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# Generalized Tepper's Identity and Its Application

**Dmitry Kruchinin [1,*], Vladimir Kruchinin [2] and Yilmaz Simsek [3]**

1 Department of Complex Information Security of Computer Systems, Tomsk State University of Control Systems and Radioelectronics, 634050 Tomsk, Russia
2 Institute of Innovation, Tomsk State University of Control Systems and Radioelectronics, 634050 Tomsk, Russia; kru@2i.tusur.ru
3 Department of Mathematics, Akdeniz University, 07070 Antalya, Turkey; ysimsek@akdeniz.edu.tr
* Correspondence: kdv@fb.tusur.ru

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**Abstract:** The aim of this paper is to study the Tepper identity, which is very important in number theory and combinatorial analysis. Using generating functions and compositions of generating functions, we derive many identities and relations associated with the Bernoulli numbers and polynomials, the Euler numbers and polynomials, and the Stirling numbers. Moreover, we give applications related to the Tepper identity and these numbers and polynomials.

**Keywords:** Tepper identity; generating function; composition of generating functions; Bernoulli numbers and polynomials; Euler numbers and polynomials; Stirling numbers; Frobenius–Euler polynomials

**MSC:** 11B68; 05A15; 11B73

## 1. Introduction

The following interesting identity had been conjectured by Tepper [1] and had proved by Long [2] and Papp [3]:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)^n = n!, \tag{1}$$

where $n$ is a positive integer and $x$ is any real number. If we consider the special case where $x = 0$, then it will be a variant of the well-known Euler's formula (Equation (2.1) in [4])

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n = n!.$$

Gould [4] wrote a detailed expository paper about Euler's formula. Using the calculus of finite differences, he gave another generalization of Equation (1):

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (A+Bk)^m = \begin{cases} 0, & 0 \le m < n; \\ B^n n!, & m = n, \end{cases}$$

where $A$ and $B$ are any constants (Equation (5.12) in [4]).

There are many other variants of generalizations of the Tepper identity. For example, Egorychev (Equation (2.2) in [5]) showed another proof for the Tepper identity and obtained the following identity:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^m = \begin{cases} 0, & 0 \le m < n; \\ n!, & m = n. \end{cases} \tag{2}$$

For $m = n$ the identity (2) corresponds to the identity (1). A similar result was obtained by Bayat and Teimoori [6,7]: For any positive integer $n$, $m$ $(0 \leq m \leq n)$ and any real number $x$,we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^m = (-1)^m m! \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta function.

Bayat and Teimoori (Corollary 5 in [8]) also presented another generalization of the Tepper identity: For any positive integer $n$ and any real numbers $x$ and $\lambda$

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (x+k)^{n|\lambda} = n!, \tag{3}$$

where the notation $x^{n|\lambda}$ is defined as follows:

$$x^{n|\lambda} = \begin{cases} x(x+\lambda)\cdots(x+(n-1)\lambda), & n > 0; \\ 1, & n = 0. \end{cases}$$

For $\lambda = 0$ the identity (3) corresponds to the identity (1). A similar result was obtained by Zhao and Wang (Theorem 4.4 in [9]): For any positive integer $n$ and any real numbers $x$ and $y$, we have

$$\sum_{k=0} n(-1)^{n-k} \binom{n}{k} \varphi_n(kx+y) = n! x^n,$$

where

$$\varphi_n(x) = x(x+1)\cdots(x+n-1).$$

Polynomials play an extremely important role in mathematics, physics and engineering. There are a lot of problems that can be solved by applying polynomials and methods for their investigations. A great contribution to the development of the theory of polynomials is made by various identities that operate with special polynomials. Several recent results can be found in [10–14]. Moreover, there are several generalizations of the Tepper identity that are related to polynomials. For example, for any arbitrary polynomial $f(x)$ with degree $m$, we have [6,8,15,16]

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+k) = (-1)^m a_m m! \delta_{n,m},$$

in which $a_m$ is the leading coefficient of the polynomial $f(x)$. The following similar result is presented in (Equation (31) in [13]):

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k) = \Delta^n f(x),$$

where $\Delta$ denote the difference operator given by

$$\Delta f(x) = f(x+1) - f(x).$$

Yang and Micek (Equation (34) in [16]) proved that for any positive integers $n$ and $m$ $(m \leq n)$ and any functions $f(x)$ and $g(x)$ with $n$-th order derivatives, we have

$$\sum_{k=0} n(-1)^{n-k} \binom{n}{k} [f^k(x) g(x)]^{(m)} f^{n-k}(x) = n! (f'(x))^n g(x) \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta function.

As a corollary, they also obtained (Equation (39) in [16]) the following result: If $g_m(x)$ is the sequence of binomial-type polynomials, then we have

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}g_m(kx+y)=n!g_1^n(x)\delta_{n,m}$$

for any positive integers $n$ and $m$ $(m \leq n)$ and any real numbers $x$ and $y$.

In addition, there are generalizations of the Tepper identity that are related to the Stirling numbers. For instance, Gould highlight (Equation (1.16 in [17]) conjectured the following formula for arbitrary complex $A$:

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(A-j)^{n+k}=\sum_{j=0}^{k}\binom{A-n}{j}(n+j)!S(n+k,n+j),$$

where $S(n,k)$ are the Stirling numbers of the second kind.

There are many notations for the Stirling numbers of the second kinds. In this paper, we will use the notation $\left\{ {n \atop k} \right\}$ given by Knuth [18]. The values of the Stirling numbers count the number of ways to partition a set of $n$ elements into $k$ nonempty subsets [18–20].

A general formula for the Stirling numbers of the second kind is given as follows [19,21]:

$$\left\{ {n \atop k} \right\}=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

The Stirling numbers of the second kind are defined by the generating function [19,22]

$$\Phi_k(x)=\sum_{n\geq k}\left\{ {n \atop k} \right\}\frac{x^n}{n!}=\frac{1}{k!}(e^x-1)^k.$$

The Tepper identity is a great tool to check mathematical methods for their application. Sun applied the Tepper identity to obtain some interesting combinatorial identities and formulas (for example, formulas for enumerating Dyck paths according to the number of udus [23], identities for the Jacobsthal and Fibonacci polynomials [24]). Using an exponential generating function and the Bell polynomials, Yang [25] obtained several identities related to the Tepper identity.

The aim of this paper is to study the Tepper identity and show a way for applying the Tepper identity to get various identities based on the coefficients of the powers of generating functions [26].

This paper is organized as follows:

In Section 2, using generating functions for the Bernoulli polynomials, we give another proof of (1). We also prove a generalization of the Tepper identity for any variable $x$ and present it in a convenient form for obtaining various identities based on the composition. Then in Section 3 we apply the obtained generalization for getting new identities for the Bernoulli and Euler numbers and polynomials. Using generating functions and our identities, we also obtain many novel formulas related to the combinatorial sums including binomial coefficients, the Bernoulli polynomials, the Euler polynomials, the Stirling numbers and the Tepper identity. Finally, in Section 4 we get various identities based on the coefficients of the powers of generating functions.

## 2. Generalized Tepper'S Identity

The Bernoulli polynomials $B_n(x)$ are defined by the following generating function [27,28]:

$$A(t,x)=\frac{t}{e^t-1}e^{xt}=\sum_{n\geq 0}B_n(x)\frac{t^n}{n!}.$$

Let a generating function $V(t,x)$ be a reciprocal generating function for the Bernoulli polynomials:

$$V(t,x) = \frac{t}{A(t,x)} = (e^t - 1)e^{-xt}. \tag{4}$$

The generating function (4) can be represented as the following formal power series:

$$V(t,x) = \sum_{n\geq 0} y_n(x)\frac{t^n}{n!}.$$

Then, for the coefficients $y_n(x)$, the following explicit formula can be obtained:

$$y_n(x) = (1-x)^n + (-1)^{n+1}x^n.$$

Let us introduce the following polynomials $K(x;n,k)$ that are defined by the $k$-th power of the generating function (4):

$$V(t,x)^k = (e^t - 1)^k e^{-xtk} = \sum_{n\geq k} K(x;n,k)\frac{t^n}{n!}. \tag{5}$$

The generating function (5) can be represented as a function of the form

$$(e^{-xt+t} - e^{-xt})^k. \tag{6}$$

For the function (6) we can apply the binomial theorem

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j$$

and the Taylor series for the exponential function

$$e^t = \sum_{n\geq 0} \frac{t^n}{n!}.$$

Hence, we get the following explicit formula for the polynomials $K(x;n,k)$:

$$K(x;n,k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j-kx)^n. \tag{7}$$

We also can calculate polynomials $K(x;n,k)$ using the next theorem:

**Theorem 1.**

$$K(x;n,k) = k! \sum_{j=k}^{n} \left\{ {j \atop k} \right\} \binom{n}{j} (-kx)^{n-j}, \tag{8}$$

*where* $\left\{ {j \atop k} \right\}$ *denotes the Stirling numbers of the second kind.*

**Proof.** The generating function (5) can be represented as the product of two generating functions $(e^t - 1)^k$ and $e^{-xtk}$. For the product of generating functions $A(t) = \sum\limits_{n\geq 0} a(n)t^n = B(t)C(t)$, where $B(t) = \sum\limits_{n>0} b(n)t^n$, $C(t) = \sum\limits_{n\geq 0} c(n)t^n$, the coefficients are

$$a(n) = \sum_{j=0}^{n} b(j)c(n-j). \tag{9}$$

Therefore, we get the coefficients of the first generating function $(e^t-1)^k$, using the well-known formula (p. 206 in [19])

$$(e^t-1)^k = \sum_{n\geq k} \frac{k!}{n!} \left\{ {n \atop k} \right\} t^n,$$

and the coefficients of the second generating function $e^{-xtk}$, using the Taylor series for the exponential function. Next we apply (9) and obtain

$$(e^t-1)^k e^{-xtk} = \sum_{n\geq k} \sum_{j=0}^{n} \frac{k!}{j!} \left\{ {j \atop k} \right\} \frac{(-kx)^{n-j}}{(n-j)!} t^n. \tag{10}$$

Using (5) and simplifying (10), we get the explicit formula (7) for the polynomials $K(x;n,k)$. □

**Corollary 1.**

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (j-kx)^n = n!. \tag{11}$$

**Proof.** Substituting $k=n$ in (8), we get

$$K(x;n,n) = n!.$$

We also can apply (7) for calculating $K(x;n,n)$. Hence, we obtain the desired result (11). □

If we consider the special case $K(\frac{x}{k};n,n)$, then we rederive the Tepper identity (1):

$$K(\frac{x}{k};n,n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)^n = n!.$$

Next we give a generalization of the Tepper identity by the following theorem:

**Theorem 2.** *For any variable $x$ there hold the following:*

$$\frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (j+x)^{n+k} = \sum_{j=0}^{k} \binom{n+k}{j} \left\{ {n+k-j \atop n} \right\} x^j. \tag{12}$$

**Proof.** Integrating the Tepper identity (1) with respect to $x$, we get

$$\frac{1}{(n+1)!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (j+x)^{n+1} = x + C.$$

For finding the value of $C$, we set $x=0$ in the above equation, then we have

$$C = \frac{1}{(n+1)!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^{n+1}.$$

Since

$$\left\{ {n+k \atop n} \right\} = \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^{n+k},$$

we get

$$C = \frac{1}{n+1} \left\{ {n+1 \atop n} \right\} = \frac{n}{2}.$$

Therefore,

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(j+x)^{n+1}=(n+1)x+\frac{n(n+1)}{2}.$$

If we repeat the above process for $k=1$, then we obtain

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(j+x)^{n+2}=\frac{(n+2)(n+1)}{2}x^2+\frac{(n+2)(n+1)n}{2}x+\left\{\begin{matrix}n+2\\ n\end{matrix}\right\}.$$

Then we can generalize it with $k$-times integrating and get

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(j+x)^{n+k}=\sum_{j=0}^{k}\binom{n+k}{j}\left\{\begin{matrix}n+k-j\\ n\end{matrix}\right\}x^j.$$

Hence, the proof of the theorem is completed. □

We can represent the left part of (12) as polynomials of degree $k$

$$T(x;n,k)=\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(j+x)^{n+k}.$$

If we take the derivative of these polynomials with respect to $x$, then we have the following derivative formula:

$$\frac{d}{dx}T(x;n,k)=(n+k)T(x;n,k-1).$$

## 3. New Identities Related to the Bernoulli and Euler Polynomials

It is known that for a polynomial $P(x)$ one can define two symbolical operators that are based on properties of the Bernoulli and Euler polynomials [29,30]. The Euler polynomials are defined by the following generating function [31,32]:

$$\frac{2e^{xt}}{1+e^t}=\sum_{n\geq 0}E_n(x)\frac{t^n}{n!}.$$

Let

$$B_n(x+h)=(B(x)+h)^n,$$
$$E_n(x+h)=(E(x)+h)^n,$$

where $B(x)^n$ and $E(x)^n$ are replaced conventionally by $B_n(x)$ and $E_n(x)$, respectively, and

$$P(B(x)+1)-P(B(x))=P'(x),$$

$$P(E(x)+1)+P(E(x))=2P(x).$$

Since

$$P(x+h)=R(x),$$

where $P(x)$ and $R(x)$ are any polynomials in $x$, we get

$$P(B(x)+h)=R(B(x)) \tag{13}$$

and

$$P(E(x)+h)=R(E(x)).$$

Next applying those symbolical operators, we arrive at the following identities including the Bernoulli polynomials and the Euler polynomials:

**Theorem 3.** *For the Bernoulli and Euler polynomials there hold the following identities:*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{n+k}(j+x)=\sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}B_j(x) \tag{14}$$

*and*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}E_{n+k}(j+x)=\sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}E_j(x). \tag{15}$$

**Proof.** Applying (13) for the generalized Tepper identity (12) and the Bernoulli polynomials, we get

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(j+B(x))^{n+k}=\sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}B(x)^j.$$

Therefore, we obtain (14).
Similarly, for the Euler polynomials we have

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(j+E(x))^{n+k}=\sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}E(x)^j.$$

Therefore, we obtain (15). So the proof of the theorem is completed. □

For the Bernoulli polynomials, the Tepper identity (12) with $k=0$ takes the following form:

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_n(j+x)=1.$$

For the Euler polynomials, the Tepper identity (12) with $k=0$ takes the following form:

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}E_n(j+x)=1.$$

The formula (12) gives us an opportunity for obtaining an explicit formula for the Bernoulli polynomials and the Euler polynomials through the Stirling numbers of the second kind. Applying (12) tofor the following well-known results:

$$B_n(x)=\sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(j+x)^n$$

and

$$E_n(x)=\sum_{k=0}^{n}\frac{1}{2^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(j+x)^n,$$

we obtain the known formulas [33,34]

$$B_n(x)=\sum_{k=0}^{n}\frac{(-1)^k k!}{k+1}\sum_{j=0}^{n-k}\binom{n}{j}\left\{{n-j \atop k}\right\}x^j$$

and

$$E_n(x) = \sum_{k=0}^{n} \frac{(-1)^k k!}{2^k} \sum_{j=0}^{n-k} \binom{n}{j} \left\{ {n-j \atop k} \right\} x^j.$$

**Theorem 4.** *Let $n$ and $k$ be nonnegative integers. Then we have*

$$\sum_{j=0}^{k} \binom{n+k}{j} \left\{ {n+k-j \atop n} \right\} B_j = \frac{n+k}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{m=1}^{j-1} m^{n+k-1}.$$

**Proof.** It is known that [35]

$$B_n(j) = B_n + n \sum_{m=1}^{j-1} m^{n-1}.$$

Let us write the formula for the generalized Tepper identity for the Bernoulli polynomials (14) with $x = 0$

$$\frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{n+k}(j) = \sum_{j=0}^{k} \binom{n+k}{j} \left\{ {n+k-j \atop n} \right\} B_j.$$

Substituting the value of $B_{n+k}(j)$ in the above formula and using

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{n+k} = 0,$$

we arrive at desired result. □

The identity (14) can also be expanded for the generalized Bernoulli polynomials. The generalized Bernoulli polynomials are defined by the following generating function [36]:

$$e^{xt} \left( \frac{t}{e^t - 1} \right)^{\alpha} = \sum_{n \geq 0} B_n^{\alpha}(x) \frac{t^n}{n!}$$

for an arbitrary parameter $\alpha$.

**Theorem 5.** *Let $n$ and $k$ be nonnegative integers. Then we have*

$$\frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{n+k}^{\alpha}(j+x) = \sum_{j=0}^{k} \binom{n+k}{j} \left\{ {n+k-j \atop n} \right\} B_j^{\alpha}(x).$$

**Proof.** For the generalized Bernoulli polynomials there hold a well-known relation

$$B_n^{\alpha}(x+h) = \sum_{k=0}^{n} \binom{n}{k} B_k^{\alpha}(x) h^{n-k}.$$

It means that the mention above two symbolical operators can be applied for the generalized Bernoulli polynomials. Therefore, the desired identity is true. □

The identity (15) can also be expanded for the Frobenius–Euler polynomials of order $\alpha$. The Frobenius–Euler polynomials of order $\alpha$ are defined by the following generating function [37,38]:

$$\left( \frac{1-\lambda}{e^t - \lambda} \right)^{\alpha} e^{xt} = \sum_{n \geq 0} H_n^{\alpha,\lambda}(x) \frac{t^n}{n!},$$

where $\lambda$ is an arbitrary parameter and $\lambda \neq 1$, $\alpha$ is a positive integer. Observe that

$$E_n(x) = H_n^{1,-1}(x)$$

(cf. [39]).

**Theorem 6.** *For the Frobenius–Euler polynomials $H_n^{\alpha,\lambda}(x)$ of order $\alpha$ there hold the following:*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}H_{n+k}^{\alpha,\lambda}(j+x) = \sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}H_j^{\alpha,\lambda}(x).$$

**Proof.** For the Frobenius–Euler polynomials there hold a well-known relation [40]

$$H_n^{\alpha,\lambda}(x+h) = \sum_{k=0}^{n}\binom{n}{k}H_k^{\alpha,\lambda}(x)h^{n-k}.$$

It means that the mention above two symbolical operators can be applied for the Frobenius–Euler polynomials. Therefore, the desired identity is true. □

## 4. Taylor Series for the Generalized Tepper Identity

The obtained generalized Tepper identity allows us to get various identities based on the coefficients of the powers of generating functions [41–43]. For that we substitute a generating function $F(x) = \sum\limits_{m\geq 0} f(m)x^m$ in the identity and get

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(j+F(x))^{n+k} = \sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}F(x)^j.$$

After evaluating, we get

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{n+k}\binom{n+k}{i}j^{n+k-i}F(x)^i = \sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}F(x)^j.$$

Equating coefficients of the Taylor series, we have

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{n+k}\binom{n+k}{i}j^{n+k-i}F(m,i) = \sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}F(m,j),$$

where

$$F(x)^i = \sum_{m\geq 0} F(m,i)x^m.$$

**Example 1.** *Let*

$$F(x) = \frac{1}{1-x}$$

*and*

$$F(m,i) = \binom{m+i-1}{m}.$$

*Then we have*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{n+k}\binom{n+k}{i}j^{n+k-i}\binom{m+i-1}{m} = \sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}\binom{m+j-1}{m}. \quad (16)$$

*Substituting $m = k = n$ in (16), we get the following identity:*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{2n}\binom{2n}{i}j^{2n-i}\binom{n+i-1}{n} = \sum_{j=0}^{n}\binom{2n}{j}\left\{{2n-j \atop n}\right\}\binom{n+j-1}{n}.$$

*Substituting $m = n$, $k = 0$ in (16), we get the following identity:*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{n}\binom{n}{i}j^{n-i}\binom{n+i-1}{n} = 0.$$

**Example 2.** *Let*

$$F(x) = e^x$$

*and*

$$F(m,i) = \frac{j^m}{m!}.$$

*Then we have*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{n+k}\binom{n+k}{i}j^{n+k-i}\frac{i^m}{m!} = \sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}\frac{j^m}{m!}. \tag{17}$$

*Substituting $m = k = n$ in (17), we get the following identity:*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{2n}\binom{2n}{i}j^{2n-i}\frac{i^n}{n!} = \sum_{j=0}^{n}\binom{2n}{j}\left\{{2n-j \atop n}\right\}\frac{j^n}{n!}.$$

*Substituting $m = n$, $k = 0$ in (17), we get the following identity:*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{n}\binom{n}{i}j^{n-i}i^n = 0.$$

**Example 3.** *Let*

$$F(x) = e^x - 1$$

*and*

$$F(m,i) = \frac{i!}{m!}\left\{{m \atop i}\right\}.$$

*Then we have*

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{n+k}\binom{n+k}{i}j^{n+k-i}\frac{i!}{m!}\left\{{m \atop i}\right\} = \sum_{j=0}^{k}\binom{n+k}{j}\left\{{n+k-j \atop n}\right\}\frac{j!}{m!}\left\{{m \atop j}\right\}.$$

## 5. Conclusions

The main results of this paper are in six theorems, one corollary and three examples. In the paper we have studied polynomials $K(x;n,k)$ related to the Tepper identity (1). Using generating functions and the Bernoulli polynomials, we have given another proof of the Tepper identity. We also have introduced a new generalization of the Tepper identity (12) that is related to Gould's formulas and we have proved the presented generalization for any variable $x$.

As an application of the presented generalization of the Tepper identity, we have obtained new identities related to the combinatorial sums including binomial coefficients, the Bernoulli and Euler numbers and polynomials, the Stirling numbers. The obtained identities can be applied for studying properties of special numbers and polynomials. They also can be used for effectively calculating the sums that are related to special numbers and polynomials.

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