot_c} = -\cot(2\varphi), \quad (r^2_{\odot_c}) \odot_c 1 = \frac{r^2 - 2r - 1}{r^2 + 2r - 1} \tag{15}$$

and the first relation (15) means that $\odot_c$ is a "shrinking" composition. The $\odot_c$-square root of 1 is the number:

$$\sqrt[c]{1} := 1 + \sqrt{2} = 2.4142135\ldots = \tan \frac{3\pi}{8}, \quad (\sqrt[c]{1})^2 - 2\sqrt[c]{1} - 1 = 0 \tag{16}$$

while the $\odot_c$-square root of $\sqrt[c]{1}$ is the number:

$$\sqrt[2c]{1} := 1 + \sqrt{2} + \sqrt{4 + 2\sqrt{2}} = 5.027339\ldots, \quad (\sqrt[2c]{1})^2_{\odot_c} = \sqrt[c]{1}. \tag{17}$$

Let us remark that $\sqrt[c]{1}$ is exactly *the silver ratio* $\Psi := 1 + \sqrt{2}$ and we point out that Ψ is a quadratic Pisot-Vijayaraghavan number considered as solution of:

$$x^2 - 2x - 1 = 0. \tag{18}$$

The conjugate of Ψ with respect to this algebraic equation is:

$$-\Psi^{-1} = 1 - \sqrt{2} = -0.44\ldots. \tag{19}$$

Let us point out that from the point of view of endomorphisms on smooth manifolds the silver mean is treated in Reference [7] (p. 16) and a fourth order square root of unit is called *almost electromagnetic structure* in Reference [8] (p. 721). The continuous fraction of these remarkable numbers are easy to compute with Mathematica; we use the standard expression for these continuous fractions:

$$\sqrt[c]{1} = [2; \bar{2}], \quad \sqrt[2c]{1} = [5; 36, 1, 1, 2, 1, 2, 1, 6, \ldots]. \tag{20}$$

The usual inverses of $\sqrt[c]{1}$ is:

$$\frac{1}{\sqrt[c]{1}} = \sqrt{2} - 1 = 0.414235\ldots = \cot\frac{3\pi}{8}. \tag{21}$$

Property 1.5 Recall that the quaternion (2) has an Euclidean norm:

$$\|q(H(r))\|^2 = 1 + a^2 + b^2 + c^2 = 1 + r^2 \geq 1 \tag{22}$$

and then the given square (15) is:

$$r^2_{\odot_c} = \frac{\|q(H(r))\|^2 - 2}{2\sqrt{\|q(H(r))\|^2 - 1}}. \tag{23}$$

Property 1.6 We extend the previous products from hyperbolas to *oriented hyperbolas* that is, pairs $\mathcal{H} := (H, \varepsilon := \pm 1)$ with $\varepsilon \in \{\pm 1\}$. Then we introduce:

$$\mathcal{H}_1 \odot_c \mathcal{H}_2 := (H_1 \odot_c H_2, \varepsilon_1 \cdot \varepsilon_2). \tag{24}$$

Remark 1.2 We can avoid the degeneration $(H(1))^2_{\odot_c} = H(0)$ by considering the para-complex algebra $\mathbb{R}[X]/(X^2 - 1)$ instead of the complex algebra. Since in this new algebra the square of k is $+1$ we arrive at a new product $\odot_{pc}$ on $\mathbb{R}^*_+ = (0, +\infty)$:

$$x \odot_{pc} y = \frac{xy + 1}{x + y}. \tag{25}$$

The product $\odot_{pc}$ is commutative with $x \odot_{pc} 1 = 1$ and:

$$x^2_{\odot_{pc}} = \frac{x^2 + 1}{2x}, \quad x \odot_{pc} y \odot_{pc} z = \frac{xyz + x + y + z}{xy + yz + zx + 1}, \quad \tan\varphi_1 \odot_{pc} \tan\varphi_2 = \frac{\cos(\varphi_2 - \varphi_1)}{\sin(\varphi_1 + \varphi_2)}. \tag{26}$$

3. The Extension via Inversion of the Quaternionic Product to Bernoulli Lemniscates and Q-Lemniscates

In Reference [9] it is proved, using purely geometrical means, that the image of an equilateral hyperbola with foci F_1 and F_2 by an inversion I_r with respect to the circle centered in O and with radius $r = |OF_1| = |OF_2|$ is a Bernoulli lemniscate with the same foci F_1 and F_2. We start this section with a complex approach in order to achieve easier the extension to the quaternionic approach.

Thus, we will prove the above assertion using complex numbers. Firstly, we associate to every point (x, y) in the Euclidean plane the complex number $z = x + iy \in \mathbb{C}$. As $z' = I_r(z) = \alpha z$ with $\alpha \in \mathbb{R}_+^*$ and $|I_r(z)| \cdot |z| = r^2$ we have $|\alpha z| \cdot |z| = r^2$, so $\alpha = \dfrac{r^2}{|z|^2} = \dfrac{r^2}{z \cdot \overline{z}}$ and therefore the equation of the inversion I_r is:

$$z' = I_r(z) = \frac{r^2}{\overline{z}}. \tag{27}$$

The equation:

$$H(a^2) : x^2 - y^2 = a^2, \tag{28}$$

of the equilateral hyperbola with foci $\left(\pm a\sqrt{2}, 0\right)$, taking into account that $x^2 - y^2 = \mathrm{Re}\left(z^2\right) = \dfrac{z^2 + \overline{z}^2}{2}$, can be written as:

$$H(a^2) : z^2 + \overline{z}^2 = 2a^2. \tag{29}$$

The image of the above equilateral hyperbola by the inversion I_r has the equation $\left(z^2 + \overline{z}^2\right) r^4 = 2a^2 \left(\overline{z} z\right)^2$. Taking into account that $z \cdot \overline{z} = x^2 + y^2$ we obtain $\left(x^2 + y^2\right)^2 = \dfrac{r^4}{a^2}\left(x^2 - y^2\right)$ which is the equation of a Bernoulli lemniscate with foci $\left(\pm \dfrac{r^2}{a\sqrt{2}}, 0\right)$.

As the equilateral hyperbola has the foci $\left(\pm a\sqrt{2}, 0\right)$ while the Bernoulli lemniscate has the foci $\left(\pm \dfrac{r^2}{a\sqrt{2}}, 0\right)$ the foci are preserved by the inversion I_r if and only if $a\sqrt{2} = \dfrac{r^2}{a\sqrt{2}}$ that is, $r = a\sqrt{2}$. More exactly, the image by the inversion I_r of the equilateral hyperbola $H\left(\dfrac{r^2}{2}\right)$ that has the foci $(\pm r, 0)$ is the Bernoulli lemniscate $L(r)$ with the same foci, having the equation $\left(x^2 + y^2\right)^2 = 2r^2\left(x^2 - y^2\right)$.

Remark 2.1 Let $L(r)$ be the Bernoulli lemniscate with parameter $r > 0$; more precisely $2r$ is the distance between the foci of $L(r)$. Then we can introduce the products of Bernoulli lemniscates $L(r_1)$ and $L(r_2)$ in the same manner as the products of equilateral hyperbolas:

$$L(r_1) \odot_c L(r_2) := L(r_1 \odot_c r_2), \quad L(r_1) \odot_{pc} L(r_2) := L(r_1 \odot_{pc} r_2).$$

All the properties proved for quaternionic products of equilateral hyperbolas are also true for quaternionic products of Bernoulli lemniscates.

Returning now to the initial equilateral hyperbola its Equation (28) is expressed as:

$$||MF_1| - |MF_2|| = 2a = const. \tag{30}$$

This property could be proved using pure geometric or analytic geometry means but we give a proof using complex numbers. Indeed, since $x^2 - y^2 = a^2$, the foci are $F_1(-a\sqrt{2}, 0)$, $F_2(+a\sqrt{2}, 0)$, the vertices are $V_1(-a, 0)$, $V_2(+a, 0)$, thus for M of affix z, we have:

$$||MF_1| - |MF_2||^2 = \left|\sqrt{\left(z + a\sqrt{2}\right)\left(\overline{z} + a\sqrt{2}\right)} - \sqrt{\left(z - a\sqrt{2}\right)\left(\overline{z} - a\sqrt{2}\right)}\right|^2 =$$

$$= \left|2z\overline{z} + 4a^2 - 2\sqrt{\left(z + a\sqrt{2}\right)\left(\overline{z} + a\sqrt{2}\right)\left(z - a\sqrt{2}\right)\left(\overline{z} - a\sqrt{2}\right)}\right| = \left|2z\overline{z} + 4a^2 - 2\sqrt{\left(z^2 - 2a^2\right)\left(\overline{z}^2 - 2a^2\right)}\right| =$$

$$= \left| 2z\bar{z} + 4a^2 - 2\sqrt{(z\bar{z})^2 - 2a^2(z^2 + \bar{z}^2) + 4a^4} \right| = \left| 2z\bar{z} + 4a^2 - 2\sqrt{(z\bar{z})^2 - 2a^2 \cdot 2a^2 + 4a^4} \right| = 4a^2.$$

In the same order of ideas, a well known property of a current point M on a Bernoulli lemniscate, given by the equation:

$$\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right) \tag{31}$$

is:

$$|MF_1| \cdot |MF_2| = |OF_1| \cdot |OF_2| = \frac{a^2}{2} = const. \tag{32}$$

The foci are $F_1\left(-\frac{a}{\sqrt{2}}, 0\right)$, $F_2\left(+\frac{a}{\sqrt{2}}, 0\right)$ and thus for M of affix z we have:

$|MF_1|^2 |MF_2|^2 = \left(z + \frac{a}{\sqrt{2}}\right)\left(\bar{z} + \frac{a}{\sqrt{2}}\right)\left(z - \frac{a}{\sqrt{2}}\right)\left(\bar{z} - \frac{a}{\sqrt{2}}\right) =$

$= \left(z^2 - \frac{a^2}{2}\right)\left(\bar{z}^2 - \frac{a^2}{2}\right) = z^2\bar{z}^2 - a^2\frac{z^2 + \bar{z}^2}{2} + \frac{a^4}{4} = \frac{a^4}{4}$, because $z^2\bar{z}^2 = a^2\frac{z^2 + \bar{z}^2}{2}$ is (31) written in a complex form.

Recall that the inversion I_r with $r = a\sqrt{2}$ preserves the foci $F_{1,2}$. In this case, a current point M on the equilateral hyperbola $H\left(\frac{r^2}{2}\right)$, with foci $F_{1,2}\,(\pm r)$ and vertices $V_{1,2}\left(\pm\frac{r}{\sqrt{2}}\right)$, has the property $||MF_1| - |MF_2|| = r\sqrt{2} = const.$ and a current point M on its image by this inversion I_r, the Bernoulli lemniscate $L(r)$ given by the equation $\left(x^2 + y^2\right)^2 = 2r^2\left(x^2 - y^2\right)$, with the same foci, has the property $|MF_1| \cdot |MF_2| = |OF_1| \cdot |OF_2| = r^2 = const.$

We are going to extend these constructions in a quaternionic setting. First we recall that $M\,(x, y, z, w) \in \mathbb{R}^4$ has the quaternionic affix $q = x + yi + zj + wk$. We say that the hyperquadric in $\mathbb{R}^4$ defined by the equation:

$$H_q(a^2) : x^2 - y^2 - z^2 - w^2 = a^2 \tag{33}$$

is a *q-equilateral hyperboloid*. We can define its foci as the points $\left(\pm a\sqrt{2}, 0, 0, 0\right)$.

Also we say that:

$$L_q(a^2) : \left(x^2 + y^2 + z^2 + w^2\right)^2 = a^2\left(x^2 - y^2 - z^2 - w^2\right)$$

is the *Bernoulli q-lemniscate* with the points $\left(\pm\frac{a}{\sqrt{2}}, 0, 0, 0\right)$ as foci.

In the following we will show that the names q-equilateral hyperboloid and Bernoulli q-lemniscate are fully justified because the main properties of equilateral hyperbola and Bernoulli lemniscate stated above in complex context are also preserved in quaternionic context.

Proposition 1. *The relation (30), specific to a hyperbola, holds also true for a q-equilateral hyperboloid.*

Proof. We proceed in a similar way as for equilateral hyperbola:

$$||MF_1| - |MF_2||^2 = \left| \sqrt{\left(q + a\sqrt{2}\right)\left(\bar{q} + a\sqrt{2}\right)} - \sqrt{\left(q - a\sqrt{2}\right)\left(\bar{q} - a\sqrt{2}\right)} \right|^2 =$$

$$= \left| 2q\bar{q} + 4a^2 - 2\sqrt{\left(q + a\sqrt{2}\right)\left(\bar{q} + a\sqrt{2}\right)\left(q - a\sqrt{2}\right)\left(\bar{q} - a\sqrt{2}\right)} \right| = \left| 2q\bar{q} + 4a^2 - 2\sqrt{(q^2 - 2a^2)\,(\bar{q}^2 - 2a^2)} \right| =$$

$$= \left| 2q\bar{q} + 4a^2 - 2\sqrt{(q\bar{q})^2 - 2a^2(q^2 + \bar{q}^2) + 4a^4} \right| = \left| 2q\bar{q} + 4a^2 - 2\sqrt{(q\bar{q})^2 - 2a^2 \cdot 2a^2 + 4a^4} \right| = 4a^2.$$

□

Remark 2.2 In a similar way with the quaternionic products on equilateral hyperbolas we can introduce the quaternionic products of q-equilateral hyperboloids $H_q(r_1)$ and $H_q(r_2)$:

$$H_q(r_1) \odot_c H_q(r_2) = H_q(r_1 \odot_c r_2), \quad H_q(r_1) \odot_{pc} H_q(r_2) = H_q(r_1 \odot_{pc} r_2).$$

Recall that the equilateral hyperbola (28) can be written in a complex form as (29) and, in an analogous way, the q-equilateral hyperboloid (33) can be written in a quaternionic form as:

$$H_q(a^2) : q^2 + \bar{q}^2 = 2a^2. \tag{34}$$

Analogously to the usual inversion (27) we can define a quaternionic inversion on $\mathbb{R}^4 \setminus \{0\}$:

$$q' = I_r(q) = \frac{r^2}{\bar{q}} \tag{35}$$

and it is easy to see that $I_r^2 := I_r \circ I_r = id$ thus, as in the case of the planar inversion, I_r is an involution.

Proposition 2. *The image of the q-equilateral hyperboloid (34) by the inversion I_r is a Bernoulli q-lemniscate.*

Proof. We have $(q')^2 + (\bar{q}')^2 = 2a^2$ or $\dfrac{r^4}{\bar{q}^2} + \dfrac{r^4}{q^2} = 2a^2$, thus:

$$(q\bar{q})^2 = \frac{r^4}{a^2}\frac{q^2 + \bar{q}^2}{2}.$$

Using the coordinates in $\mathbb{R}^4$ the above equation has the form:

$$\left(x^2 + y^2 + z^2 + w^2\right)^2 = \frac{r^4}{a^2}\left(x^2 - y^2 - z^2 - w^2\right), \tag{36}$$

which is the equation of the Bernoulli q-lemniscate with foci $\left(\pm\dfrac{r^2}{a\sqrt{2}}, 0, 0, 0\right)$. $\square$

Remark 2.3 Since the q-equilateral hyperboloid has the foci $\left(\pm a\sqrt{2}, 0, 0, 0\right)$ while its image by the inversion I_r is the Bernoulli q-lemniscate with the foci $\left(\pm\dfrac{r^2}{a\sqrt{2}}, 0, 0, 0\right)$, the foci are preserved by the inversion I_r if and only if $a\sqrt{2} = \dfrac{r^2}{a\sqrt{2}}$, that is, $r = a\sqrt{2}$. More exactly, the image by the inversion I_r of the q-equilateral hyperboloid that has the foci $(\pm r, 0, 0, 0)$, that is, having the equation $x^2 - y^2 - z^2 - w^2 = \dfrac{r^2}{2}$ is the Bernoulli q-lemniscate with the same foci, having the equation $\left(x^2 + y^2 + z^2 + w^2\right)^2 = 2r^2\left(x^2 - y^2 - z^2 - w^2\right)$.

Proposition 3. *The relation (32), specific to a Bernoulli lemniscate, holds also true for a Bernoulli q-lemniscate.*

Proof. We proceed in a similar way as previously for the Bernoulli lemniscate but now in a quaternionic setting. We have:

$$|MF_1|^2 |MF_2|^2 = \left(q + \frac{a}{\sqrt{2}}\right)\left(\bar{q} + \frac{a}{\sqrt{2}}\right)\left(q - \frac{a}{\sqrt{2}}\right)\left(\bar{q} - \frac{a}{\sqrt{2}}\right) =$$

$$= \left(q^2 - \frac{a^2}{2}\right)\left(\bar{q}^2 - \frac{a^2}{2}\right) = q^2\bar{q}^2 - a^2\frac{q^2 + \bar{q}^2}{2} + \frac{a^4}{4} = \frac{a^4}{4}$$

since $q^2\overline{q}^2 = a^2\dfrac{q^2+\bar{q}^2}{2}$ is the Equation (36) in quaternionic form. Thus, since $|MF_1|\,|MF_2| = 2a^2 = |OF_1|\,|OF_2|$ the conclusion follows. $\square$

Remark 2.4 In a similar way with the quaternionic products of Bernoulli lemniscates we can introduce two quaternionic products of Bernoulli q-lemniscates:

$$L_q(r_1) \odot_c L_q(r_2) = L_q(r_1 \odot_c r_2), \quad L_q(r_1) \odot_{pc} L_q(r_2) = L_q(r_1 \odot_{pc} r_2).$$

4. The Extension of the Quaternionic Product on Conics

Let us consider a pure imaginary quaternion $q_0 = ai + bj + dk$ and the set:

$$Q_{q_0} = \{c + \alpha q_0; c, \alpha \in \mathbb{R}\}.$$

If $q_0 \neq 0$ then we can identify Q_{q_0} with $\mathbb{R}^2$ and even with $\mathbb{C}$, as we see below.

Let us consider $q_1 = c_1 + \alpha_1 q_0$ and $q_2 = c_2 + \alpha_2 q_0 \in Q_{q_0}$. It follows, by a straightforward computation, that:

$$q_1 \cdot q_2 = c_3 + \alpha_3 q_0,$$

where:

$$c_3 = c_1c_2 - \alpha_1\alpha_2(a^2+b^2+d^2) = c_1c_2 - \alpha_1\alpha_2\Delta_0, \text{with } \Delta_0 = a^2+b^2+d^2, \alpha_3 = \alpha_1c_2 + \alpha_2c_1. \tag{37}$$

Remark 3.1 Taking into account the skew-symmetry of the multiplication of quaternionic units i, j, k they do not appear in the expression of q_0^2: $q_0^2 = (ai+bj+dk)^2 = -\Delta_0 \in \mathbb{R}$. Therefore, if q_1, $q_2 \in Q_{q_0}$ with $q_1 = c_1 + \alpha_1 q_0$ and $q_2 = c_2 + \alpha_2 q_0$ then we have $q_1 + q_2 = (c_1+c_2) + (\alpha_1+\alpha_2)q_0 \in Q_{q_0}$ and $q_1 \cdot q_2 \in Q_{q_0}$, thus Q_{q_0} is stable at the sum and multiplication defined this way.

Moreover, the quaternionic product induces on $Q_{q_0}^* = Q_{q_0} \backslash \{0\}$ a group structure isomorphic with the multiplicative group on $\mathbb{C}^*$. Since $Q_{q_0} \subset Q$ is a vector subspace, generated by $\{1, q_0\}$, we can consider also the additive group structure on Q_{q_0}. Thus, using these two operations, Q_{q_0} is a field isomorphic with the field $\mathbb{C}$.

Remark 3.2 The isomorphism is given by $f : Q_{q_0} \longrightarrow \mathbb{C}$ with $f(c + \alpha q_0) = c + \alpha\sqrt{\Delta_0}i$ for every $q = c + \alpha q_0 \in Q_{q_0}$. Note that $q_0 = ai + bj + dk$ is arbitrarily chosen, but fixed, therefore a, b and d are fixed, thus Δ_0 is fixed. Taking into account these considerations, every $q = c + \alpha q_0 \in Q_{q_0}$ can be written as the pair $q = (c, \alpha)$ and hence f can be written more simple as $f(c, \alpha) = c + \alpha\sqrt{\Delta_0}i$.

Let Γ be a conic in the Euclidean plane given by:

$$\Gamma : x^2 + dy^2 + ax + by + c = 0.$$

We associate to Γ the quaternion $q(\Gamma) = c + ai + bj + dk = (c, a, b, d) \in \mathbb{R}^4$.

Considering two conics:

$$\Gamma_1 : x^2 + \alpha_1 dy^2 + \alpha_1 ax + \alpha_1 by + c_1 = 0, \quad \Gamma_2 : x^2 + \alpha_2 dy^2 + \alpha_2 ax + \alpha_2 by + c_1 = 0,$$

where a, b, $d, c_1, c_2, \alpha_1, \alpha_2 \in \mathbb{R}$ we can associate a conic $\Gamma_3 = \Gamma_1 \odot_c \Gamma_2$ corresponding to the product and a conic $\Gamma_4 = \Gamma_1 \oplus \Gamma_2$ corresponding to the sum of the corresponding quaternions $q_1 = q(\Gamma_1)$ and $q_2 = q(\Gamma_2)$:

$$\begin{aligned} \Gamma_3 = \Gamma_1 \odot_c \Gamma_2 = q^{-1}\left(q\left(\Gamma_1\right) \cdot q\left(\Gamma_2\right)\right) : x^2 + \alpha_3 dy^2 + \alpha_3 ax + \alpha_3 by + c_3 = 0, \\ \Gamma_4 = \Gamma_1 \oplus \Gamma_2 = q^{-1}\left(q\left(\Gamma_1\right) + q\left(\Gamma_2\right)\right) : x^2 + \alpha_4 dy^2 + \alpha_4 ax + \alpha_4 by + c_4 = 0, \end{aligned}$$

where c_3 and α_3 are given by formulas (37) and $\alpha_4 = \alpha_1 + \alpha_2, \quad c_4 = c_1 + c_2$.

Thus, we can consider now the conic $\Gamma_0 \ : \ x^2 + dy^2 + ax + by \ = \ 0$ and also the set of associated conics:

$$\mathcal{Q}_{\Gamma_0} = \left\{ \Gamma : x^2 + \alpha dy^2 + \alpha ax + \alpha by + c = 0; c, \alpha \in \mathbb{R} \right\}.$$

Remark 3.3 With a straightforward computation one can prove that $\odot_c$ is a commutative and associative law and has a neutral element; namely the element corresponding to $c = 1$, $\alpha = 0$, therefore it is the (imaginary) conic $x^2 + 1 = 0$.

Remark 3.4 As we note before q_0 can be arbitrarily chosen, but then it is fixed, therefore a, b and d are fixed. But once q_0 is fixed, the family is unique; so with this hypothesis, for a conic $\Gamma \in \mathcal{Q}_{\Gamma_0}$, the corresponding c and α are unique. Of course, a given conic can be seen as belonging to several families, but once the conical family is fixed, the corresponding c and α are unique; therefore the above operations on an arbitrary, but fixed family $\mathcal{Q}_{\Gamma_0}$ are well defined, as they are defined on this given family (as for example for the case of a natural number, which can be seen as belonging to several classes of congruence modulo k where k can be chosen arbitrarily, but fixed, and operations are defined on this given congruence class). This approach has an important advantage because any conic can be considered.

Property. The triple $\left(\mathcal{Q}_{\Gamma_0}, \oplus, \odot_c\right)$ is a field isomorphic to the field of complex numbers.

Remark 3.5 Let us look more on the product defined above, considering (c, α) as parameters in Q_{q_0} or $\mathcal{Q}_{\Gamma_0}$. We have that the product of (c_1, α_1) and (c_2, α_2) corresponds to the parameters $(c_1c_2 - \alpha_1\alpha_2\Delta_0, \alpha_1c_2 + \alpha_2c_1)$, where $\Delta_0 = -q_0^2$. It is easy to see that the product factorizes to the projective space P^1 that is, we can define

$$[c_1, \alpha_1] \odot_{c,\Delta_0} [c_2, \alpha_2] = [c_1c_2 - \alpha_1\alpha_2\Delta_0, \alpha_1c_2 + \alpha_2c_1].$$

The corresponding group structure is isomorphic with the multiplicative circular group S^1.

Remark 3.6 The neutral element for $\odot_{c,\Delta_0}$ is $(1, 0)$.

Remark 3.7 Let us consider $\alpha_1c_2 + \alpha_2c_1$, α_1, $\alpha_2 \neq 0$. Thus we obtain that the product of $[c_1, \alpha_1] = \left[\frac{c_1}{\alpha_1}, 1\right]$ and $[c_2, \alpha_2] = \left[\frac{c_2}{\alpha_2}, 1\right]$ corresponds to $[c_1c_2 - \alpha_1\alpha_2\Delta_0, \alpha_1c_2 + \alpha_2c_1] = \left[\frac{c_1c_2 - \alpha_1\alpha_2\Delta_0}{\alpha_1c_2 + \alpha_2c_1}, 1\right] = \left[\frac{\frac{c_1}{\alpha_1}\frac{c_2}{\alpha_2} - \Delta_0}{\frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2}}, 1\right]$.

Therefore the product $\odot_c$ defined in the first section comes from the product $\odot_{c,1}$ when restricted to the classes $\{[r, 1]\,; r > 0\}$.

Remark 3.8 If $\Delta_0 = 1$ then we can consider restrictions of the product $\odot_{c,1}$ from $P^1 = P_1^1 \cup P_2^1$ to P_1^1 or P_2^1, where $P_1^1 = \{[r, 1]\,; r \in \mathbb{R}\}$ and $P_2^1 = \{[1, r]\,; r \in \mathbb{R}\}$. We have to note that the products restricted to P_1^1 and P_2^1 are partial. Indeed, for example, if $r \in \mathbb{R}$, then $[r, 1] \odot_{c,1} [-r, 1] = \left[-r^2 - 1, 0\right] = [1, 0] \in P_2^1$. One can explain now why $\odot_c$ does not have a neutral element when it is restricted to the classes $\{[r, 1]\,; r > 0\}$ or even to P_1^1, since the neutral element $[1, 0] \in P_2^1$ does not belong to these sets. Notice also that the sum of parameters do not factorize to an additive law in the projective space P^2.

Remark 3.9 More particulary, for $\alpha_1 = \alpha_2 = 1$, $a = b = 0$ and $d = -1$, we obtain the composition law associated to the family of equilateral hyperbolas considered also in first section.

Remark 3.10 Let us consider the determinants:

$$\delta = \begin{vmatrix} 1 & 0 \\ 0 & \alpha d \end{vmatrix} = \alpha d, \quad \Delta = \begin{vmatrix} 1 & 0 & \frac{a\alpha}{2} \\ 0 & \alpha d & \frac{b\alpha}{2} \\ \frac{a\alpha}{2} & \frac{b\alpha}{2} & c \end{vmatrix} = -\frac{1}{4}\alpha\left(b^2\alpha - 4cd + a^2d\alpha^2\right).$$

associated to a conic in a family $\mathcal{Q}_{\Gamma_0}$. It is easy to see that the family does not always have only one type of conic. For example, in the case $a = b = 0$, $d = -1$, we have $\delta = \alpha$ and $\Delta = -c$. If $\alpha c \neq 0$ then all the conics are non-degenerated with the center in origin; for $\alpha > 0$ all the conics are hyperbolas; for $\alpha < 0$ all the conics are ellipses; they are all real for $c < 0$ and all imaginary for $c > 0$.

Remark 3.11 For $x = (c, a, b, \alpha) \in \mathbb{R}^4$ we associate the quaternion $q(x) = c + (\alpha a)\, i + (\alpha b)\, j$ and the conic $P_x : x^2 + \alpha a x + \alpha b y + c = 0$.

For $(a, b) \in \mathbb{R}^2$ we consider $\mathcal{Q}_{(a,b)} = \{q(x) = c + (\alpha a)\, i + (\alpha b)\, j : c, \alpha \in \mathbb{R}\}$ and $\mathcal{P}_{(a,b)} = \{P_x : c, \alpha \in \mathbb{R}\}$. If $x_1 = (c_1, a, b, \alpha_1)$, $x_2 = (c_2, a, b, \alpha_2) \in \mathbb{R}^4$ then:

$$q(x_1) \cdot q(x_2) = (c_1 + (\alpha_1 a)\, i + (\alpha_1 b)\, j)\,(c_2 + (\alpha_2 a)\, i + (\alpha_1 b)\, j) =$$

$$= \left(c_1 c_2 - \alpha_1 \alpha_2 \left(a^2 + b^2\right)\right) + a\,(\alpha_1 c_2 + \alpha_2 c_1)\, i + b\,(\alpha_1 c_2 + \alpha_2 c_1)\, j = q(x_3)$$

where:

$$x_3 = \left(c_1 c_2 - \alpha_1 \alpha_2 \left(a^2 + b^2\right), a, b, \alpha_1 c_2 + \alpha_2 c_1\right),$$

thus the quaternionic product is a composition law that is internal on $\mathcal{Q}_{(a,b)}$. It induces also an internal composition law on $\mathcal{P}_{(a,b)}$.

Remark 3.12 We have $P_{x_1} \odot_{c,\Delta} P_{x_2} = P_{x_3}$ where $\Delta = a^2 + b^2$ and $P_{x_1}, P_{x_2}, P_{x_3} \in \mathcal{P}_{(a,b)}$.

Property. The quaternionic product on $\mathcal{Q}_{(a,b)}$ and the induced composition law on $\mathcal{P}_{(a,b)}$ are commutative, associative, but has not always neutral elements.

5. Using the Inversion to Extend the Quaternionic Product on $\mathcal{Q}_{\Gamma_0}$ to Other Curves

Let us consider a more general case, that is, the following equation $x^2 + dy^2 + c = 0$, $c \neq 0$. As above, to every point (x, y) in the Euclidean plane we associate $z = x + iy \in \mathbb{C}$. The inversion I_r with respect to the circle centered in O and with radius r is given by $z' = I_r(z) = \dfrac{r^2}{\overline{z}}$. We analyze now two different cases.

$\circ$ If $d < 0$ then we have $d = -\delta^2$, so the equation $x^2 - (\delta y)^2 + c = 0$ is the equation of a hyperbola and can be written as $(z + \overline{z})^2 + \delta^2 (z - \overline{z})^2 + 4c = 0$. Therefore, the image of this hyperbola by the inversion I_r has the equation:

$$\left(\frac{r^2}{\overline{z}} + \frac{r^2}{z}\right)^2 + \delta^2 \left(\frac{r^2}{\overline{z}} - \frac{r^2}{z}\right)^2 + 4c = 0 \iff \frac{(z + \overline{z})^2}{z^2 \overline{z}^2} + \delta^2 \frac{(z - \overline{z})^2}{z^2 \overline{z}^2} + \frac{4c}{r^4} = 0 \iff$$

$$(z^2 + \overline{z}^2)(1 + \delta^2) + 2z\overline{z}(1 - \delta^2) + \frac{4c}{r^4} z^2 \overline{z}^2 = 0 \iff$$

$$2(x^2 - y^2)(1 + \delta^2) + 2(x^2 + y^2)(1 - \delta^2) + \frac{4c}{r^4}\left(x^2 + y^2\right)^2 = 0. \tag{38}$$

For $\delta = \pm 1$ the Equation (38) is $(x^2 - y^2) + \dfrac{c}{r^4}\left(x^2 + y^2\right)^2 = 0 \iff \left(x^2 + y^2\right)^2 = \dfrac{r^4}{-c}\left(x^2 - y^2\right)$, which is the equation of a Bernoulli lemniscate. Let us note that for $c < 0$ we have an usual equation of a Bernoulli lemniscate because $\dfrac{r^4}{-c} > 0$; for $c > 0$ the equation can be written as $\left(y^2 + x^2\right)^2 = \dfrac{r^4}{c}\left(y^2 - x^2\right)$ where $\dfrac{r^4}{c} > 0$, therefore, with a change of coordinates, we have also an equation of a Bernoulli lemniscate.

For $\delta \neq \pm 1$ the Equation (38) is $(x^2 - \delta^2 y^2) + \dfrac{c}{r^4}\left(x^2 + y^2\right)^2 = 0 \iff \left(x^2 + y^2\right)^2 = \dfrac{r^4}{-c}\left(x^2 - \delta^2 y^2\right)$ which is (with the above discussion for $c < 0$, but also for $c > 0$) the equation of a generalized lemniscate.

Thus, if $d < 0$ for every type of above lemniscates L_1, L_2 and L_3, taking into account $\odot_{c,\Delta}$ introduced in the previous section, we have $L_1 \odot_{c,\Delta} L_2 = L_3$, where $\Delta = \delta^4$.

Remark 4.1 For $\alpha_1 = \alpha_2 = 1$ and $\delta = \pm 1$ the above product has the same form as the product $\odot_c$ on the family of Bernoulli lemniscates, considered in Section 3.

$\circ$ If $d > 0$ then we have $d = \delta^2$, so the equation is $x^2 + (\delta y)^2 + c = 0$, which is the equation of an ellipse when $\delta \neq \pm 1$ or of a circle when $\delta = \pm 1$ and can be written as $(z + \overline{z})^2 - \delta^2 (z - \overline{z})^2 + 4c = 0$. Therefore, the image of this curve (ellipse or circle) by the inversion I_r has the equation:

$$\left(\frac{r^2}{\overline{z}} + \frac{r^2}{z}\right)^2 - \delta^2 \left(\frac{r^2}{\overline{z}} - \frac{r^2}{z}\right)^2 + 4c = 0 \iff$$

$$\frac{(z+\overline{z})^2}{z^2\overline{z}^2} - \delta^2 \frac{(z-\overline{z})^2}{z^2\overline{z}^2} + \frac{4c}{r^4} = 0 \iff (z^2+\overline{z}^2)(1-\delta^2) + 2z\overline{z}(1+\delta^2) + \frac{4c}{r^4} z^2\overline{z}^2 = 0 \iff$$

$$2(x^2-y^2)(1-\delta^2) + 2(x^2+y^2)(1+\delta^2) + \frac{4c}{r^4}\left(x^2+y^2\right)^2 = 0. \tag{39}$$

For $\delta = \pm 1$ the Equation (39) is $(x^2+y^2) + \frac{c}{r^4}\left(x^2+y^2\right)^2 = 0 \iff \left(x^2+y^2\right) = \frac{r^4}{-c}, x^2+y^2 \neq 0$, which is the equation of a real circle (for $c < 0$) or an imaginary circle (for $c > 0$); for $x^2 + y^2 = 0$ the circle is degenerated in a point (the origin). Therefore, the image of the circle by the inversion I_r is also a circle.

Using $\odot_{c,\Delta}$ introduced in previous section, we have $C_1 \odot_{c,\Delta} C_2 = C_3$, where $\Delta = \delta^4$ and C_1, C_2 and C_3 are circles.

Remark 4.2 For $\alpha_1 = \alpha_2 = 1$ and $\delta = \pm 1$ we obtain the composition law associated to the above family of circles (see Reference [3]).

For $\delta \neq \pm 1$ the Equation (39) is $\left(x^2+y^2\right)^2 = \frac{r^4}{-c}\left(x^2+\delta^2 y^2\right)$, which is: – for $c < 0$, it is the equation of a Booth lemniscate (an oval of Booth with 0 as an isolated point, for $\delta \neq 0$, or a pair of externally tangent circles for $\delta = 0$) or, – for $c > 0$, it is the equation of a curve degenerated in a double point.

Therefore for $\delta \neq \pm 1$, the image of the ellipse by the inversion I_r is a Booth lemniscate or a curve degenerated in a point.

We have $L_1 \odot_{c,\Delta} L_2 = L_3$, where $\Delta = \delta^4$ and L_1, L_2 and L_3 are lemniscates as above.

Remark 4.3 If $d = 0$ then the equation $x^2 + ax + by + c = 0$ of parabolic type form can be written as: $(z+\overline{z})^2 + 2a(z+\overline{z}) - 2b(z-\overline{z})i + 4c = 0$. Therefore, the image of this curve by the inversion I_r has the equation:

$$\frac{\left(\frac{r^2}{\overline{z}} + \frac{r^2}{z}\right)^2}{4} + a\frac{\frac{r^2}{\overline{z}} + \frac{r^2}{z}}{2} - b\frac{\frac{r^2}{\overline{z}} - \frac{r^2}{z}}{2}i + c = 0 \iff \frac{r^4\,(z+\overline{z})^2}{4z^2\overline{z}^2} + \frac{ar^2\,(z+\overline{z})}{2z\overline{z}} - \frac{br^2\,(z-\overline{z})}{2z\overline{z}}i + c = 0 \iff$$

$$r^4\,(z+\overline{z})^2 + 2z\overline{z}\,(z+\overline{z})\,ar^2 - 2z\overline{z}\,(z-\overline{z})\,br^2 i + 4cz^2\overline{z}^2 = 0 \iff r^4x^2 + r^2\,(ax+by)\left(x^2+y^2\right) + c\left(x^2+y^2\right)^2 = 0.$$

For $a = c = 0$ corresponding to the canonic form of the parabolic type form equation we have:

$$r^2x^2 + by\left(x^2+y^2\right) = 0 \iff y = \frac{-r^2x^2}{b\,(x^2+y^2)} \iff y\left(x^2+y^2\right) = 2\left(\frac{-r^2}{2b}\right)x^2,$$

which is the equation of a *cissoid of Diocles*. We have $D_1 \odot_{c,\Delta} D_2 = D_3$, where $\Delta = b^2$ and D_1, D_2, D_3 are cissoids of Diocles.

6. An Extension to Octonionic Product for Pairs of Hyperbolas

Recall that an octonion $o \in \mathbb{O}$ can be thought as a pair of quaternions $o := (q_1, q_2)$ and their non-associative product is:

$$o_1 \cdot o_2 = (p_1, p_2) \cdot (q_1, q_2) := (p_1q_1 - \bar{q}_2p_2, q_2p_1 + p_2\bar{q}_1) \tag{40}$$

with bar for the usual conjugation of quaternions. It follows that a pair of hyperbolas $\mathcal{P} = (H_1, H_2)$ can be considered as an octonion $o(\mathcal{P}) := (q(H_1), q(H_2))$ and we define the product:

$$\mathcal{P}_1 \odot_o \mathcal{P}_2 = o(\mathcal{P}_1) \cdot o(\mathcal{P}_2). \tag{41}$$

If $H_i = H(r_i)$, $1 \leq i \leq 4$ then a long but straightforward computation yields:

$$(r_1, r_2) \odot_o (r_3, r_4) := \left(\frac{r_1 r_3 - r_2 r_4 - 2}{r_1 + r_2 + r_3 - r_4}, \frac{r_1 r_4 + r_2 r_3}{r_1 + r_3 + r_4 - r_2} \right) \tag{42}$$

with the conditions:

$$r_1 + r_2 + r_3 \neq r_4, \quad r_1 + r_3 + r_4 \neq r_2. \tag{43}$$

Remark 5.1 (i) The quaternionic product is not commutative but the product $\odot_c$ is commutative. The octonionic product $\odot_o$ is also non-commutative.
(ii) Having the model of the first section we can introduce an octonionic product on pairs of oriented hyperbolas with $(\varepsilon_1 \varepsilon_3, \varepsilon_2 \varepsilon_4)$ on the second slot.

Examples 5.1 (i) Considering the unit hyperbola on the first pair it results:

$$(1,1) \odot_o (r_3, r_4) = \left(\frac{r_3 - r_4 - 2}{r_3 + r_4 + 2}, 1 \right), \quad r_3 + r_4 + 2 \neq 0. \tag{44}$$

(ii) Considering the unit hyperbola in the second pair we have:

$$(r_1, r_2) \odot_o (1,1) = \left(\frac{r_1 - r_2 - 2}{r_1 + r_2}, \frac{r_1 + r_2}{r_1 - r_2 + 2} \right), \quad r_1 + r_2 \neq 0, r_1 - r_2 \neq -2. \tag{45}$$

(iii) The squares are given by:

$$(r_1, r_2)^2_{\odot_o} = \left(\frac{r_1^2 - r_2^2 - 2}{2r_1}, r_2 \right), \quad r_1 \neq 0. \tag{46}$$

For example $(2,1)^2_{\odot_o} = (\frac{1}{4}, 1)$.
(iii) If the unit hyperbola is distributed in both factors we have:

$$(r_1, 1) \odot_o (r_3, 1) = \left(\frac{r_1 r_3 - 3}{r_1 + r_3}, 1 \right), \quad r_1 + r_3 \neq 0, \tag{47}$$

$$(1, r_2) \odot_o (1, r_4) = \left(\frac{-r_2 r_4 - 1}{r_2 - r_4 + 2}, \frac{r + 2 + r_4}{r_4 - r_2 + 2} \right), \quad r_2 - r_4 \notin \{\pm 2\}. \tag{48}$$

The last products with $H(1)$ are:

$$\begin{cases} (r_1, 1) \odot_o (1, r_4) = \left(\frac{r_1 - r_4 - 2}{r_1 - r_4 + 2}, \frac{r_1 r_4 + 1}{r_1 + r_4} \right), & r_1 - r_4 \neq -2, r_1 + r_4 \neq 0, \\ (1, r_2) \odot_o (r_3, 1) = \left(\frac{r_3 - r_2 - 2}{r_2 + r_3}, \frac{r_2 r_3 + 1}{r_3 - r_2 + 2} \right), & r_2 + r_3 \neq 0, r_3 + 2 \neq r_2. \end{cases} \tag{49}$$

From (47) it results the squares:

$$(r, 1)^2_{\odot_o} = \left(\frac{r^2 - 3}{2r}, 1 \right), \quad r \neq 0. \tag{50}$$

7. Applications

In this section we consider three applications of the given product.

Application 6.1 We define a 2-valued composition law on the main sheet of the reduced equilateral hyperbola:

$$H_e : xy = 1, \quad x, y \in (0, +\infty). \tag{51}$$

For a point $P \in H_e$ let:

$$\max(P) := \max\{x_P, y_P\} \geq 1. \tag{52}$$

We define a product on $H_e \setminus \{E(1,1)\}$:

$$P_1 \odot_c P_2 = \{A, B \in H_e;\ x_A = \max(P_1) \odot_c \max(P_2) = y_B\}. \tag{53}$$

This product is available only for $P_1 \neq P_2$ and its form is:

$$P_1 \odot_c P_2 = \left\{ A\left(\max(P_1) \odot_c \max(P_2), \frac{1}{\max(P_1) \odot_c \max(P_2)} \right), B\left(\frac{1}{\max(P_1) \odot_c \max(P_2)}, \max(P_1) \odot_c \max(P_2) \right) \right\}. \tag{54}$$

When $x_A = \max(P_1) = r_1$ and $x_B = \max(P_2) = r_2$ then the product has the explicit form:

$$P_1 \odot_c P_2 = \left(r_1, \frac{1}{r_1} \right) \odot_c \left(r_2, \frac{1}{r_2} \right) = \left\{ A\left(\frac{r_1 r_2 - 1}{r_1 + r_2}, \frac{r_1 + r_2}{r_1 r_2 - 1} \right), B\left(\frac{r_1 + r_2}{r_1 r_2 - 1}, \frac{r_1 r_2 - 1}{r_1 + r_2} \right) \right\}. \tag{55}$$

For example, $\left(2, \frac{1}{2}\right) \odot_c \left(3, \frac{1}{3}\right) = \left(\frac{1}{2}, 2\right) \odot_c \left(\frac{1}{3}, 3\right) = \left(2, \frac{1}{2}\right) \odot_c \left(\frac{1}{3}, 3\right) = \left(\frac{1}{2}, 2\right) \odot_c \left(3, \frac{1}{3}\right) =$ $\{E(1,1)\}$, hence the point $E(1,1) \in H_e$ belongs to the image of this composition law; more general, E is obtained for $r_2 = \frac{r_1+1}{r_1-1}$, when $r_1 = \max(P_1)$ and $r_2 = \max(P_2)$, but, as it can be seen in this example, the pair of points is not unique.

Remark 6.1 We can define another 2-valued composition law on the main sheet of H_e, in an analogous way, by replacing the product $\odot_c$with the product $\odot_{pc}$.

Application 6.2 Another multi-valued product can be introduced on the set of hyperbolic matrices following the approach of Section 5 from Reference [3]. A matrix $\gamma \in SL_2(\mathbb{R})$ is called *hyperbolic* if its eigenvalues are real and distinct; let us denotes $SL_2^H(\mathbb{R})$ their set. Since the characteristic polynomial of arbitrary γ is:

$$f_\gamma(x) = x^2 - tr(\gamma)x + \det(\gamma) = x^2 - tr(\gamma)x + 1 \tag{56}$$

it follows that $\gamma \in SL_2^H(\mathbb{R})$ if and only if $|tr(\gamma)| > 2$ and then its eigenvalues are reciprocal numbers. Let $e(\gamma)$ be the eigenvalue whose absolute value is larger than 1 and define the norm of γ as:

$$N(\gamma) := e(\gamma)^2. \tag{57}$$

We introduce a product on $SL_2^H(\mathbb{R})$:

$$\gamma_1 \odot_c \gamma_2 = \left\{ \gamma \in SL_2^H(\mathbb{R}); e(\gamma) = e(\gamma_1) \odot_c e(\gamma_2) \right\}. \tag{58}$$

From (9) the norm of an arbitrary $\gamma \in \gamma_1 \odot_c \gamma_2$ is:

$$\begin{aligned} N(\gamma) &= N(\gamma_1 \odot_c \gamma_2) = (e(\gamma_1 \odot_c \gamma_2))^2 = \left(\frac{e(\gamma_1)e(\gamma_2) - 1}{e(\gamma_1) + e(\gamma_2)} \right)^2 \\ &= \frac{e(\gamma_1)^2 e(\gamma_2)^2 + 1 - 2e(\gamma_1)e(\gamma_2)}{e(\gamma_1)^2 + e(\gamma_2)^2 + 2e(\gamma_1)e(\gamma_2)} = \frac{N(\gamma_1)N(\gamma_2) + 1 - 2\sqrt{N(\gamma_1)N(\gamma_2)}}{N(\gamma_1) + N(\gamma_2) + 2\sqrt{N(\gamma_1)N(\gamma_2)}}. \end{aligned} \tag{59}$$

For example fix $\gamma \in SL_2^H(\mathbb{R})$ of diagonal form:

$$\gamma = \gamma(R) = diag\left(R, \frac{1}{R}\right), \quad R > 1. \tag{60}$$

We have to note that $\gamma^2_{\odot_c} = \left\{\gamma' \in SL_2^H(\mathbb{R}); e(\gamma') = e(\gamma) \odot_c e(\gamma)\right\} \neq \varnothing \iff \frac{R^2-1}{2R} > 1$ that is, $R > 1+\sqrt{2}$. The first relation (14) yields the norm of an arbitrary $\gamma' \in \gamma(R)^2_{\odot_c}$, when $R > 1+\sqrt{2}$:

$$N(\gamma') = e(\gamma')^2 = (e(\gamma(R)) \odot_c e(\gamma(R)))^2 = \left(\frac{R^2-1}{2R}\right)^2 < R^2 = N(\gamma(R)). \tag{61}$$

Notice that for $R \in (1, 1+\sqrt{2})$ the set $\gamma^2_{\odot_c} = \varnothing$, thus we can not consider $N(\gamma')$.

Remark 6.2 We introduce here a matrix intermezzo in relationship with the matrix product (7). We associate a 2×2 matrix to the hyperbola $H(R)$ through:

$$m(H(R)) := \begin{pmatrix} -R & 1 \\ -1 & -R \end{pmatrix} = -RI_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -RI_2 + m(k) \tag{62}$$

and then, as is expected:

$$-(R_1+R_2)m(H(R_1 \odot_c R_2)) = m(H(R_1)) \cdot m(H(R_2)). \tag{63}$$

The elements of this correspondence are:

$$tr(m(H(R))) = -2R, \quad \det(m(H(R))) = R^2+1, \quad f_{m(H(R))}(x) = (x+R)^2+1. \tag{64}$$

Remark 6.3 We can make in an analogous way all the above constructions, replacing $\odot_c$ product by $\odot_{pc}$ product.

Remark 6.4 The relations (64) and the analogues ones for the product $\odot_{pc}$ are useful to obtain expressions for $\odot_c$ and for $\odot_{pc}$ respectively in terms of trace and/or determinant of the corresponding matrices.

Application 6.3 In this application we associate a p-label to each vertex of a polygon $\mathcal{P} = P_1...P_n$ with $p \in \{c, pc\}$. We denote the length $l_i = \|P_iP_{i+1}\| \in M = (0, +\infty)$ and then the p-number of the vertex P_i is defined as:

$$p_i := l_{i-1} \odot_p l_i. \tag{65}$$

For example, let the right triangle ΔABC with legs $\|AB\| = 3$ and $\|AC\| = 4$. Then $c_A = l_{\|CA\|} \odot_c l_{\|AB\|} = 4 \odot_c 3 = \frac{11}{7}$, $c_B = \frac{7}{4}$, $c_C = \frac{19}{9}$ and the c-chain of ΔABC is:

$$c(\Delta ABC) := (c_A, c_B, c_C) = \left(\frac{11}{7}, \frac{7}{4}, \frac{19}{9}\right), \tag{66}$$

$pc_A = l_{\|CA\|} \odot_{pc} l_{\|AB\|} = 4 \odot_{pc} 3 = \frac{13}{7}$, $pc_B = 2$, $pc_C = \frac{7}{3}$ and the pc-chain of ΔABC is:

$$pc(\Delta ABC) := (pc_A, pc_B, pc_C) = \left(\frac{13}{7}, 2, \frac{7}{3}\right). \tag{67}$$

Also, a (regular) polygon with sides of length 1, as $c_i = 1 \odot_c 1 = 0$, $i = \overline{1,n}$, has a vanishing c-chain and, as $pc_i = 1 \odot_{pc} 1 = 1$, $i = \overline{1,n}$, a constant pc-chain $(1, \ldots, 1)$.

Remark 6.5 Conversely, knowing the c-chain or the pc-chain of a polygon $\mathcal{P}$ we can deduce the length of some of its sides.

If the c-chain of ΔABC is $(0,0,0)$ we have $l_1l_2 = l_2l_3 = l_3l_1 = 1 \implies l_1 = l_2 = l_3 = 1$ and ΔABC is equilateral with sides of length 1.

If a quadrilateral $ABCD$ has a vanishing c-chain we have $l_1l_2 = l_2l_3 = l_3l_4 = l_4l_1 = 1 \implies l_1 = l_3 = l$ and $l_2 = l_4 = \frac{1}{l}$, therefore $ABCD$ is a parallelogram with opposite sides of equal length, l and $\frac{1}{l}$ respectively.

If a polygon $\mathcal{P}$ has $n = 2k+1$ sides and a vanishing c-chain we have $l_1l_2 = l_2l_3 = \ldots = l_{2k}l_{2k+1} = l_{2k+1}l_1 = 1 \implies l_1 = l_2 = \ldots = l_{2k} = 1$, therefore $\mathcal{P}$ is a polygon with all sides of length 1.

we have $l_1l_2 = l_2l_3 = \ldots = l_{2k-1}l_{2k} = l_{2k}l_1 = 1 \implies l_1 = l_3 = \ldots = l_{2k-1} = l$ and $l_2 = l_4 = \ldots = l_{2k} = \frac{1}{l}$, therefore $\mathcal{P}$ is a polygon with odd sides of length l and even sides of length $\frac{1}{l}$.

If the pc-chain of a polygon $\mathcal{P}$ is a constant pc-chain $(1,\ldots,1)$, we have $l_il_{i+1} + 1 = l_i + l_{i+1} \iff (l_i - 1)(l_{i+1} - 1) = 0, i = \overline{1,n}, l_{n+1} \equiv l_1 \iff$

$(l_1 - 1)(l_2 - 1) = \ldots = (l_{n-1} - 1)(l_n - 1) = (l_n - 1)(l_1 - 1) = 0$. We deduce the following properties.

If the pc-chain of ΔABC is $(1,1,1)$, we have $(l_1 - 1)(l_2 - 1) = (l_2 - 1)(l_3 - 1) = (l_3 - 1)(l_1 - 1) = 0$; if $l_1 \neq 1$, then $l_2 = l_3 = 1$ and if $l_1 = 1$, then at least one of l_2 or l_3 has the length equal to 1, thus ΔABC is isosceles with two sides of length 1.

If a quadrilateral $ABCD$ has a constant pc-chain $(1,1,1,1)$, we have $(l_1 - 1)(l_2 - 1) = (l_2 - 1)(l_3 - 1) = (l_3 - 1)(l_4 - 1) = (l_4 - 1)(l_1 - 1) = 0$; if $l_1 \neq 1$, then at least $l_2 = l_4 = 1$ and if $l_1 = 1$, then at least two of l_2, l_3 or l_4 have the length equal to 1, therefore $ABCD$ is a quadrilateral with two opposite sides of length 1.

If a polygon $\mathcal{P}$ has a constant pc-chain $(1,\ldots,1)$ and $n = 2k+1$ sides, then, in the same way, we deduce that $\mathcal{P}$ has at least $k+1$ sides of length 1 and if $\mathcal{P}$ has $n = 2k$ sides, then $\mathcal{P}$ has at least k sides of length 1. Thus, if a polygon $\mathcal{P}$ has n sides and a constant pc-chain $(1,\ldots,1)$, then $\mathcal{P}$ has at least $\left[\frac{n+1}{2}\right]$ sides of length 1.

Thus, it is useful to know the c-chain or the pc-chain of a polygon $\mathcal{P}$ because we can deduce relations involving the length of its sides.

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