

# Silting Complexes and Partial Tilting Modules

Jiaqun Wei <sup>†</sup>

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China; weijiaqun@njnu.edu.cn

<sup>†</sup> Current address: Institute of Mathematics, School of Mathematics Sciences, Nanjing Normal University, Nanjing 210023, China

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**Abstract:** It is well known that a partial tilting module may not be completed to a tilting module. However, it is still unknown whether a partial tilting module can be completed to a silting complex. The affirmative answer to this question will give an affirmative answer to the well-known rank question for tilting modules. In this paper, we prove that a partial tilting simple module can always be completed to a silting complex. More generally, we give the sufficient conditions for a partial tilting module to be completed to a silting complex.

**Keywords:** partial tilting module; silting complex; complement; simple module

## 1. Introduction

In this paper, we always let  $R$  be an artin algebra. As usual,  $\text{mod}R$  denotes the category of all finitely generated left  $R$ -modules,  $\mathcal{D}^b(\text{mod}R)$  denotes the correspondent bounded derived category, and  $\mathcal{K}^b(\mathcal{P}_R)$  is the bounded homotopy category of finitely generated projective modules. Finally, the Grothendieck group of  $R$  is denoted by  $K_0(R)$ .

Let  $T$  be an  $R$ -module. As usual, we denote this using  $\text{add}T$ , the class of all direct summands of direct sums of copies of  $T$ . Recall that a module  $T \in \text{mod}R$  is said to be *tilting* [1,2] if it satisfies the following conditions:

- (i) The projective dimension of  $T$  is finite;
- (ii)  $\text{Ext}_R^i(T, T) = 0$  for all  $i > 0$ ;
- (iii) There is a finite coresolution of  $R$  by objects of  $\text{add}T$ , i.e., an exact sequence  $0 \rightarrow R \rightarrow T_0 \rightarrow \cdots \rightarrow T_n \rightarrow 0$  with each  $T_i \in \text{add}T$ , for some  $n$ .

Modules satisfying the first two conditions are called *partial tilting modules*.

A *classical tilting module* is just a tilting module of a projective dimension of no more than 1. In [3], Bongartz proved that, for classical tilting modules, the third condition can be replaced by the following rank-condition:

(rank-condition): the number of distinct indecomposable direct summands of  $T$  is equal to the rank of  $K_0(R)$ .

As to the general tilting modules of finite projective dimension, it was often asked whether the analogue of the Bongartz' result holds to [4], which is the following question.

**Rank Question for tilting modules:** *Is a partial tilting module that satisfies the rank-condition always tilting?*

We note that every tilting module satisfies the rank-condition, see, for instance, [1]. Therefore, the question is whether the converse holds for partial tilting modules.

This question has been studied for thirty years and there are some partial answers, as presented in [4]. However, it is still open now and no new answers have been found.

The idea that Bongartz used for classical tilting modules was to show that a partial tilting module of a projective dimension no more than 1, say,  $T$ , can always be completed to a classical tilting module, i.e.,  $T \oplus T'$  is an example of a classical tilting module  $T'$ . Therefore, according to his idea, the following question is presented.

**Complement Question for partial tilting modules:** *Can a partial tilting module always be completed to a tilting module? In another words, for a partial tilting module  $T$ , is there a module  $T'$  such that  $T \oplus T'$  becomes a tilting module?*

It is easy to see that a positive answer to the Complement Question for partial tilting modules implies a positive answer to the Rank Question for tilting modules. In [4], Rickard and Schofield proved that the Complement Question for partial tilting modules has a positive answer for algebra of finite representation type. However, they also gave a counter-example to the question. Thus, the Complement Question for partial tilting modules has a negative answer in general.

Note that the counter-example by Rickard and Schofield is a partial tilting simple module of projective dimension 2 over a finite dimensional algebra of global dimension 4, so the Complement Question for partial tilting modules is not a direct corollary of the famous finitistic dimension conjecture. We do not know if there are further connections between them.

The above two questions were extended to tilting complexes by Rickard [5]. Let  $T \in \mathcal{D}^b(\text{mod} R)$ . Recall that  $T$  is *tilting* [5] if it satisfies the following conditions:

- (1)  $T \in \mathcal{K}^b(\mathcal{P}_R)$ ;
- (2)  $\text{Hom}_{\mathcal{D}}(T, T[i]) = 0$  holds for each  $i \neq 0$ ;
- (3)  $\mathcal{K}^b(\mathcal{P}_R)$  is generated by  $T$ , i.e.,  $\langle \text{add} T \rangle = \mathcal{K}^b(\mathcal{P}_R)$ , where  $\langle \text{add} T \rangle$  denotes the smallest triangulated subcategory containing  $\text{add} T$  in the derived category.

A complex  $T \in \mathcal{D}^b(\text{mod} R)$  is said to be *partial tilting* if it satisfies the first two conditions.

Tilting complexes are natural generalizations of tilting modules in the derived categories. In fact, a tilting module is just a module which is a tilting complex in the derived category. More importantly, tilting complexes entirely characterize Morita theory for the derived categories [5]. It was also shown that tilting complexes satisfy the rank-condition, i.e., the number of distinct indecomposable direct summands of a tilting complex is just the rank of  $K_0(R)$  [5]. Motivated by the above two questions for tilting modules, the following, similar questions were asked by Rickard [5].

**Rank Question for tilting complexes:** *Is a partial tilting complex that satisfies the rank-condition always silting?*

**Complement Question for partial tilting complexes:** *Can a partial tilting complex always be completed to a tilting complex? That is, for a partial tilting complex  $T$ , is there a complex  $T' \in \mathcal{D}^b(\text{mod} R)$  such that  $T \oplus T'$  becomes a tilting complex?*

It is also easy to see that a positive answer to the Complement Question for partial tilting complexes implies a positive answer to the Rank Question for tilting complexes, and the latter implies a positive answer to the Rank Question for tilting modules.

Unfortunately, Rickard [5] also gave a simple counter-example to the Complement Question for partial tilting complexes. Moreover, it was also shown in [6] that the counter-example by Rickard and Schofield [4] is also a counter-example to this question.

In this short note, we will suggest a new idea to consider the Rank Question for tilting modules via silting complexes and show that the above mentioned counter-examples will not be counter-examples to our new questions.

## 2. Silting Complexes

In studying tilting complexes, Keller and Vossieck [7] first introduced silting complexes. Later, silting complexes were recognized in [8,9]. Let  $T \in \mathcal{D}^b(\text{mod} R)$ . Recall that  $T$  is *silting* [7–9] if it satisfies the following conditions:

- (1)  $T$  is isomorphic to an object in  $\mathcal{K}^b(\mathcal{P}_R)$ ;
- (2)  $\text{Hom}_{\mathcal{D}}(T, T[i]) = 0$  for all  $i > 0$ ;
- (3)  $T$  generates  $\mathcal{K}^b(\mathcal{P}_R)$ , i.e.,  $\langle \text{add} T \rangle = \mathcal{K}^b(\mathcal{P}_R)$ , where  $\langle \text{add} T \rangle$  denotes the smallest triangulated subcategory containing  $\text{add} T$  in the derived category.

A complex  $T \in \mathcal{D}^b(\text{mod} R)$  is said to be *partial silting* if it satisfies the first two conditions.

Thus, silting complexes are a small generalization of tilting complexes and are larger generalizations of tilting modules in the derived category. Some important results of tilting modules have been extended to silting complexes, see, for instance [8–12] etc. A good survey on the theory of silting complexes is [13], by Angeleri-Hügel. In particular, it was proved in [8] that silting complexes satisfy the rank-condition, i.e., the number of distinct indecomposable direct summands of a silting complex is equal to the rank of  $K_0(R)$ .

Now, we extend the above two questions on tilting modules to silting complexes.

**Rank Question for silting complexes:** *Is a partial silting complex that satisfies the rank-condition always silting?*

**Complement Question for partial silting complexes:** *Can a partial silting complex always be completed to a silting complex? That is, for a partial silting complex  $T$ , is there a complex  $T' \in \mathcal{D}^b(\text{mod} R)$  such that  $T \oplus T'$  becomes a silting complex?*

It is easy to see that a positive answer to the Complement Question for partial silting complexes implies a positive answer to the Rank Question for silting complexes, and the latter implies a positive answer to the Rank Question for tilting complexes, which finally implies a positive answer to the Rank Question for tilting modules.

As proven in [9], the Complement Question for partial 1-silting complexes has a positive answer. Here, a partial silting complex is *partial 1-silting* provided that, up to shifts, it is isomorphic to a complex of the form  $0 \rightarrow P_1 \rightarrow P_0 \rightarrow 0$  in the derived category, where  $P_1, P_0$  are finitely generated projective modules. The notion of 1-silting complex is defined similarly. In particular, classical tilting modules are 1-silting complexes when considered as a complex in the derived category. Theorem 2.10 in [14] shows that every 1-silting module has a Bongartz complement.

We do not know the answers to the above questions in general.

Now, let us return to the Rank Question for tilting modules. Note that partial tilting modules are partial silting complexes when considered in the derived category, so we can also ask the following question.

**Silting Complement Question for partial tilting modules:** *Can a partial tilting module always be completed to a silting complex?*

We have the following observation.

**Observation** *If the Silting Complement Question for partial tilting modules has a positive answer, then the Rank Question for tilting modules also has a positive answer.*

**Proof.** Given a partial tilting module satisfying the rank-condition, say  $T$ , then the number of distinct indecomposable direct summands of  $T$  is just the rank of  $K_0(R)$ . By assumption, there is a complex  $T' \in \mathcal{D}^b(\text{mod} R)$  such that  $T \oplus T'$  becomes a silting complex. Therefore, the number of distinct indecomposable direct summands of  $T \oplus T'$  are also the same as the rank of  $K_0(R)$ . It follows that the number of distinct indecomposable direct summands of  $T$  is equal to the number of distinct indecomposable direct summands of  $T \oplus T'$ , and, consequently, indecomposable direct summands of  $T'$  are also indecomposable direct summands of  $T$ , up to isomorphisms. Thus,  $T$  and  $T \oplus T'$  are equivalent, and then  $T$  is a silting complex too. Hence,  $T$  is a tilting module, as we mentioned before.  $\square$

According to the above observation, a new idea to consider in the Rank Question for tilting modules is the Complement Question for partial silting complexes, or the Silting Complement Question for partial tilting modules.

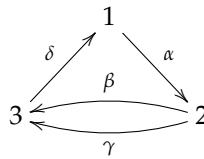
From the above, one must wonder whether Rickard and Schofield's counter-example is a counter-example to the Complement Question for partial silting complexes or the Complement Question for partial tilting modules. We now deal with this question in the following section.

It should be noted that in Example 2.6, in the paper [15] by Breaz, a similar phenomenon is exhibited to those studied in our paper, but for a different problem.

### 3. Partial Tilting Simple Modules

Recall that the counter-example to the Complement Question for partial tilting modules, given by Rickard and Schofield [4], is as follows.

Let  $R$  be the path algebra defined by the quiver as follows



and be bounded by relations  $\alpha\beta = \gamma\delta = \delta\alpha = 0$ .

The global dimension of this algebra is 4. Let  $S$  be the simple module correspondent to vertex 1. Then,  $S$  has projective dimension 2 and satisfies that  $\text{Ext}_R^i(S, S) = 0$  for  $i > 0$ . Moreover, it was shown in [4] that there is no module  $X$  such that all three equalities  $\text{Ext}_A^1(X, X) = 0$ ,  $\text{Ext}_A^2(S, X) = 0$  and  $\text{Ext}_A^2(X, S) = 0$  hold together. Thus, the partial tilting simple module  $S$  has no complements in tilting modules.

The next theorem, in particular, shows that this is different for the Silting Complement Question for simple partial tilting modules.

**Theorem 1.** *Let  $R$  be an artin algebra and  $S$  be a simple module, which is partial tilting. Then,  $S$  always has a complement,  $S' \in \mathcal{K}^b(\mathcal{P}_R)$ , such that  $S \oplus S'$  is a silting complex.*

**Proof.** We may assume that  $R$  is basic and that a complete set of primitive orthogonal idempotent elements is  $\{e_1, \dots, e_n\}$ . Then, the indecomposable projective module is of the form  $Re_i$  for some  $1 \leq i \leq n$ .

As the simple module  $S$  is partial tilting, it has a finite projective dimension and satisfies  $\text{Ext}_R^i(S, S) = 0$  for all  $i > 0$ . We see that its minimal projective dimension is of the form

$$0 \rightarrow P_s \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow S \rightarrow 0,$$

for some integer  $s \geq 0$ . One can get that these assumptions imply the following two facts. One is that the projective module  $P_0$  in the minimal projective resolution is indecomposable, and hence we may assume that  $P_0 \simeq Re_k$ , for some  $1 \leq k \leq n$ . The other is that the remained projective modules  $P_1, \dots, P_s$  in the minimal projective resolution lie in  $\text{add} R(1 - e_k)$ .

Now, setting  $S' = R(1 - e_k)[s] \in \mathcal{K}^b(\mathcal{P}_R)$ , we will show that  $S \oplus S'$  is a silting complex.

We know that  $S \oplus S' \in \mathcal{K}^b(\mathcal{P}_R)$  and that  $\text{Hom}_{\mathcal{D}}(S, S'[i]) = 0 = \text{Hom}_{\mathcal{D}}(S', S'[i])$  for all  $i > 0$ . Note that  $S$  is isomorphic to the truncated complex

$$P_S : 0 \rightarrow P_s \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow 0,$$

in the derived category. It is easy to see that  $\text{Hom}_{\mathcal{D}}(S, S'[i]) \simeq \text{Hom}_{\mathcal{D}}(P_S, R(1 - e_k)[s][i]) = 0$  for all  $i > 0$ , since  $S' = R(1 - e_k)[s]$  is a stalk complex at the  $(-s)$ -th position and  $P_S$  is a complex with non-zero terms only at  $i$ -th positions for  $-s \leq i \leq 0$ . Similarly, we also obtain  $\text{Hom}_{\mathcal{D}}(S', S[i]) \simeq \text{Hom}_{\mathcal{D}}(R(1 - e_k)[s], P_S[i]) = 0$  for all  $i > s$ . As homologies of  $P_S$  at  $i$ -th positions, where  $-s \leq i \leq -1$ , are zero, and as  $R(1 - e_k)$  is projective, we obtain that  $\text{Hom}_{\mathcal{D}}(R(1 - e_k)[s], P_S[i]) = 0$  for all  $1 \leq i \leq s - 1$ . By our assumption,  $S \simeq \text{top}(P_0) = \text{top}(Re_k)$ , so we also obtain  $\text{Hom}_{\mathcal{D}}(R(1 - e_k)[s], P_S[s]) \simeq \text{Hom}_R(R(1 - e_k), S) = 0$ . Together, we obtain  $\text{Hom}_{\mathcal{D}}(S', S[i]) \simeq \text{Hom}_{\mathcal{D}}(R(1 - e_k)[s], P_S[i]) = 0$  for all  $i > 0$ . Then, it follows that  $\text{Hom}_{\mathcal{D}}(S \oplus S', S \oplus S'[i]) = 0$  for all  $i > 0$ .

It remains be seen whether  $S \oplus S'$  generates  $\mathcal{K}^b(\mathcal{P}_R)$  by the definition of silting complexes. However, this follows from the natural triangle  $Q \rightarrow P_0 \rightarrow P_S \rightarrow$  arising from the complex  $P_S$ , where  $Q$  is the shifted truncated complex

$$Q : 0 \rightarrow P_s \rightarrow \dots \rightarrow P_1 \rightarrow 0,$$

with  $P_1$  at 0-th position. Indeed, since  $S' = R(1 - e_k)[s]$  and  $P_1, \dots, P_s \in \text{add}R(1 - e_k)$ , we obtain  $Q \in \langle S' \rangle$ . Together with the fact that  $P_5 \simeq S$  in the derived category, we obtain  $P_0 \in \langle S \oplus S' \rangle$ . Then we further prove that all indecomposable projective modules are contained in  $\langle S \oplus S' \rangle$ . It follows that  $\langle S \oplus S' \rangle$  generates  $\mathcal{K}^b(\mathcal{P}_R)$ .  $\square$

Regarding the counter-example of Rickard and Schofield, as above, we see that  $(Re_2 \oplus Re_3)[2] \oplus S$  is a silting complex from the above theorem.

Let  $P^\bullet := \{P_i\}$  be a (cochain) complex and  $t$  an integer. We denote by  $P^{\bullet < t}$  the brutal truncation complex of  $P^\bullet$  consisting of terms  $P_i^\bullet$  with  $i < t$ . Similarly, we denote by  $P^{\bullet \geq t}$  the brutal truncation complex of  $P^\bullet$  consisting of terms  $P_i^\bullet$  with  $i \geq t$ . Recall that  $\langle \mathcal{C} \rangle$  denotes the smallest thick triangulated subcategory containing  $\mathcal{C}$ .

We have the following, more general result, which gives a condition for a partial tilting module when it can be completed to a silting complex.

**Theorem 2.** Let  $R$  be an artin algebra. Assume that  $T$  is a partial tilting module with its projective dimension  $d$ . Denote by  $R_T$  ( $R_0$ , resp.) the direct sum of indecomposable projective modules  $P$  with  $\text{Hom}_R(P, T)$  nonzero (zero, resp.). Let  $P_T^\bullet$  (as a cochain complex) be the minimal projective resolution of  $T$ . If there is an integer  $t$ , where  $0 \leq t \leq d$ , such that  $P_T^{\bullet < -t}$  consists of projective modules in  $\text{add}R_0$  and  $R_T \in \langle R_0, P_T^{\bullet \geq -t} \rangle$ , then  $T \oplus R_0[d]$  is a silting complex.

**Proof.** Clearly,  $T \oplus R_0[d] \in \mathcal{K}^b(\text{proj})$  and  $R_0[d]$  is partial silting.

From the definition of  $R_0$ , we obtain that  $\text{Hom}_{\mathcal{D}}(R_0, T) \simeq \text{Hom}_R(R_0, T) = 0$ . Together with the knowledge that  $R_0$  is projective and  $T, R_0$  are modules, we see that  $\text{Hom}_{\mathcal{D}}(R_0, T[i]) = 0$  for all integer  $i$ . In particular,  $\text{Hom}_{\mathcal{D}}(R_0[d], T[k]) = 0$  for all  $k > 0$ . On the other hand, since  $d$  is the projective dimension of  $T$ , we can also see that  $\text{Hom}_{\mathcal{D}}(T, (R_0[d])[k]) = \text{Hom}_{\mathcal{D}}(T, R_0[d+k]) \simeq \text{Ext}_R^{d+k}(T, R_0) = 0$  for all  $k > 0$ . It follows that  $T \oplus R_0[d]$  is partial silting.

It remains to prove that  $R \in \langle T, R_0[d] \rangle$  by the definition of silting complexes. Obviously, we already have  $R_0 \in \langle T, R_0[d] \rangle$ . By the assumption,  $R_T \in \langle R_0, P_T^{\bullet \geq t} \rangle$ . Note that there is an obvious triangle  $P_T^{\bullet < t}[-1] \rightarrow P_T^{\bullet \geq t} \rightarrow T \rightarrow$ ; we can see that  $P_T^{\bullet \geq t}$  is contained in  $\langle P_T^{\bullet < t}, T \rangle$  and that  $P_T^{\bullet < t} \in \langle R_0 \rangle$  by assumptions, and so we obtain  $R_T \in \langle R_0, T \rangle$ . It follows  $R = R_0 \oplus R_T \in \langle R_0, T \rangle$ .

Altogether, we demonstrate that  $T \oplus R_0[d]$  is a silting complex.  $\square$

Finally, we present some examples where our results apply.

**Example 1.** (1) Let  $R$  be the path algebra which is defined by the following quiver

$$5 \xleftarrow{\alpha} 4 \xleftarrow{\beta} 1 \xrightarrow{\gamma} 2 \xrightarrow{\delta} 3$$

with relations  $\beta\alpha = \gamma\delta = 0$ .

Let  $T = \begin{smallmatrix} 1 \\ 222 \\ 33 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3$ . Then,  $T$  is partial tilting and its projective dimension is 2. The minimal projective resolution  $P_T^\bullet$  of  $T$  is as follows:

$$0 \rightarrow 5 \rightarrow \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \rightarrow \begin{smallmatrix} 1 \\ 4222 \\ 33 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3 \rightarrow 0,$$

In terms of the above theorem, we show that  $R_0 := P_4 \oplus P_5$  and  $R_T := P_1 \oplus P_2 \oplus P_3$ , where  $P_i$  is the indecomposable projective module corresponding to the point  $i$ . Obviously,  $P_T^{\bullet < 0}$  consists of terms in  $R_0$  and  $R_T \in \text{add}P_T^{\bullet \geq 0}$ . Thus, by the above theorem,  $T \oplus (5 \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix})[2]$  is a silting complex.

(2) Let  $R$  be the path algebra given by the following quiver

$$6 \xleftarrow[\gamma]{\beta} 5 \xrightarrow{\delta} 2 \xrightarrow{\zeta} 1 \xrightarrow{\eta} 3 \xrightarrow{\theta} 4$$

$\begin{matrix} \nearrow \rho \\ \searrow \end{matrix}$

bounded by the ideal  $I = \langle \gamma\beta, \beta\gamma, \rho^2, \delta\zeta, \eta\theta \rangle$ .

Let  $T = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus 2$ . Then,  $T$  is partial tilting and its projective dimension is 2. The minimal projective resolution of the direct summand  $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$  is as follows:

$$0 \rightarrow 4 \rightarrow \begin{smallmatrix} 3 & 2 \\ 4 & \end{smallmatrix} \rightarrow 1 \rightarrow 0,$$

and the minimal projective resolution of the direct summand 2 is as follows:

$$0 \rightarrow \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \rightarrow 1 \rightarrow 0.$$

The projective resolution  $P_T^\bullet$  of  $T$  is just the direct sum of the above two projective resolutions. In terms of the above theorem, one can easily see that  $R_0 := P_3 \oplus P_4 \oplus P_5 \oplus P_6$  and  $R_T := P_1 \oplus P_2$ , where  $P_i$  is the indecomposable projective module corresponding to the point  $i$ . Obviously,  $P_T^{\bullet < -1}$  consists of terms in  $R_0$ . By the

first resolution complex, we obtain that  $P_2(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) \in \langle R_0 \rangle$ . Then, by the second resolution complex, we further

obtain that  $P_1(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}) \in \langle P_T^{\bullet \geq -1}, P_2 \rangle \subseteq \langle P_T^{\bullet \geq -1}, R_0 \rangle$ . Hence, by the above theorem, we see that  $T \oplus R_0[2]$  is a silting complex.

Finally, we remark that any finitely generated partial tilting module over an artin algebra can be completed to an infinitely generated tilting module, see [16]. We refer to [17] for recent progress on infinitely generated tilting modules. One can also compare the tilting theory to the theory of semi-dualizing complexes (or semi-dualizing modules), see for instance [18] for some results in later theory.

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