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Multiple Solutions for a Class of New $p(x)$-Kirchhoff Problem without the Ambrosetti-Rabinowitz Conditions

Bei-Lei Zhang, Bin Ge * and Xiao-Feng Cao

College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China;
18737010665@163.com (B.-L.Z.); xftsao@163.com (X.-F.C.)
* Correspondence: gebin791025@hrbeu.edu.cn

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Abstract: In this paper, we consider a nonlocal $p(x)$-Kirchhoff problem with a p^+-superlinear subcritical Caratheodory reaction term, which does not satisfy the Ambrosetti–Rabinowitz condition. Under some certain assumptions, we prove the existence of nontrivial solutions and many solutions. Our results are an improvement and generalization of the corresponding results obtained by Hamdani et al. (2020).

Keywords: variable exponent; nonlocal Kirchhoff equation; variational method; existence of nontrivial solutions; multiple solutions; fountain theorem; dual fountain theorem

1. Introduction

This paper is concerned with the following nonlocal $p(x)$-Kirchhoff problem

$$\begin{cases} -\left(a - b\int_\Omega |\nabla u|^{p(x)}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u + g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $a \geq b > 0$, $p \in C(\overline{\Omega})$ with $1 < p(x) < N$, $\lambda > 0$ is a real number, and $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function whose potential satisfies some conditions which will be stated later on.

The Kirchhoff type equations involving variable exponent growth conditions have been a very interesting topic in recent years, and we have seen the publication of a great number of manuscripts dealing with this subject (see, for example, [1–15] and references therein). Problems of this type arise in mathematical models of various physical and biological phenomena. We mention the works of Shahruz et al. [16] (in physics systems), Chipotv and Rodrigues [17] (in biological systems). Since the left-hand side in (1) contains an integral over Ω, it is no longer a pointwise identity, and therefore, it is often called a nonlocal problem. It was proposed by Kirchhoff in 1883 as a generalization of the well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0. \tag{2}$$

For free vibrations of elastic strings, see [18]. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. The parameters in (2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and p_0 is the initial tension.

Recently, Hamdani, Harrabi, Mtiri and Repovs established in [19] the existence of nontrivial solutions for problem 1 by assuming the following conditions:

(g_1) $g \in C(\overline{\Omega} \times \mathbb{R})$, and positive constant C, such that

$$|g(x,t)| \leq C(1 + |t|^{q(x)-1}), \ \forall (x,t) \in \Omega \times \mathbb{R},$$

where $q \in C(\overline{\Omega})$ with

$$\begin{aligned} 1 < & p^- := \min_{x\in\overline{\Omega}} p(x) < p(x) \\ < & p^+ := \max_{x\in\overline{\Omega}} p(x) < 2p^- \\ < & q^- = \min_{x\in\overline{\Omega}} q(x) < q(x) < p^*(x), \end{aligned}$$

$p^*(x) = \frac{Np(x)}{N-p(x)}$ is the critical exponent;

(g_2) $\lim\limits_{t\to 0} \frac{g(x,t)}{|t|^{p(x)-2}t} = 0$ uniformly in $x \in \Omega$;

(g_3) there exist $s_A > 0$, $\theta \in (p^+, \frac{2(p^-)^2}{p^+})$ such that for all $|t| \geq s_A$ and $x \in \Omega$,

$$0 < \theta G(x,t) \leq t g(x,t),$$

where $G(x,t) = \int_0^t g(x,s)ds$;

(g_4) $g(x,-t) = -g(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.

Then by the Mountain Pass theorem and Fountain Theorem, the following result was presented.

Theorem 1 (Theorem 1.1 [19])**.** *Suppose that (g_1), (g_2) and (g_3) hold. Then for any $\lambda < \lambda_*$ (λ_* is defined in Lemma 4.1 [19]), the problem (1) has a nontrivial weak solution.*

Theorem 2 (Theorem 1.2 [19])**.** *Suppose that (g_1), (g_2), (g_3) and (g_4) hold. Then for any $\lambda < \lambda_*$ (λ_* is defined in Lemma 4.1 [19]) the problem (1) has infinitely many solutions $\{u_n\}$ such that $I(u_n) \to +\infty$ as $n \to +\infty$.*

It is well known that, the condition (g_3) is originally due to Ambrosetti and Rabinowitz [20]. This is a tool to study superlinear problems, it is a natural and useful condition not only to ensure that the Euler–Lagrange functional associated to problem (1) has a mountain pass geometry, but also to guarantee that the Palais–Smale sequence of the Euler–Lagrange functional is bounded. However, condition (g_3) is too restrictive and eliminates many nonlinearities. Clearly, the condition (g_3) implies condition

$$G(x,t) \geq c_1|t|^\theta - c_2, \ \forall (x,t) \in \Omega \times \mathbb{R}, \tag{3}$$

where c_1, c_2 are two positive constants. However, there are many functions which are superlinear at infinity, but do not satisfy the condition (g_3), for example,

$$g(x,t) = |t|^{p^+-2} t \ln(|t| + 1).$$

At this purpose, we would note that from (3) and the fact that $\theta > p^+$, it follows that

(g_5) $\lim\limits_{|t|\to\infty} \frac{G(x,t)}{|t|^{p^+}} = +\infty$, uniformly a.e. $x \in \Omega$.

Moreover, condition (g_5) characterizes the nonlinearity g to be p^+-superlinear at infinity.

In this paper, we consider problem (1) in the case when the nonlinear term $g(x,t)$ is p^+-superlinear at infinity but does not satisfy condition (g_3). More precisely, we shall study the existence and multiplicity of weak solutions of problem (1) under the suitable conditions. To state our results, we make the following assumption on g:

(g_6) there exists a positive constant $C_0 > 0$ such that

$$\mathcal{G}(x,t) \leq \mathcal{G}(x,s) + C_0$$

for any $x \in \Omega$, $0 < t < s$ or $s < t < 0$, where $\mathcal{G}(x,t) := tg(x,t) - p^+ G(x,t)$.

We remark that the condition (g_6) is a consequence of the following condition $(g_6)'$, which was firstly introduced by Miyagaki and Souto [21] and developed by G. Li et al. [22] and C. Ji [23]:

$(g_6)'$ There exists $t_0 > 0$ such that for $\forall x \in \Omega$,

$$\frac{g(x,t)}{|t|^{p^+-2}t} \text{ is increasing in } t \geq t_0 \text{ and decreasing in } t \leq -t_0.$$

The readers may consult the proof and comments on this assertion in the papers [21–23] and the references cited there. Now, we give an example to illustrate the feasibility of assumptions $(g_1) - (g_2)$ and $(g_4) - (g_6)$. Let

$$g(x,t) = |t|^{p^+-2} t \ln(1+|t|) + \frac{1}{p^+} \frac{|t|^{p^+-1} u}{1+|u|}. \tag{4}$$

by a straightforward computation, we deduce that

$$G(x,t) = \frac{1}{p^+} |t|^{p^+} \ln(1+|u|).$$

So, it is easy to check that $g(x,t)$ satisfies our conditions (g_1) (when $(q(x) \equiv p^+ + 1)$, (g_2) and $(g_4) - (g_6)$, but it does not satisfy the condition (g_3).

We are now in the position to state our main results.

Theorem 3. *Suppose that $p \in MIP(\Omega)$ ($MIP(\Omega)$ is defined in Section 2), $(g_1) - (g_2)$, (g_5) and (g_6) hold. Then, problem (1) has at least one nontrivial weak solution in $W_0^{1,p(x)}(\Omega)$ for all $\lambda < \lambda_0$ (λ_0 is given in Section 3).*

Theorem 4. *Suppose that $p \in MIP(\Omega)$ ($MIP(\Omega)$ is defined in Section 2), $(g_1) - (g_2)$ and $(g_4) - (g_6)$ hold. Then, the problem (1) has infinitely many solutions in $W_0^{1,p(x)}(\Omega)$ for all $\lambda < \lambda_0$ (λ_0 is given in Section 3).*

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on space $W_0^{1,p(x)}(\Omega)$. In Section 3, we establish the variational framework associated with problem (1), and we also state the critical point theorems needed for the proofs of our main results. We complete the proofs of Theorems 3 and 4 in Sections 4 and 5, respectively.

2. Preliminaries

Firstly, we introduce some definitions and basic properties of the Lebesgue–Sobolev spaces with variable exponents. The detailed result can be found in [24–29]. Let Ω be a bounded domain of $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^- = \min_{x \in \overline{\Omega}} h(x), \quad h^+ = \max_{x \in \overline{\Omega}} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space:

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} | \ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < +\infty\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\tau > 0 : \int_\Omega |\frac{u(x)}{\tau}|^{p(x)} dx \le 1\},$$

and define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. With these norms, the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces: see [27] for details.

Denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, then the Hölder type inequality

$$\int_\Omega |uv| dx \le (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(x)}|v|_{p'(x)},\ u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega) \tag{5}$$

holds. Furthermore, define mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by $\rho(u) = \int_\Omega |u|^{p(x)} dx$, then, the following relations hold

$$\begin{aligned} &|u|_{p(x)} < 1(=1, >1) \Leftrightarrow \rho(u) < 1(=1, >1), \\ &|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}, \\ &|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}. \end{aligned} \tag{6}$$

Proposition 1 ([30])**.** *Assume that $h \in L^\infty_+(\Omega)$, $p \in C_+(\overline{\Omega})$. If $|u|^{h(x)} \in L^{p(x)}(\Omega)$, then we have*

$$\min\{|u|^{h^-}_{h(x)p(x)}, |u|^{h^+}_{h(x)p(x)}\} \le \big||u|^{h(x)}\big|_{p(x)} \le \max\{|u|^{h^-}_{h(x)p(x)}, |u|^{h^+}_{h(x)p(x)}\}.$$

Now, we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Then, we have

Proposition 2 ([27])**.** *(1) Poincaré inequality in $W_0^{1,p(x)}(\Omega)$ holds, that is, there exists a positive constant C_0 such that*

$$|u|_{p(x)} \le C_1|\nabla u|_{p(x)}, \forall u \in W_0^{1,p(x)}(\Omega).$$

(2) If $h \in C(\overline{\Omega})$ and $1 \le h(x) \le p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W_0^{1,p(x)}(\Omega)$ to $L^{h(x)}(\Omega)$ is continuous. In particular, if $1 \le h(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega)$ is compact.

By (1) of Proposition 2, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussions.

Remark 1. *Although the Poincaré inequality holds, we must point out that the modular inequality*

$$\int_\Omega |u|^{p(x)} dx \le C_2 \int_\Omega |\nabla u|^{p(x)} dx,\ \forall u \in W_0^{1,p(x)}(\Omega) \tag{7}$$

not always holds (see Theorem 3.1 [31]). It is known that (7) holds in each of these cases:

(i) $N > 1$ and there exists a $x_0 \notin \overline{\Omega}$, such that for any $w \in \mathbb{R}^N$ with $\|w\| = 1$, the function $f(t) := p(x_0 + tw)$ is monotone (Theorem 3.4 [31]) with $x_0 + tw$ with an appropriate setting in Ω;

(ii) there exists a function $\xi \ge 0$, such that $\nabla p \cdot \nabla \xi \ge 0$, $|\nabla \xi| \ne 0$, see (Theorem 1 [32]) for details;

(iii) there exists $a : \Omega \to \mathbb{R}^N$ bounded such that $\mathrm{div} a(x) \ge a_0 > 0$ and $a(x) \cdot \nabla p(x) = 0$ for all $x \in \Omega$, see (Theorem 1 [33]) for details.

To the best of our knowledge, necessary and sufficient conditions in order to ensure that

$$\lambda_{p(x)} =: \inf_{u\in W_0^{1,p(x)}(\Omega)\backslash\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx} > 0$$

has not been obtained yet, except in the case $N = 1$, (Theorem 3.2 [31]). To overcome this difficulty, the following definition is given.

Definition 1. *We say that $p(\cdot)$ belongs to the modular Poincaré inequality, $MPI(\Omega)$, if there exists necessary conditions to ensure that (7) holds.*

Finally, in order to discuss the problem (1), we need to define a functional in $W_0^{1,H}(\Omega)$:

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx.$$

We know that (see [34]), $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and the $p(x)$-Laplacian operator $-\mathrm{div}(|\nabla u|^{p(x)-2})$ is the derivative operator of J in the weak sense. We define $L = J' : W_0^{1,H}(\Omega) \to (W_0^{1,H}(\Omega))^*$, then

$$\langle L(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v dx$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. Here $(W_0^{1,p(x)}(\Omega))^*$ denotes the dual space of $W_0^{1,p(x)}(\Omega)$ and $\langle\cdot,\cdot\rangle$ denotes the pairing between $W_0^{1,p(x)}(\Omega)$ and $(W_0^{1,p(x)}(\Omega))^*$. Then, we have the following proposition

Proposition 3 (Theorem 3.1 [28])**.** *Set $E = W_0^{1,p(x)}(\Omega)$, L is as above, then*

(1) $L : E \to E^*$ a continuous, bounded and strictly monotone operator;

(2) $L : E \to E^*$ is a mapping of type $(S)_+$, i.e., if $u_n \rightharpoonup u$ in E and $\limsup_{n\to+\infty}\langle L(u_n), u_n - u\rangle \le 0$, implies $u_n \to u$ in E;

(3) $L : E \to E^*$ is a homeomorphism.

3. Variational Setting and Some Preliminary Lemmas

To prove our theorems, we recall the variational setting corresponding to the problem (1). Firstly, we introduce the energy functional $\varphi_\lambda(u) : W_0^{1.p(x)}(\Omega) \to \mathbb{R}$ associated with problem (1), defined by

$$\begin{aligned}\varphi_\lambda(u) =& a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 \\ & - \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega G(x,u)dx.\end{aligned}$$

From the hypotheses on g, it is standard to check that $\varphi_\lambda \in C^1(W_0^{1.p(x)}(\Omega), \mathbb{R})$ and its Gateaux derivative is

$$\begin{aligned}\langle \varphi_\lambda'(u), v\rangle =& \Big(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx \\ & - \lambda\int_\Omega |u|^{p(x)-2}uv dx - \int_\Omega g(x,u)v dx\end{aligned}$$

for any $u, v \in W_0^{1.p(x)}(\Omega)$.

Thus, the critical points of φ_λ are precisely the weak solutions of problem (1). First of all, notice that φ_λ verifies the mountain pass geometry, in a uniform way on compact sets:

Lemma 1. *Suppose that $p \in MPI(\Omega)$, (g_1), (g_2) and (g_5) hold. Then*

(a) φ_λ is unbounded from below;
(b) there exists a $\lambda_0 > 0$ such that $u = 0$ is a strict local minimum of φ_λ for all $\lambda < \lambda_0$.

Proof. (a) From (g_5), it follows that, $\forall M > 0, \exists C_M > 0$, such that

$$G(x,t) \geq M|t|^{p^+} - C_M, \forall x \in \Omega,\ \forall t \in \mathbb{R}. \tag{8}$$

Take $\phi \in W_0^{1,p(x)}(\Omega)$ with $\phi > 0$, from (8) we have

$$\begin{aligned}
\varphi_\lambda(t\phi) =& a\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\phi|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\phi|^{p(x)}dx\Big)^2 \\
&- \lambda\int_\Omega \frac{t^{p(x)}}{p(x)}|\phi|^{p(x)}dx - \int_\Omega G(x,t\phi)dx \\
\leq& at^{p^+}\int_\Omega \frac{1}{p(x)}|\nabla\phi|^{p(x)}dx - \frac{b}{2}t^{2p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla\phi|^{p(x)}dx\right)^2 \\
&- \lambda t^{p^+}\int_\Omega \frac{1}{p(x)}|\phi|^{p(x)}dx - Mt^{p^+}\int_\Omega \phi^{p^+}dx + C_M|\Omega| \\
=& t^{p^+}\left(a\int_\Omega \frac{1}{p(x)}|\nabla\phi|^{p(x)}dx - \lambda\int_\Omega \frac{1}{p(x)}|\phi|^{p(x)}dx - M\int_\Omega \phi^{p^+}dx\right) \\
&- \frac{b}{2}t^{2p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla\phi|^{p(x)}dx\right)^2 + C_M|\Omega| \\
\to& -\infty, \text{ as } t \to +\infty,
\end{aligned}$$

since $2p^- > p^+$. Therefore, φ_λ is unbounded from below.

(b) Firstly, from (g_1) and (g_2), it follows that, for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$F(x,t) \leq \varepsilon|t|^{p(x)} + C_\varepsilon|t|^{q(x)},\ \forall(x,t) \in \Omega\times\mathbb{R}.$$

Thus, for $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| \leq 1$, using Proposition 2, (6) and (7), we have

$$\begin{aligned}
\varphi_\lambda(u) =& a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 \\
&- \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega G(x,u)dx \\
\geq& \begin{cases}
\frac{a}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{b}{2(p^-)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\
-\varepsilon\int_\Omega |u|^{p(x)}dx - C_\varepsilon\int_\Omega |u|^{q(x)}dx, & \text{if } \lambda \leq 0, \\
\frac{a}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{b}{2(p^-)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\
-(\frac{\lambda}{p^-}+\varepsilon)\int_\Omega |u|^{p(x)}dx - C_\varepsilon\int_\Omega |u|^{q(x)}dx, & \text{if } \lambda > 0,
\end{cases} \\
\geq& \begin{cases}
(\frac{a}{p^+} - C_2\varepsilon)\int_\Omega |\nabla u|^{p(x)}dx - \frac{b}{2(p^-)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\
-C_\varepsilon(|u|_{q(x)}^{q^-} + |u|_{q(x)}^{q^+}), & \text{if } \lambda \leq 0, \\
\frac{a}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{b}{2(p^-)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\
-(\frac{\lambda}{p^-}+\varepsilon)C_2\int_\Omega |\nabla u|^{p(x)}dx - C_\varepsilon(|u|_{q(x)}^{q^-} + |u|_{q(x)}^{q^+}), & \text{if } \lambda > 0,
\end{cases}
\end{aligned}$$

$$
\geq \begin{cases} \left(\frac{a}{p^+} - C_2\varepsilon\right)\|u\|^{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^-} \\ - C_\varepsilon(C_1^{q^-}\|u\|^{q^-} + C_1^{q^+}\|u\|^{q^+}), & \text{if } \lambda \leq 0, \\ \left(\frac{a}{p^+} - (\frac{\lambda}{p^-} + \varepsilon)C_2\right)\|u\|^{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^-} \\ - C_\varepsilon(C_1^{q^-}\|u\|^{q^-} + C_1^{q^+}\|u\|^{q^+}), & \text{if } \lambda > 0, \end{cases}
$$

$$
\geq \begin{cases} \left(\frac{a}{p^+} - C_2\varepsilon\right)\|u\|^{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^-} \\ - C_\varepsilon(C_1^{q^-} + C_1^{q^+})\|u\|^{q^-}, & \text{if } \lambda \leq 0, \\ \left(\frac{a}{p^+} - (\frac{\lambda}{p^-} + \varepsilon)C_2\right)\|u\|^{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^-} \\ - C_\varepsilon(C_1^{q^-} + C_1^{q^+})\|u\|^{q^-}, & \text{if } \lambda > 0, \end{cases}
$$

$$
\geq \begin{cases} \frac{a}{2p^+}\|u\|^{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^-} \\ - C_\varepsilon(C_1^{q^-} + C_1^{q^+})\|u\|^{q^-}, & \text{if } \lambda \leq 0, \\ \left(\frac{a}{2p^+} - \frac{\lambda C_2}{2p^-}\right)\|u\|^{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^-} \\ - C_\varepsilon(C_1^{q^-} + C_1^{q^+})\|u\|^{q^-}, & \text{if } \lambda > 0, \end{cases}
$$

$$
= \begin{cases} \|u\|^{p^+}\Big[\frac{a}{2p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^- - p^+} \\ - C_\varepsilon(C_1^{q^-} + C_1^{q^+})\|u\|^{q^- - p^+}\Big], & \text{if } \lambda \leq 0, \\ \|u\|^{p^+}\Big[\left(\frac{a}{2p^+} - \frac{\lambda C_2}{2p^-}\right) - \frac{b}{2(p^-)^2}\|u\|^{2p^- - p^+} \\ - C_\varepsilon(C_1^{q^-} + C_1^{q^+})\|u\|^{q^- - p^+}\Big], & \text{if } \lambda > 0. \end{cases}
$$

From this, and the fact that $p^+ < 2p^- < q^-$, we can choose $r > 0$ and

$$
\lambda_0 = \frac{a(p^-)^2 - bp^+ r^{2p^- - p^+} - 2p^+(p^-)^2 C_\varepsilon(C_1^{q^-} + C_1^{q^+})r^{q^- - p^+}}{p^- p^+ C_2}
$$

such that

$$
\frac{a}{2p^+} - \frac{b}{(2p^-)^2}r^{2p^-} - C_\varepsilon r^{q^- - p^+}(C_1^{q^-} + C_1^{q^+}) > 0
$$

and

$$
\left(\frac{a}{2p^+} - \frac{\lambda C_2}{2p^-}\right) - \frac{b}{2(p^-)^2}r^{2p^- - p^+} - C_\varepsilon(C_1^{q^-} + C_1^{q^+})r^{q^- - p^+} > 0, \ \forall \lambda \in (0, \lambda_0).
$$

thus, there exists $\delta > 0$ such that $\varphi_\lambda(u) \geq \delta > 0$ for every $u \in W_0^{1,p(x)}(\Omega)$ and $\|u\| = r$. This proves (b). So far, we complete the proof. □

Definition 2. *Let $(X, \|\cdot\|)$ be a real Banach space, $I \in C^1(X, \mathbb{R})$. We say that $I \in C^1(E, \mathbb{R})$ satisfies $(C)_c$-condition if any sequence $\{u_n\} \subset E$ satisfying*

$$
I(u_n) \to c \text{ and } \|I'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0 \tag{9}
$$

contains a convergent subsequence. If this condition is satisfied at every level $c \in \mathbb{R}$, then we say that I satisfies (C)-condition.

Now, we present the following Lemmas which will play a crucial role in the proof of Main Theorems. First of all, let us recall the mountain pass theorem, which we use in the proof of Theorem 3.

Lemma 2 (Theorem 1 [35])**.** *Let X be a real Banach space, let $I : X \to \mathbb{R}$ be a functional of class $C^1(X,\mathbb{R})$ that satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$, $I(0) = 0$, and the following conditions hold:*

(1) There exist positive constants ρ and α such that $I(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$.

(2) There exists a function $e \in X$ such that $\|e\| > \rho$ and $I(e) \leq 0$.

Then, the functional I has a critical value $c \geq \alpha$, that is, there exists $u \in X$ such that $I(u) = c$ and $I'(u) = 0$ in X^*.

In order to prove the Theorem 4, we will use the following symmetric mountain pass theorem of Rabinowitz [36]. It is remarked that the symmetric mountain pass theorem is established under the (PS) condition. Since the deformation theorem is still valid under the $(C)_c$-condition ([37]), we see that the symmetric mountain pass theorem also holds under the $(C)_c$-condition (see [38]).

Lemma 3 ([38])**.** *Assume that X is an infinite dimensional Banach space, and let $I : X \to \mathbb{R}$ be an even functional of class $C^1(X,\mathbb{R})$ that satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$, $I(0) = 0$, and the following conditions hold:*

(1) There exist two constants $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$;

(2) for all finite dimensional subspace $\widetilde{X} \subset X$, there exists $R = R(\widetilde{X}) > 0$ such that $I(u) \leq 0$ for any $u \in \widetilde{X}$ with $\|u\| = \rho$.

Then, I possesses an unbounded sequence of critical values characterized by a minimax argument.

4. The Proof of Theorem 3

In this section, we will prove Theorem 3. Firstly, we show that (C)-condition holds. The proof idea is mainly due to Hamdani, Harrabi, Mtiri and Repovs [19], where the Palais–Smale compactness condition was obtained.

Lemma 4. *Assume that (g_1), (g_2), (g_5) and (g_6) hold. Then, the functional φ_λ satisfies the $(C)_c$ condition at any level $c < \frac{a^2}{2b}$.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence. Firstly, we claim that the sequence $\{u_n\}$ is bounded in E. Indeed, if $\|u_n\| \leq 1$, we have done. If $\|u_n\| > 1$, then from (g_6), (9) and $2p(x) \geq 2p^- > p^+$, we have that

$$
\begin{aligned}
& p^+ c + o_n(1) \geq p^+ \varphi_\lambda(u_n) - \langle \varphi'_\lambda(u_n), u_n \rangle \\
=& a \int_\Omega \Big(\frac{p^+}{p(x)} - 1\Big)|\nabla u_n|^{p(x)} dx - \lambda \int_\Omega \Big(\frac{p^+}{p(x)} - 1\Big)|u_n|^{p(x)} dx + \int_\Omega \mathcal{G}(x,u_n) dx \\
& - b \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \Big(\int_\Omega \Big[\frac{p^+}{2p(x)} - 1\Big] |\nabla u_n|^{p(x)} dx \Big) \\
\geq& b\Big(\frac{1}{p^+} - \frac{1}{2p^-}\Big)\Big(\int_\Omega |\nabla u_n|^{p(x)} dx\Big)^2 - \lambda \int_\Omega \Big(\frac{p^+}{p(x)} - 1\Big)|u_n|^{p(x)} dx \\
& + \int_\Omega (\mathcal{G}(x,0) - C_0) dx \\
\geq& b\Big(\frac{1}{p^+} - \frac{1}{2p^-}\Big)\|u_n\|^{2p^-} - \lambda \int_\Omega \Big(\frac{p^+}{p(x)} - 1\Big)|u_n|^{p(x)} dx \\
& + \int_\Omega (\mathcal{G}(x,0) - C_0) dx \\
\geq& \begin{cases} b\Big(\frac{1}{p^+} - \frac{1}{2p^-}\Big)\|u_n\|^{2p^-} - C_0|\Omega|, & \text{if } \lambda \leq 0, \\ b\Big(\frac{1}{p^+} - \frac{1}{2p^-}\Big)\|u_n\|^{2p^-} - \lambda \int_\Omega \Big(\frac{p^+}{p(x)} - 1\Big)|u_n|^{p(x)} dx - C_0|\Omega|, & \text{if } \lambda > 0. \end{cases}
\end{aligned}
$$

From this, we conclude that

$$
\begin{aligned}
& b\Big(\frac{1}{p^+}-\frac{1}{2p^-}\Big)\|u_n\|^{2p^-} \\
\le & \begin{cases} p^+c + C_0|\Omega| + o_n(1), & \text{if } \lambda \le 0, \\ (\frac{p^+}{p^-}-1)\lambda(c_{p(x)}^{p^+}\|u_n\|^{p^+} + c_{p(x)}^{p^-}\|u_n\|^{p^-}) & \\ + p^+c + C_0|\Omega| + o_n(1), & \text{if } \lambda > 0, \end{cases} \\
\le & \begin{cases} p^+c + C_0|\Omega| + o_n(1), & \text{if } \lambda \le 0, \\ (\frac{p^+}{p^-}-1)\lambda(c_{p(x)}^{p^+} + c_{p(x)}^{p^-})\|u_n\|^{p^+} & \\ + p^+c + C_0|\Omega| + o_n(1), & \text{if } \lambda > 0. \end{cases}
\end{aligned} \tag{10}
$$

It follows from (10) and $2p^- > p^+ > p^-$ that $\{u_n\}$ is bounded in E. Therefore, going if necessary to a subsequence, we may assume that

$$
\begin{aligned}
& u_n \rightharpoonup u_0 \text{ in } E, \\
& u_n \to u_0 \text{ in } L^{s(x)}(\Omega),\ 1 \le s(x) < p^*(x), \\
& u_n(x) \to u_0(x) \text{ a.e. on } \Omega.
\end{aligned} \tag{11}
$$

It is easy to compute directly that

$$
\begin{aligned}
& \int_\Omega |g(x,u_n)||u_n-u_0|dx \\
\le & \int_\Omega C(1+|u_n|^{q(x)-1})|u_n-u_0|dx \\
\le & |u_n-u_0|_1 + 2C\big||u_n|^{q(x)-1}\big|_{q'(x)}|u_n-u_0|_{q(x)} \\
\le & |u_n-u_0|_1 + 2C\max\{|u_n|_{q(x)}^{q^--1}, |u_n|_{q(x)}^{q^+-1}\}|u_n-u_0|_{q(x)} \\
\to & 0,\ \text{as } n\to\infty
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
& \int_\Omega |u_n|^{p(x)-2}u_n(u_n-u_0)dx \\
\le & \int_\Omega |u_n|^{p(x)-1}|u_n-u_0|dx \\
\le & 2\big||u_n|^{p(x)-1}\big|_{p'(x)}|u_n-u_0|_{p(x)} \\
\le & 2\max\{|u_n|_{p(x)}^{p^--1}, |u_n|_{p(x)}^{p^+-1}\}|u_n-u_0|_{p(x)} \\
\to & 0,\ \text{as } n\to\infty,
\end{aligned} \tag{13}
$$

where $\frac{1}{q(x)}+\frac{1}{q'(x)}=1$ and $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.

Noting that

$$
\begin{aligned}
& \Big(a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big)\langle L(u_n), u_n-u_0\rangle = \langle \varphi_\lambda'(u_n), u_n-u_0\rangle \\
& + \lambda\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u_0)dx + \int_\Omega g(x,u_n)(u_n-u_0)dx.
\end{aligned} \tag{14}
$$

Moreover, by (9), one yields

$$
\lim_{n\to\infty}\langle \varphi_\lambda'(u_n), u_n-u_0\rangle = 0. \tag{15}
$$

Finally, the combination of (12)–(15) implies

$$\lim_{n\to\infty}\left(a - b\int_{\Omega}\frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\langle L(u_n), u_n - u_0\rangle = 0. \tag{16}$$

Similar to the proof of Lemma 3.1 in [19], we can deduce that the sequence $\{a - b\int_{\Omega}\frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\}$ is bounded, and

$$a - b\int_{\Omega}\frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \not\to 0,\ n \to +\infty.$$

This fact combined with (16) implies that

$$\lim_{n\to\infty}\langle L(u_n), u_n - u_0\rangle = 0.$$

Since L is of type $(S)_+$ by Proposition 3, we obtain $u_n \to u_0$ in E. The proof is complete. □

Now, we are ready to prove Theorem 3.

Proof of Theorem 1. Let $X = E$ and $I = \varphi_\lambda$. Obviously, $\varphi_\lambda(0) = 0$, and Lemma 4 implies that φ_λ satisfies the (C)-condition for any $c < \frac{a^2}{2b}$. In view of Lemma 1, φ_λ satisfies the mountain pass geometry for any $\lambda < \lambda_0$. Therefore, all the assumptions of Lemma 2 are satisfied, so that, for each $\lambda < \lambda_0$, the problem (1) admits at least one nontrivial solution in E. This completes the proof. □

5. The Proof of Theorem 4

In this section, we will show that (1) has many pairs of solutions by using Lemma 3. To prove the Theorem 4, we will need the following Lemma 5.

Lemma 5. *Assume that* (g_1), (g_2), (g_6) *and* (g_7) *hold. Then, for any finite dimensional subspace* $\widetilde{E} \subset E$, *there holds*

$$\varphi_\lambda(u) \to -\infty,\ \|u\| \to +\infty,\ u \in \widetilde{E}.$$

Proof. Arguing indirectly, assume that there exists a sequence $\{u_n\} \subset \widetilde{E}$ such that

$$\|u_n\| \to +\infty,\ n \to +\infty \text{ and } \varphi_\lambda(u_n) \geq M,\ \forall n \in N, \tag{17}$$

where $M \in \mathbb{R}$ is a fixed constant not depending on $n \in N$.

Let $v_n = \frac{u_n}{\|u_n\|}$. Then, it is obvious that $\|v_n\| = 1$. Since $\dim\widetilde{E} < +\infty$, there exists $v \in \widetilde{E} \setminus \{0\}$ such that up to a subsequence, $\|v_n - v\| \to 0$ and $v_n(x) \to v(x)$ a.e. $x \in \Omega$ as $n \to +\infty$.

If $v(x) \neq 0$, then $|u_n(x)| \to +\infty$ as $n \to +\infty$. By virtue of (g_5), we get $\lim\limits_{k\to+\infty}\frac{G(x,u_n(x))}{\|u_n\|^{p^+}} = \lim\limits_{k\to+\infty}\frac{G(x,u_n(x))}{|u_n(x)|^{p^+}}|v_n(x)|^{p^+} = +\infty$, for all $x \in \Omega_0 := \{x \in \Omega : v(x) \neq 0\}$. Moreover, by virtue of condition (g_6), we can find $t_0 > 0$, such that

$$\frac{G(x,t)}{|t|^{p^+}} > 1,\ \forall x \in \Omega \text{ and } |t| > t_0. \tag{18}$$

On the other hand, Hypothesis (g_1) implies that there exists a positive constant C_4 such that

$$|G(x,t)| \leq C_4,\ \forall (x,t) \in \Omega \times [-t_0, t_0]. \tag{19}$$

Then, by (18) and (19), we deduce that there is a constant $C_5 \in \mathbb{R}$, such that

$$G(x,t) \geq C_5,\ \forall (x,t) \in \Omega \times \mathbb{R}. \tag{20}$$

From this, we conclude that

$$\frac{G(x,u_n)-C_5}{\|u_n\|^{p^+}} \geq 0,\ \forall x\in\Omega,\ \forall n\in N,$$

which implies that

$$\frac{G(x,u_n)}{|u_n(x)|^{p^+}}|v_n(x)|^{p^+} - \frac{C_5}{\|u_n\|^{p^+}} \geq 0,\ \forall x\in\Omega,\ \forall n\in N. \tag{21}$$

Therefore, using (7), (17) and (22), we have

$$\begin{aligned}
0 &\leq \lim_{n\to\infty}\frac{\varphi_\lambda(u_n)}{||u_n||^{p^+}}\\
&\leq \lim_{n\to\infty}\Big[\frac{a\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \lambda\int_\Omega \frac{1}{p(x)}|u_n|^{p(x)}dx}{||u_n||^{p^+}} - \int_\Omega \frac{G(x,u_n)}{||u_n||^{p^+}}dx\Big]\\
&\leq \begin{cases} \dfrac{a-\lambda C_2}{p^-} - \lim\limits_{n\to\infty}\displaystyle\int_\Omega \frac{G(x,u_n)}{||u_n||^{p^+}}dx, & \text{if } \lambda\leq 0,\\ \dfrac{a}{p^-} - \lim\limits_{n\to\infty}\displaystyle\int_\Omega \frac{G(x,u_n)}{||u_n||^{p^+}}dx, & \text{if } \lambda > 0.\end{cases}\\
&\leq \begin{cases} \dfrac{a-\lambda C_2}{p^-} - \lim\limits_{n\to\infty}\displaystyle\int_\Omega \frac{G(x,u_n)-C_5}{||u_n||^{p^+}}dx, & \text{if } \lambda\leq 0,\\ \dfrac{a}{p^-} - \lim\limits_{n\to\infty}\displaystyle\int_\Omega \frac{G(x,u_n)-C_5}{||u_n||^{p^+}}dx, & \text{if } \lambda > 0.\end{cases}\\
&\leq \begin{cases} \dfrac{a-\lambda C_2}{p^-} - \liminf\limits_{n\to\infty}\displaystyle\int_{\Omega_0} \frac{G(x,u_n)-C_5}{||u_n||^{p^+}}dx, & \text{if } \lambda\leq 0,\\ \dfrac{a}{p^-} - \liminf\limits_{n\to\infty}\displaystyle\int_{\Omega_0} \frac{G(x,u_n)-C_5}{||u_n||^{p^+}}dx, & \text{if } \lambda > 0.\end{cases}\\
&\leq \begin{cases} \dfrac{a-\lambda C_2}{p^-} - \liminf\limits_{n\to\infty}\displaystyle\int_{\Omega_0} \frac{G(x,u_n)}{|u_n(x)|^{p^+}}|v_n(x)|^{p^+}dx, & \text{if } \lambda\leq 0,\\ \dfrac{a}{p^-} - \liminf\limits_{n\to\infty}\displaystyle\int_{\Omega_0} \frac{G(x,u_n)}{|u_n(x)|^{p^+}}|v_n(x)|^{p^+}dx, & \text{if } \lambda > 0.\end{cases}\\
&\to -\infty,
\end{aligned} \tag{22}$$

which is contradiction. The proof of Lemma 5 is complete. □

Now, we are ready to prove Theorem 4.

Proof of Theorem 2. Let $X = E$ and $I = \varphi_\lambda$. Obviously, $\varphi_\lambda(0) = 0$. Thanks to Lemma 4, φ_λ satisfies the (C)-condition for any $c < \frac{a^2}{2b}$. Similar to the proof of Lemma 1(b), we can deduce that φ_λ satisfies condition (2) of Lemma 3. Thus, it follows from Lemma 5 that all conditions of Lemma 3 are satisfied. Therefore, problem (1) possesses many nontrivial solutions. □

6. Conclusions

In this paper, we have discussed the $p(x)$-Kirchhoff problem without the Ambrosetti–Rabinowitz conditions. The Ambrosetti–Rabinowitz conditions provide a major tool to study superlinear problems, it is a natural and useful condition, not only to ensure that the Euler–Lagrange functional associated with problem (1) has a mountain pass geometry, but also to guarantee that Palais–Smale sequence of the Euler-Lagrange functional is bounded. However, this condition is too restrictive and eliminates many nonlinearities. The novelty of this study is the existence of nontrivial solutions of (1) under a weaker condition than the Ambrosetti–Rabinowitz conditions. To the best of our knowledge, there are few related results on elliptic equations involved with a new non-local term $a - b\int_\Omega |\nabla u|^{p(x)}dx$ under some weaker assumptions on f. To deal with the difficulty caused by the noncompactness due to the Kirchhoff function term, we must estimate precisely the value of c and give a threshold value (see Lemma 4) under which the Cerami condition at the level c for φ_λ is satisfied. So, the variational technique for problem (1) becomes more delicate. Furthermore, under an additional assumption of symmetry, the infinitely many solutions are shown, formulated in the paper as Theorem 4. One example is given to show the effectiveness of our results.

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