mathematics

Article

On the $(29,5)$-Arcs in $\mathrm{PG}(2,7)$ and Some Generalized Arcs in $\mathrm{PG}(2,q)$

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Received: 27 January 2020; Accepted: 25 February 2020; Published: 2 March 2020

Abstract: Using an exhaustive computer search, we prove that the number of inequivalent $(29,5)$-arcs in $\mathrm{PG}(2,7)$ is exactly 22. This generalizes a result of Barlotti (see Barlotti, A. Some Topics in Finite Geometrical Structures, 1965), who constructed the first such arc from a conic. Our classification result is based on the fact that arcs and linear codes are related, which enables us to apply an algorithm for classifying the associated linear codes instead. Related to this result, several infinite families of arcs and multiple blocking sets are constructed. Lastly, the relationship between these arcs and the Barlotti arc is explored using a construction that we call transitioning.

Keywords: projective plane; arc; blocking set; linear code; Griesmer code

MSC: 94B27; 94B05; 51E20; 05B25

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements, q a prime power. We denote by $\mathrm{PG}(2,q)$ the projective plane over $\mathbb{F}_q$. Two subsets K_1 and K_2 in $\mathrm{PG}(2,q)$ are *projectively equivalent* (denoted by $K_1 \sim K_2$) if there exists a projectivity τ such that $\tau(K_1) = K_2$. An *(n,r)-arc* K in $\mathrm{PG}(2,q)$ is an n-set in $\mathrm{PG}(2,q)$ such that each line contains at most r points of K and some lines contain exactly r points of K. An $(n,2)$-arc in $\mathrm{PG}(2,q)$ is simply called an n-arc. A fundamental problem of (n,r)-arcs in $\mathrm{PG}(2,q)$ is the following.

Problem. *Let r be an integer with $2 \le r \le q-1$.*

(1) *Find $m_r(2,q)$, the maximum value of n for which an (n,r)-arc exists in $\mathrm{PG}(2,q)$.*
(2) *Classify (n,r)-arcs in $\mathrm{PG}(2,q)$ for $n = m_r(2,q)$ up to projective equivalence.*

In the cases $3 \le q \le 16$, the values of $m_r(2,q)$ are known as given in Table 1, see [1,2]. It is also known that every $(q+1)$-arc is projectively equivalent to a conic $V(x_1^2 - x_0x_2)$ when q is odd, and that every $(q+2)$-arc is projectively equivalent to a conic plus a point (called the *nucleus*) when $q = 4$ or 8 [3]. Marcugini et al. [4,5] proved with the aid of a computer that $(15,3)$-arcs in $\mathrm{PG}(2,7)$ are unique, and Hill and Love [6] showed that there are three $(22,4)$-arcs in $\mathrm{PG}(2,7)$ up to projective equivalence, see Table 2 (see also [7,8] for $q = 8,9$). In unpublished work, the authors of [7] have classified the $(33,5)$-arcs in $\mathrm{PG}(2,8)$.

Table 1. The known values and bounds on $m_r(2,q)$, $3 \le q \le 16$.

$r \setminus q$	**3**	**4**	**5**	**7**	**8**	**9**	**11**	**13**	**16**
2	4	6	6	8	10	10	12	14	18
3		9	11	15	15	17	21	23	28
4			16	22	28	28	32	38–40	52
5				29	33	37	43–45	49–53	65
6				36	42	48	56	64–66	78–82
7					49	55	67	79	93–97
8						65	78	92	120
9							89–90	105	129–130
10							100–102	118–119	142–148
11								132–133	159–164
12								145–147	180–181
13									195–199
14									210–214
15									231

Table 2. The number of inequivalent $(m_r(2,q), r)$-arcs.

$r \setminus q$	**3**	**4**	**5**	**7**	**8**	**9**
2	1	1	1	1	1	1
3		3	2	1	19	4
4			6	3	1	?
5				**22**	6	?
6				**194**	5	1
7					?	?
8						?

For 5-arcs in $\mathrm{PG}(2,7)$, it has been known that the maximal size is $n = 29$ [9,10], but it was not known how many inequivalent arcs of maximal size exist. In [11], the second author presented 13 inequivalent $(29,5)$-arcs in $\mathrm{PG}(2,7)$. Subsequently, Professor M. Grassl, attending the same conference, found in total 22 inequivalent linear codes corresponding to $(29,5)$-arcs with the help of the computer algebra system Magma [12]. Those results have been presented in [13]. Finally, using the package Q-EXTENSION developed by the first author [14] (see Section 2), the following extended result has been confirmed.

Theorem 1. *There are exactly* 22 *inequivalent* $(29,5)$*-arcs in* $\mathrm{PG}(2,7)$ *as listed in Table 3.*

Table 3. The $(29, 5)$-arcs in $\mathrm{PG}(2, 7)$.

Arc	a_0	a_1	a_2	a_3	a_4	a_5	$\lvert\mathrm{Aut}\rvert$	Construction
K_1	0	5	3	9	6	24	6	
K_2	0	8	0	0	21	28	336	Barlotti [10]
K_3	1	3	4	8	8	33	2	
K_4	1	4	3	5	13	31	2	
K_5	2	1	5	7	10	32	2	
K_6	2	1	6	4	13	31	1	
K_7	2	2	2	10	9	32	2	
K_8	2	2	4	4	15	30	2	
K_9	2	2	4	4	15	30	2	
K_{10}	2	3	2	4	17	29	2	
K_{11}	3	0	3	9	11	31	1	Theorem 4 (3)
K_{12}	3	0	3	9	11	31	6	
K_{13}	3	0	4	6	14	30	2	Theorem 3
K_{14}	3	1	1	9	13	30	1	Theorem 4 (2)
K_{15}	3	1	2	6	16	29	1	
K_{16}	3	1	2	6	16	29	1	
K_{17}	3	1	2	6	16	29	2	
K_{18}	3	1	3	3	19	28	3	
K_{19}	3	2	0	6	18	28	2	
K_{20}	3	2	0	6	18	28	6	Theorem 5
K_{21}	4	0	0	8	17	28	8	Example 2.3 in [15]
K_{22}	4	0	1	5	20	27	2	Theorem 7

In Section 2, we explain the algorithms in the package Q-EXTENSION, which is used for classification of linear codes with many different parameters over different fields including codes with given restrictions (self-orthogonal, self-complementary, etc.). Many published results are based on calculations with Q-EXTENSION and most of them are verified with other software, programs and theoretical proofs (for example [16]).

An $[n, k]_q$ code $\mathcal{C}$ is a linear code over $\mathbb{F}_q$ of length n and dimension k. $\mathcal{C}$ is called an $[n, k, d]_q$ code if it has minimum weight d. A $k \times n$ matrix over $\mathbb{F}_q$ whose rows form a basis of $\mathcal{C}$ is called a generator matrix of $\mathcal{C}$. A code $\mathcal{C}$ is *projective* if any two columns of G are linearly independent. Consequently, G has no all-zero column if $\mathcal{C}$ is projective. Two q-ary codes are *equivalent* if one can be obtained from the other by a sequence of the following transformations: (1) a permutation of the coordinate positions of all codewords; (2) a multiplication of a coordinate of all codewords with a nonzero element of $\mathbb{F}_q$; (3) a field automorphism. The set of all automorphisms of $\mathcal{C}$ forms the automorphism group of $\mathcal{C}$, denoted by $\mathrm{Aut}(\mathcal{C})$. Two codes are *monomially equivalent* if one can be obtained from the other by a sequence of the transformations (1) and (2). The equivalent codes over a prime field are monomially equivalent. For $i = 1, 2$, let G_i be a generator matrix of a projective $[n, k, d]_q$ code $\mathcal{C}_i$ and let K_i be the n-set in $\mathrm{PG}(k-1, q)$ corresponding to the n columns of G_i. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are monomially equivalent if and only if $K_1 \sim K_2$.

For a projective $[n, 3, d]_q$ code $\mathcal{C}$ with a generator matrix G, the n columns of G can be considered as an $(n, n-d)$-arc in $\mathrm{PG}(2, q)$ and vice versa. For more details about the equivalence between sets in projective spaces over a finite field and linear codes, see Chapter 16 in [17].

An (f, m)-*blocking set* B in $\mathrm{PG}(2, q)$ is an f-set such that each line contains at least m points of B and some lines contain exactly m points of B. If $n + f = q^2 + q + 1$ and $r + m = q + 1$, then the complement K^c of an (n, r)-arc K in $\mathrm{PG}(2, q)$ is an (f, m)-blocking set. Thus, (n, r)-arcs and (f, m)-blocking sets are equivalent objects.

Lemma 1 ([17,18])**.** *Let $\mathcal{C}$ be a projective $[n, 3]_q$ code with a generator matrix G and let K be the n-set in $PG(2, q)$ given by the n columns of G. Then, $\mathcal{C}$ has minimum weight d if and only if K is an $(n, n - d)$-arc (equivalently, K^c is a $(q^2 + q + 1 - n, q + 1 - (n - d))$-blocking set) in* $\mathrm{PG}(2, q)$*.*

For an $[n, k, d]_q$ code, we have $n \geq d + \lceil \frac{d}{q} \rceil + \cdots + \lceil \frac{d}{q^{k-1}} \rceil$, which is called the *Griesmer bound*. A linear code attaining the Griesmer bound is called a *Griesmer code*. Since Griesmer $[n, k, d]_q$ codes with $d \leq q^{k-1}$ are projective [18], the $(29, 5)$-arcs in $\mathrm{PG}(2, 7)$ and the $[29, 3, 24]_7$ codes are equivalent objects. For a set K in $\mathrm{PG}(2, q)$, a line is called an *i-line* if it meets K in exactly i points. We denote $a_i(K)$ (or simply a_i when no confusion arises) the number of i-lines of K. The list $\{a_i\}$ is called the *spectrum* of K. Let $\mathrm{Aut}(K)$ be the automorphism group of K, that is, the set of projectivities τ in $PGL(3, q)$ with $\tau(K) = K$. The spectra, together with the order of the automorphism group for the 22 projectively inequivalent $(29, 5)$-arcs in $\mathrm{PG}(2, 7)$ are given in Table 3.

In Section 3, we construct the arcs $K_1, \ldots, K_{22}$ in Table 3 without computer. We show how to construct these arcs from the well-known arc found by Barlotti [10] by exchanging some points, an operation called *transition*. From the geometrical point of view, we show how to distinguish K_8 and K_9 (also K_{15} and K_{16}), which can not be distinguished from their spectra and automorphism group orders.

In Section 4, we generalize some of the $(29, 5)$-arcs given in Section 3 to $(q^2 - 3q + 1, q - 2)$-arcs (equivalently, $(4q, 3)$-blocking sets) in $\mathrm{PG}(2, q)$.

Remark 1. *We have also confirmed that there are exactly* 194 *inequivalent* $(36, 6)$*-arcs in* $\mathrm{PG}(2, 7)$ *by exhaustive search using the package* Q-EXTENSION *as in Table 2. For the* 194 *inequivalent* $(36, 6)$*-arcs in* $\mathrm{PG}(2, 7)$*, see* *http://mars39.lomo.jp/opu/36_3_30.txt.*

Remark 2. *A similar interesting problem in the real projective plane is so called the real configuration problem. A configuration of lines and points is called an (n_k) configuration if it consists of n lines and n points, each of which is incident to exactly k of the other type. It is called geometric if these are points and lines in the real projective plane. Especially, the problem concerning the existence of geometric (n_4) configurations remains open only for the case $n = 23$, see [19].*

Remark 3. *The $(29, 5)$-arc K_2 in Table 3 is given as $K_2 = C \cup \mathcal{I}(C)$, in Section 3, where C is a conic and $\mathcal{I}(C)$ is the interior of C. In [20], they gave a realization of the configuration (21_4) and Coxeter's coordinates of them in the plane $PG(2, 7)$, which is equals to $\mathcal{I}(C)$ where C is the conic defined by the equation $x^2 + y^2 + z^2 = 0$.*

2. Algorithms in the Package Q-EXTENSION

We have proved that there are exactly 22 inequivalent $(29, 5)$-arcs (and also 194 inequivalent $(36, 6)$-arcs) in $\mathrm{PG}(2, 7)$ by exhaustive computer search using the package Q-EXTENSION [14]. It is available on the web page http://www.moi.math.bas.bg/~iliya/Q_ext.htm of the first author for fields with $q \leq 5$ elements (for larger fields write to Iliya Bouyukliev). In this section we briefly describe the algorithms in the package. We present the explanation in terms of linear codes because Q-EXTENSION is a software for construction and investigations of linear codes. We discuss the background and give the main ideas.

Each linear code is completely determined by its generator matrix. The main problem we solve is how to construct generator matrices of all inequivalent linear codes with length n, dimension k, and minimum distance d over the field $\mathbb{F}_p$, where p is a prime. If we know a part of the generator matrix, the problem will be much easier. This previously given part (submatrix) can be a generator matrix of a residual code or the identity matrix of size k, since any code has a generator matrix in systematic form.

Let G be a generator matrix of an $[n,k,d]_p$ code $\mathcal{C}$. Then the *residual code* $Res(\mathcal{C},c)$ of $\mathcal{C}$ with respect to a codeword c is the code generated by the restriction of G to the columns where c has zero entries.

Lemma 2 ([21]). *Let $\mathcal{C}$ be an $[n,k,d]$ code over $\mathbb{F}_p$ and let $c \in \mathcal{C}$ be a codeword of weight $w < (p/(p-1))d$. Then $Res(\mathcal{C},c)$ is an $[n-w,k-1,d']$ code with $d' \geq d - w + \lceil w/p \rceil$.*

Let $\mathcal{C}$ be an $[n,k,d]_p$ code with generator matrix G and let G_0 be a $k \times (n-m)$ matrix with rows $g_1,\ldots,g_k$ such that $G = (G_0\ X)$. The main idea of our approach is to construct all inequivalent codes with given parameters on the current step and to use these codes (their generator matrices) in the next step of the extension.

Let Ω_1 be the set consisting of the codes generated by a matrix in the form

$$G_1 = \left(\begin{array}{c|c} g_1 & a_{11}\ldots a_{1m} \\ \hline g_2 & \\ \vdots & O \\ g_k & \end{array}\right),$$

where O is the zero matrix, and $a = (a_{11},\ldots,a_{1m})$ is such a vector that the first row of G has weight $\geq d$. We can assume that $a_{11} = \ldots = a_{1j} = 0$ and $a_{1,j+1} = \ldots, a_{1m} = 1$. The codes from Ω_1 form the root of our search tree. We define Ω_s to be the set of all codes which have a generator matrix of the form

$$G_s = \left(\begin{array}{c|ccc} g_1 & a_{11} & \ldots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ g_s & a_{s1} & \ldots & a_{sm} \\ \hline g_{s+1} & & & \\ \vdots & & O & \\ g_k & & & \end{array}\right),$$

such that the first s rows of G_s generate an $[n,s]_p$ code whose minimum weight is at least d.

We use the equivalence of codes in terms of group action on a proper set. We consider the action of the group M_n of all monomial matrices of size n on the set Ω of linear codes with length n over the field $\mathbb{F}_p$. This action induces an equivalence relation in Ω as two codes $\mathcal{C}_1, \mathcal{C}_2 \in \Omega$ are equivalent if and only if they belong to the same orbit. Hence the equivalence classes for the defined relation are the orbits with respect to this action. The set of matrices $\sigma \in M_n$ such that $\mathcal{C}\sigma = \mathcal{C}$ form the automorphism group $\mathrm{Aut}(\mathcal{C})$ of the linear code $\mathcal{C}$.

The nodes in our search tree are objects from the search space $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$. From a linear code $A \in \Omega_{s-1}$ corresponding to the node $\overline{A}$, we obtain linear codes from Ω_s which are children of the code A. We denote the set of inequivalent children by $Child(A)$. The elements of $Child(A)$ correspond to the nodes of the next level which are connected to $\overline{A}$ by edges. To find only the inequivalent children in Ω_s, $s < k$, we use a special type of equivalence which we call *equivalence up to extension*. This type of equivalence is defined considering the action of the subgroup of M_n consisting of the monomial matrices of block-diagonal form with two blocks of sizes $n-m$ and m, respectively. Obviously, such a matrix acts on the first $n-m$ coordinates and the last m coordinates of the given linear code $\mathcal{C}$ separately.

It is easy to see that two equivalent codes up to extension have equivalent children. Practically, the rule $A \to Child(A)$, which connects all children to a code, defines our search tree. The execution of the algorithm can be considered as traversing the search tree and visiting all nodes through the edges. This can be done by a depth first search.

We use the algorithm only in the case when the search tree is not so big. That is why our isomorph rejection approach is very natural, and it is known as isomorph rejection with recorded objects [22]. The basic idea of this technique is to keep a global record R of the objects seen so far during traversal of a search tree. Whenever an object $\mathcal{C}$ is constructed, it is checked for equivalence against the recorded objects in R. If $\mathcal{C}$ is equivalent to an object in R, then the subtree rooted at $\mathcal{C}$ is pruned. This approach is fast enough for us because of concepts of canonical form. The use of canonical form reduces the problem of equivalence test of codes to comparison of codes (for more details see [22]).

We obtain the automorphism group and a canonical form of a given code $\mathcal{C}$ using a modification of the algorithm presented in [23]. This algorithm gives the order of the group, a set of generating elements, and a canonical permutation. One of the advantages of the algorithm is the effective construction of child codes. We give an example.

Example 1. *Let us try to construct all* $[13,3,9]_3$ *codes taking their generator matrices in a systematic form:*

$$G = \left(\begin{array}{c|c} 100 & \\ 010 & X \\ 001 & \end{array} \right).$$

Any row in the unknown matrix X must have at least eight nonzero coordinates. For a row a in X, consider the triple (x_0, x_1, x_2)*, where* x_i *is the number of coordinates in a equal to i,* $i = 0,1,2$*. We are looking for all triples with* $x_0 + x_1 + x_2 = 10$ *and* $x_1 + x_2 \geq 8$*. The number of nonnegative triples which satisfy these constrains is* 30*. They form the set* S_1 *of possible solutions.*

Without loss of generality we can take the first row of X to be (0011111111), (0111111111) *or* (1111111111)*. This means that the root* Ω_1 *consists of three codes with generator matrices*

$$\left(\begin{array}{c|c} 100 & 0011111111 \\ 010 & 0000000000 \\ 001 & 0000000000 \end{array} \right), \quad \left(\begin{array}{c|c} 100 & 0111111111 \\ 010 & 0000000000 \\ 001 & 0000000000 \end{array} \right), \quad \left(\begin{array}{c|c} 100 & 1111111111 \\ 010 & 0000000000 \\ 001 & 0000000000 \end{array} \right).$$

Consider in detail the first node. We can divide the matrix X into two parts, $X = (X_1\ X_2)$ *where* X_1 *and* X_2 *have 2 and 8 columns respectively. Using the possible solutions from* S_1*, the algorithm finds by exhaustive search all possible solutions for the next rows as tuples of triples* $\{(x_{00}, x_{01}, x_{02}), (x_{10}, x_{11}, x_{12})\}$ *such that* $x_{0i} = |\{b_j = i, 1 \leq j \leq 2\}|$, $x_{1i} = |\{b_j = i, 3 \leq j \leq 10\}|$*, where* $b = (b_1, b_2, \ldots, b_{10})$ *is the second or third row of X. The constrains for* x_{ij} *are*

$$x_{00} + x_{01} + x_{02} = 2,\ x_{10} + x_{11} + x_{12} = 8,\ x_{0i} + x_{1i} = x_i,\ i = 0,1,2.$$

We denote by S_2 *the set of all possible solutions in this step. Furthermore, the code generated by the vectors* (1000111111111) *and* $(010|b)$ *must have minimum distance at least 9 which reduces the possibilities and the algorithm obtains*

$$S_2 = \{\{(0,2,0),(2,3,3)\},\{(0,1,1),(2,3,3)\},\{(0,0,2),(2,3,3)\}\}.$$

This gives us the information that the first two coordinates of b are not zeros, but two of the other 8 *coordinates are zeros, three of them are* 1 *and three are equal to* 2*. It turns out that, up to a permutation, the second row of X must be* (1100111222), (1200111222) *or* (2200111222)*. The codes which correspond to these solutions have generator matrices*

$$\left(\begin{array}{c|c} 100 & 0011111111 \\ 010 & 1100111222 \\ 001 & 0000000000 \end{array} \right), \quad \left(\begin{array}{c|c} 100 & 0011111111 \\ 010 & 1200111222 \\ 001 & 0000000000 \end{array} \right), \quad \left(\begin{array}{c|c} 100 & 0011111111 \\ 010 & 2200111222 \\ 001 & 0000000000 \end{array} \right).$$

Obviously, these three codes are equivalent, therefore there is only one node in the second level of the tree connected with the considered root node.

Now we divide the matrix X into four parts (cells), $X = (X_1\ X_{20}\ X_{21}\ X_{22})$ *where* X_{20}, X_{21} *and* X_{22} *have 2, 3 and 3 columns respectively. One possible solution for the matrix G is*

$$\left(\begin{array}{c|c} 100 & 0011111111 \\ 010 & 1100111222 \\ 001 & 1212012012 \end{array}\right).$$

This solution turns out to be unique up to equivalence. The solution for the third row comes from $\{(0,1,1),(2,3,3)\} \in S_2$. *The algorithm also provides that the code* $\mathcal{C}$ *must be projective. It is easy to prove that the matrix G in systematic form cannot have rows with weights 9 and 10 (the algorithm proves this by exhaustive search).*

The same code can be constructed using as a prescribed part its residual code with respect to a codeword of weight 9. *The residual code has parameters* $[4,2,3]_3$. *There is a unique ternary code with these parameters, so we are looking for a generator matrix of* $\mathcal{C}$ *in the form*

$$\left(\begin{array}{c|c} 0000 & 111111111 \\ 1011 & \dots \\ 0112 & \dots \end{array}\right).$$

The algorithm in this case works in a similar way and it is even more effective, but it is a little bit more complicated to explain.

Consider now the construction of the $[29,3,24]_7$ codes. If $\mathcal{C}$ is a code with these parameters, then its residual code with respect to a codeword of weight 24 is a $[5,2,4]_7$ code. It is easy to prove (even by hand) that the code with a generator matrix $\begin{pmatrix} 10111 \\ 01123 \end{pmatrix}$ is, up to equivalence, the only $[5,2,4]_7$ code. Therefore we are looking for a generator matrix of $\mathcal{C}$ of the form

$$G = \left(\begin{array}{c|c} 00000 & 111111111111111111111111 \\ 10111 & \dots \\ 01123 & \dots \end{array}\right).$$

The program finds all inequivalent solutions in about 23 hours on a computer with Intel Xeon E5-2640 processor. The constructed tree consists of 8 nodes on the second level and 22 nodes on the third level. In this case the tree is not big, but finding solutions for the last level is computationally expensive. Since this search is exhaustive, we can conclude that there are exactly 22 $[29,3,24]_7$ codes up to equivalence, which proves Theorem 1.

Remark 4. *Many problems for classification of combinatorial objects are solved using computer programs. But there are only a few more general software packages for such classification, for example "Split" by David Jaffe [24] and "Orbiter" by Anton Betten [25].* Q-Extension *is convenient for our purposes, we have experience with it and therefore we use exactly this package in our classification.*

3. Construction of $(29,5)$-Arcs in $\mathrm{PG}(2,7)$

We start with the conic

$$C = \{\mathbf{P}(1,a,a^2) \mid a \in \mathbb{F}_q\} \cup \{\mathbf{P}(0,0,1)\}.$$

A line ℓ in $\mathrm{PG}(2,q)$ is called *external*, *tangent* or *secant* to C if $|C \cap \ell| = 0$, 1 or 2, respectively. For odd q, a point $P \notin C$ in $\mathrm{PG}(2,q)$ is called *internal* or *external* if the number of tangent lines on

P is 0 or 2, respectively. Let $\mathcal{I}(C)$ (resp. $\mathcal{E}(C)$) be the set of all internal (resp. external) points of C. Then, $|\mathcal{I}(C)| = q(q-1)/2$, $|\mathcal{E}(C)| = q(q+1)/2$, see ([3] Chapter 8). The following construction of a $(\frac{q^2+q+2}{2}, \frac{q+3}{2})$-arc in $\mathrm{PG}(2,q)$ is due to [10].

Theorem 2. *For q odd, let $K = \mathcal{I}(C) \cup C$. Then*

(1) *K forms a $(\frac{q^2+q+2}{2}, \frac{q+3}{2})$-arc in $\mathrm{PG}(2,q)$ with spectrum*

$$(a_1, a_{(q+1)/2}, a_{(q+3)/2}) = (q+1, q(q-1)/2, q(q+1)/2).$$

(2) $\mathrm{Aut}(K) \cong PGL(2,q)$ *and* $|\mathrm{Aut}(K)| = q(q^2-1)$.

Proof. The tangents, the secants and the external lines of C are 1-lines, $(q+3)/2$-lines and $(q+1)/2$-lines for the arc K, respectively. Recall that $\mathrm{Aut}(C) \cong PGL(2,q)$ [3]. Since any automorphism σ of K satisfies $\sigma(\mathcal{E}(C)) = \mathcal{E}(C)$, σ maps any tangent line of C to a tangent line. Then, $\sigma(C) = C$ and $\sigma(\mathcal{I}(C)) = \mathcal{I}(C)$. Hence, we get the assertion. □

Let K_2 be the above arc K for $q = 7$. We denote the line $\{\mathbf{P}(x,y,z) \in \mathrm{PG}(2,q) \mid ax + by + cz = 0\}$ by $[a,b,c]$ or $[abc]$. There is another simple construction of a $(29,5)$-arc in $\mathrm{PG}(2,7)$.

Lemma 3 (Example 2.3 in [15])**.** *Let B_0 be the set of points on the lines* $[100]$, $[010]$, $[001]$, $[111]$ *together with the points* $\mathbf{P}(-1,1,1)$, $\mathbf{P}(1,-1,1)$*. Then, the complement of B_0 forms a $(q^2-3q+2, q-2)$-arc if q is even and a $(q^2-3q+1, q-2)$-arc if q is odd.*

The arc K_{21} given below is such an arc obtained by the above lemma for $q = 7$. We give another construction for K_{21} later. See Section 4 for the spectra of the arcs in Lemma 3.

In the following, we show how to construct the arcs in Table 3 from $K_2 = C \cup \mathcal{I}(C)$. For two $(29,5)$-arcs K_i and K_j, we define the *distance* between them as

$$d(K_i, K_j) = \min_{K' \sim K_j} (29 - |K_i \cap K'|).$$

We also define the *transition number* of K_i as

$$t(K_i) = \min_{K_j \not\sim K_i} d(K_i, K_j).$$

Then, one can obtain some arc $K_j (\not\sim K_i)$ from K_i by exchanging $t(K_i)$ or $t(K_j)$ points, denoted by $K_i \to K_j$, that is, $K_j = (K_i \setminus D) \cup A$ for some disjoint t-sets $D \subset K_i$ and $A \subset \mathrm{PG}(2,7) \setminus K_i$ with $t = t(K_i)$ or $t(K_j)$, see Table 4. In what follows, we discuss how to find the set D to be deleted from the arc K_i and the set A to be added to get K_j in Table 4. Here by xyz we denote the point $\mathbf{P}(x,y,z)$ in $\mathrm{PG}(2,7)$. For two points P and Q, $\langle P, Q \rangle$ denotes the line through P and Q. Table 4 follows from the following lemmas.

Table 4. Transition $K_i \to K_j = (K_i \setminus D) \cup A$, together with the values $t_i = t(K_i)$ and $t_j = t(K_j)$.

K_i	t_i	D	A	K_j	t_j	K_i	t_i	D	A	K_j
K_2	3	001, 100, 111	131, 145, 153	K_{20}	2	K_{22}	1	146	120	K_{14}
K_2	3	136, 146, 151	131, 145, 153	K_1	3	K_{22}	1	133	120	K_{15}
K_{20}	2	115, 163, 165	015, 103, 106	K_3	3	K_{22}	1	165	120	K_{16}
K_{20}	2	101, 131	015, 130	K_4	2	K_{22}	1	101	120	K_{17}
K_{20}	2	124, 131	015, 114	K_{10}	2	K_{22}	1	152	120	K_{19}
K_{20}	2	131, 132	015, 122	K_{18}	2	K_{22}	1	131	111	K_{21}
K_{20}	2	132, 154	152, 164	K_{22}	1	K_{17}	1	134	014	K_7
K_{10}	2	102, 144	016, 141	K_6	2	K_{17}	1	113	014	K_8
K_{22}	1	113, 155	103, 105	K_9	2	K_{14}	1	126	130	K_{11}
K_9	2	145, 153	123, 135	K_5	2	K_{14}	1	104	130	K_{13}
K_{21}	1	102, 146	120, 140	K_{12}	2					

Lemma 4. $K_{20} = (K_2 \setminus \{001, 100, 111\}) \cup \{131, 145, 153\}$ *and* $t(K_2) = 3$.

Proof. Since every external point Q of the conic C is on the three 5-lines (the secants through Q), we have $t(K_2) \geq 3$. We construct K_{20} from K_2 by three point exchanges, which implies $t(K_2) = 3$. Note that the tangents of C are the 1-lines for K_2. Take three points P_1, P_2, P_3 on the conic C. Since $\mathrm{Aut}(C)$ is 3-transitive, we may assume that $P_1 = 161$, $P_2 = 142$, $P_3 = 124$. Let ℓ_i be the tangent of C at P_i for $i = 1, 2, 3$, i.e., $\ell_1 = [121]$, $\ell_2 = [134]$, $\ell_3 = [162]$, and let $Q_k = \ell_i \cap \ell_j$ for $\{i, j, k\} = \{1, 2, 3\}$. For $i = 1, 2, 3$, the line $\ell'_i = \langle P_i, Q_i \rangle$ is a secant meeting C in P_i and P_{i+3} say. Then, $Q_1 = 131$, $Q_2 = 145$, $Q_3 = 153$, $\ell'_1 = [106]$, $\ell'_2 = [150]$, $\ell'_3 = [013]$, $P_4 = 111$, $P_5 = 001$, $P_6 = 100$. Taking $D = \{P_4, P_5, P_6\}$ and $A = \{Q_1, Q_2, Q_3\}$, one obtains the transition $K_2 \to K_{20}$. Actually, the tangents at P_4, P_5, P_6 are the 0-lines and the tangents at $P_7 = 132$ and $P_8 = 154$ are the 1-lines for K_{20}, where $C = \{P_1, P_2, \ldots, P_8\}$. □

We confirmed that $|\mathrm{Aut}(K_{20})| = 6$ (and similar for the other values of $|\mathrm{Aut}|$ in Table 3 by computer. In what follows, let $C = \{P_1, P_2, \ldots, P_8\}$, ℓ'_1, ℓ'_2, ℓ'_3, Q_1, Q_2, Q_3, be as in the proof of Lemma 4 and let ℓ_i be the tangent at P_i to C for $1 \leq i \leq 8$ and $\ell_{ij} = \langle P_i, P_j \rangle$ for $1 \leq i < j \leq 8$. Then, $\ell_4 = [151]$, $\ell_5 = [100]$, $\ell_6 = [001]$, $\ell_7 = [144]$, $\ell_8 = [112]$, $\ell_{12} = [165]$, $\ell_{13} = [143]$, $\ell_{14} = l'_1 = [106]$, $\ell_{18} = [154]$, $\ell_{28} = [126]$, $\ell_{23} = [111]$, $\ell_{24} = [142]$, $\ell_{25} = l'_2 = [150]$, $\ell_{36} = l'_3 = [013]$, $\ell_{37} = [156]$, $\ell_{38} = [105]$, $\ell_{45} = [160]$, $\ell_{46} = [016]$, $\ell_{56} = [010]$, $\ell_{57} = [120]$, $\ell_{68} = [014]$, $\ell_{78} = [161]$. Note that $\ell_{14}, \ell_{57}, \ell_{68}$ are the secants through Q_1. Let $Q_{ij} = \ell_i \cap \ell_j$ apart from $Q_1 = Q_{23}$, $Q_2 = Q_{13}$, $Q_3 = Q_{12}$. For any point R and for a given arc K_i, we state that R is of type $i_1^{j_1} i_2^{j_2} \ldots$ if there exist j_1 i_1-lines and j_2 i_2-lines and so on through R for K_i.

Lemma 5. $K_1 = (K_2 \setminus \{136, 146, 151\}) \cup \{131, 145, 153\}$ *and* $t(K_1) = 3$.

Proof. Let $R_i = \ell'_i \cap \ell_{jk}$ for $\{i, j, k\} = \{1, 2, 3\}$. Then, $R_1 = 151$, $R_2 = 146$, $R_3 = 136$. Taking $D = \{R_1, R_2, R_3\}$ and $A = \{Q_1, Q_2, Q_3\}$, one can get the transition $K_2 \to K_1$. The tangents at P_1, P_2, P_3 are 3-lines for K_1, while the other five tangents remain 1-lines. The three external lines $\langle R_i, R_j \rangle$ ($1 \leq i < j \leq 3$) to C form the 2-lines for K_1. Now, take $R \notin K_1$. Then the possible types of R are: $1^1 2^1 3^2 5^4$, $1^1 3^3 4^1 5^3$, $1^1 2^1 3^1 4^2 5^3$, $1^2 2^1 5^5$, $1^2 3^1 4^1 5^4$, $2^2 3^2 4^1 5^3$, $1^2 4^3 5^3$. So, $t(K_1) \geq 3$. Now, by the transition $K_2 \to K_1$, we get K_1 by exchanging three points. Hence, we determine $t(K_1) = 3$. □

For a point $R \notin K_{20}$, the possible types of R are: $1^1 3^2 4^3 5^2$, $0^1 3^1 4^4 5^2$, $0^1 3^2 4^2 5^3$, $0^2 4^1 5^5$, $0^1 1^1 4^2 5^4$, $0^1 3^2 4^2 5^3$, $1^2 4^3 5^3$. Since every point out of K_{20} is on at least two 5-lines, we get $t(K_{20}) \geq 2$. By exhaustive computer search, we get the following.

Lemma 6. $t(K_i) \geq 2$ *for* $i = 4, 5, 6, 9, 10, 12, 18, 20$.

Lemma 7. $K_4 = (K_{20} \setminus \{101, 131\}) \cup \{015, 130\}$ *and* $t(K_4) = t(K_{20}) = 2$.

Proof. Let $S_1 = \ell_{57} \cap \ell_1$, $S_2 = \ell_{68} \cap \ell_1$ and $R_4 = \ell_{56} \cap \ell'_1$. Then, $S_1 = 130$, $S_2 = 015$, $R_4 = 101$. Taking $D = \{Q_1, R_4\}$ and $A = \{S_1, S_2\}$, one can get the transition $K_{20} \to K_4$. Since the two tangents ℓ_5 and ℓ_6 contain the points S_2 and S_1, respectively, the two tangents at ℓ_5, ℓ_6 are 1-lines for K_4, while the tangent at ℓ_4 remains a 0-line. The two 1-lines for K_{20} remain 1-lines for K_4. Now, the transition $K_{20} \to K_4$ yields $t(K_4) = t(K_{20}) = 2$ by Lemma 6. □

Lemma 8. $K_{18} = (K_{20} \setminus \{131, 132\}) \cup \{015, 122\}$ *and* $t(K_{18}) = 2$.

Proof. Take $Q_{18} = \ell_1 \cap \ell_8 = 122$. Setting $D = \{Q_1, P_7\}$ and $A = \{S_2, Q_{18}\}$, we get the transition $K_{20} \to K_{18}$. As for the 0-lines ℓ_4, ℓ_5, ℓ_6 for K_{20}, two lines ℓ_4 and ℓ_6 are also 0-lines for K_{18}, but ℓ_5 is a 1-line for K_{18}. Two 1-lines ℓ_7 and ℓ_8 for K_{20} are a 0-line and a 2-line for K_{18}, respectively. The other 2-lines for K_{18} are ℓ_2 and ℓ_3. The transition $K_{20} \to K_{18}$ yields $t(K_{18}) = 2$ by Lemma 6. □

Lemma 9. $K_{10} = (K_{20} \setminus \{124, 131\}) \cup \{015, 114\}$ *and* $t(K_{10}) = 2$.

Proof. Let $T_1 = \ell_{37} \cap \ell_1$, $T_2 = \ell_{38} \cap \ell_1$. Then, $T_1 = S_2 = 015$, $T_2 = 114$. Taking $D = \{Q_1, P_3\}$ and $A = \{T_1, T_2\}$, one can get the transition $K_{20} \to K_{10}$. Since the tangent ℓ_5 contains T_1, it is a 1-line for K_{10}, while the tangents ℓ_4 and ℓ_6 remain 0-lines. The other 1-lines for K_{10} are the tangents ℓ_3 and ℓ_8. The transition $K_{20} \to K_{10}$ yields $t(K_{10}) = 2$ by Lemma 6. □

Lemma 10. $K_{22} = (K_{20} \setminus \{132, 154\}) \cup \{052, 164\}$.

Proof. Take $Q_{24} = \ell_2 \cap \ell_4 = 164$ and $Q_{34} = \ell_3 \cap \ell_4 = 152$. Setting $D = \{P_7, P_8\}$ and $A = \{Q_{24}, Q_{34}\}$, we get the transition $K_{20} \to K_{22}$. The tangent ℓ_4 is a 0-line for K_{20}, but a 2-line for K_{22}, while the tangents ℓ_5 and ℓ_6 remain 0-lines. The tangents ℓ_7 and ℓ_8 are 1-lines for K_{20}, but 0-lines for K_{22}. □

Lemma 11. $K_6 = (K_{10} \setminus \{102, 144\}) \cup \{016, 141\}$ *and* $t(K_6) = 2$.

Proof. Let $V_1 = \langle Q_2, P_4 \rangle \cap \ell_{23}$, $V_2 = \langle Q_2, P_4 \rangle \cap \ell_{13}$, $V_3 = \langle Q_2, P_5 \rangle \cap \ell_{46}$, $V_4 = \langle Q_2, P_5 \rangle \cap \ell_{14}$. Then, $V_1 = 016$, $V_2 = 102$, $V_3 = 144$, $V_4 = 141$. Taking $D = \{V_2, V_3\}$ and $A = \{V_1, V_4\}$, one can get the transition $K_{10} \to K_6$. The 0-lines for K_{10} are also 0-lines for K_6. The tangent ℓ_3 remains a 1-line, but the other 1-lines for K_{10} are 2-lines for K_6. The transition $K_{10} \to K_6$ yields $t(K_6) = 2$ by Lemma 6. □

Lemma 12. $K_9 = (K_{22} \setminus \{113, 155\}) \cup \{103, 105\}$ *and* $t(K_9) = 2$.

Proof. Take $W_1 = \ell_{56} \cap \ell_2 = 103$, $W_2 = \ell_{56} \cap \ell_3 = 105$, $W_3 = \ell_{13} \cap \ell_{45} = 113$, $W_4 = \ell_{12} \cap \ell_{46} = 155$. Setting $D = \{W_3, W_4\}$ and $A = \{W_1, W_2\}$, we get the transition $K_{22} \to K_9$. The tangents ℓ_5, ℓ_6 are 0-lines for K_9 and the tangents ℓ_7, ℓ_8 are 1-lines for K_9. We note that three of the four 2-lines ℓ_4, ℓ_{45}, ℓ_{46} and [131] are concurrent at the point P_4. The transition $K_{22} \to K_9$ yields $t(K_9) = 2$ by Lemma 6. □

Lemma 13. $K_5 = (K_9 \setminus \{145, 153\}) \cup \{123, 135\}$ *and* $t(K_5) = 2$.

Proof. Take the two points $\ell_{13} \cap \ell_4 = 135$ and $\ell_{12} \cap \ell_4 = 123$ for the set A to be deleted from K_9. Setting $D = \{Q_2, Q_3\}$, we get the transition $K_9 \to K_5$. The 0-lines for K_9 are also 0-lines for K_5. The tangents ℓ_7 and ℓ_8 are 2-lines for K_5 and the unique 1-line for K_5 is ℓ_1. The transition $K_9 \to K_5$ yields $t(K_5) = 2$ by Lemma 6. □

Lemma 14. $K_{14} = (K_{22} \setminus \{146\}) \cup \{120\}$, $K_{15} = (K_{22} \setminus \{133\}) \cup \{120\}$, $K_{16} = (K_{22} \setminus \{165\}) \cup \{120\}$, $K_{17} = (K_{22} \setminus \{101\}) \cup \{120\}$, $K_{19} = (K_{22} \setminus \{152\}) \cup \{120\}$.

Proof. Take the point $Q_{34} = \ell_3 \cap \ell_4 = 152$ and let $L = \langle Q_{34}, R_4 \rangle = [136]$, $U = L \cap \ell_2 = 120$ and $K' = K \cup \{U\}$. Then, K' has spectrum $(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = (3, 1, 1, 5, 14, 32, 1)$ and L is the unique 6-line for K'. The 1-points of L other than U are $U_1 (= R_2) = 146$, $U_2 = 133$, $U_3 = 165$, $U_4 (= R_4) = 101$, $U_5 (= Q_{34}) = 152$. Then, $K_{13+j} = K' \setminus \{U_j\}$ for $1 \leq j \leq 4$ and $K_{19} = K' \setminus \{U_5\}$. As can be seen in Table 3, K_{15}, K_{16} and K_{17} have the same spectrum. Nevertheless, one can distinguish them as follows. The 3-lines for K_{15} are ℓ_1, ℓ_{18}, ℓ_{45}, ℓ_{56}, ℓ_{57}, ℓ_{78}. And there are two points on three 3-lines; the point 106 on ℓ_1, ℓ_{56}, ℓ_{78} and the point P_5 on ℓ_{45}, ℓ_{56}, ℓ_{57}. As for K_{16}, the 3-lines are ℓ_{24}, ℓ_{45}, ℓ_{46}, ℓ_{56}, [123], ℓ_1 and there is only one point on three 3-lines; the point P_4 on the three lines ℓ_{24}, ℓ_{45}, ℓ_{46}. Meanwhile, the 3-lines for K_{17}: ℓ_1, ℓ_{28}, ℓ_{37}, ℓ_{45}, ℓ_{46}, ℓ_{78} form a 6-arc of lines (no three of which are concurrent). Thus, K_{15}, K_{16} and K_{17} are projectively inequivalent. K_{19} and K_{20} can be also distinguished similarly as follows. Let $\mathcal{L}$ and $\mathcal{L}'$ be the sets of 3-lines for K_{19} and K_{20}, respectively. Then, $\mathcal{L}' = \{\ell_1, \ell_2, \ell_3, \ell_{45}, \ell_{46}, \ell_{56}\}$ is a 6-arc of lines, but $\mathcal{L} = \{\ell_1, \ell_3, \ell_{45}, \ell_{46}, \ell_{56}, \ell_{78}\}$ is not, for $106 = l_1 \cap \ell_{56} \cap \ell_{78}$ and $110 = l_3 \cap \ell_{45} \cap \ell_{78}$. Hence, K_{19} and K_{20} are projectively inequivalent. □

Lemma 15. $K_{21} = (K_{22} \setminus \{131\}) \cup \{111\}$.

Proof. Taking $D = \{Q_1\}$ and $A = \{P_4\}$, we get the transition $K_{22} \to K_{21}$. The tangents ℓ_5, ℓ_6, ℓ_7, ℓ_8, no three of which are concurrent, remain 0-lines. The two 0-points out of the 0-lines are $\ell_{23} \cap \ell_1 = 106$ and Q_1. It turns out that K_{21} is projectively equivalent to the arc constructed in Lemma 3. The tangent ℓ_4 is the 2-line for K_{22}, but a 3-line for K_{21}. □

Lemma 16. $K_7 = (K_{17} \setminus \{134\}) \cup \{014\}$, $K_8 = (K_{17} \setminus \{113\}) \cup \{014\}$.

Proof. Take the points $Q_{24} = \ell_2 \cap \ell_4 = 164$ and $X = \langle Q_{24}, R_4 \rangle \cap \ell_3 = 014$, where $R_4 = \ell_{56} \cap \ell_1'$, and let $K_{17}' = K_{17} \cup \{X\}$. Then, the arc K_{17}' has spectrum $(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = (2, 2, 2, 6, 10, 34, 1)$ and the unique 6-line for K' is $L_1 = \langle X, U \rangle = [131]$. The 1-points of L_1 other than X are $X_1 = 134$, $X_2 = 113$, $X_3 = 162$, $X_4 = 155$, $X_5 = 120$. Then, $K_7 = K_{17}' \setminus \{X_1\}$ and $K_8 = K_{17}' \setminus \{X_2\}$. $K_{17}' \setminus \{X_3\}$, $K_{17}' \setminus \{X_4\}$, $K_{17}' \setminus \{X_5\}$ are projectively equivalent to K_7, K_8, K_{17}, respectively. We can distinguish K_8 and K_9 as follows. The 2-lines for K_8 are ℓ_4, ℓ_{28}, ℓ_{45}, ℓ_{56}, which form a 4-arc of lines. On the other hand, the 2-lines for K_9 are ℓ_4, ℓ_{45}, ℓ_{46}, L_1, the first three of which are concurrent at the point P_4. □

Lemma 17. $K_{11} = (K_{14} \setminus \{126\}) \cup \{130\}$, $K_{13} = (K_{14} \setminus \{104\}) \cup \{130\}$.

Proof. Let $E = l_1' \cap \ell_2' \cap \ell_3' = 141$, $L_2 = \langle Q_{34}, E \rangle$ and $Y = L_2 \cap \ell_1 = 130$, where $Q_{34} = 152$. Then, $K_{14}' = K_{14} \cup \{Y\}$ has spectrum $(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = (3, 0, 2, 8, 9, 34, 1)$ and the unique 6-line for K_{14}' is $L_2 = [125]$. The 1-points of L_2 other than Y are $Y_1 = 126$, $Y_2 = 104$, $Y_3 = 163$, $Y_4 = 115$, $Y_5 = 130$. Then, $K_{11} = K_{14}' \setminus \{Y_1\}$ and $K_{13} = K_{14}' \setminus \{Y_2\}$. $K_{14}' \setminus \{Y_3\}$, $K_{14}' \setminus \{Y_4\}$, $K_{14}' \setminus \{Y_5\}$ are projectively equivalent to K_{11}, K_{13}, K_{14}, respectively. □

Lemma 18. $K_{12} = (K_{21} \setminus \{102, 146\}) \cup \{120, 140\}$ *and* $t(K_{12}) = 2$.

Proof. Take the two points $\ell_{13} \cap \ell_2' (= R_2) = 146$ and $\ell_{13} \cap \ell_{56} (= V_2) = 102$ for the 2-set D to be deleted and take $A = \{Q_{26} = 120, Q_{46} = 140\}$. Then, we get the transition $K_{21} \to K_{12}$. The 0-line ℓ_6 for K_{21} becomes a 2-line for K_{12}, while the tangents ℓ_5, ℓ_7, ℓ_8 remain 0-lines. The arcs K_{11} and K_{12} are projectively inequivalent since their automorphism group orders are different. In addition, we can distinguish K_{11} and K_{12} as follows. The 2-lines for K_{11} are [101], ℓ_4, ℓ_6, having no common point. On the other hand, the 2-lines for K_{12} are [101], ℓ_5, ℓ_6, which are concurrent at the point 010. The transition $K_{21} \to K_{12}$ yields $t(K_{12}) = 2$ by Lemma 6. □

Lemma 19. $K_3 = (K_{20} \setminus \{115, 163, 165\}) \cup \{015, 103, 106\}$ *and* $t(K_3) = 3$.

Proof. Take the points $Z_1 = \ell_{68} \cap \ell_1 (= S_2) = 015$, $Z_2 = \ell_{68} \cap \ell_{23} = 115$, $Z_3 = \ell_{23} \cap \ell_1 = 106$, $Z_4 = \ell_{24} \cap \ell_3 (= W_1) = 103$, $Z_5 = \ell_{24} \cap \ell_{78} = 165$, $Z_6 = \ell_{37} \cap \langle Z_3, Q_1 \rangle = 163$. Setting $D = \{Z_2, Z_5, Z_6\}$ and $A = \{Z_1, Z_3, Z_4\}$, we get the transition $K_{20} \to K_3$. The 0-lines ℓ_4, ℓ_5 for K_{20} are 1-lines for K_3, while the ℓ_6 remains a 0-line. The 1-lines ℓ_7 and ℓ_8 for K_{20} are a 1-line and a 2-line for K_3, respectively. Now, there are only two points out of K_3, namely 122 and 165, which are on at most two 5-lines. The types of 122 and 165 are $2^1 3^3 4^2 5^2$ and $2^1 3^3 4^2 5^2$, respectively. Hence $t(K_3) \geq 2$. Suppose $t(K_3) = 2$. Then, we have a transition $K_3 \to (K_3 \setminus \{Z, Z'\}) \cup \{122, 165\}$ for some points $Z, Z' \in K_3$. Since the five lines $\langle 122, 143 \rangle = [103]$, $\langle 122, 106 \rangle = [121]$, $\langle 165, 142 \rangle = [142]$, $\langle 165, 106 \rangle = [161]$, $\langle 122, 165 \rangle = [155]$ are 6-lines for $K_3 \cup \{122, 165\}$, the points Z and Z' must be on the five lines, which is impossible. Thus, $t(K_3) \geq 3$. Now, by the transition $K_{20} \to K_3$, we determine $t(K_3) = 3$. □

In summary, we have determined the transition numbers of each K_i as in Table 4. Considering the graph with vertex set $\{K_i \colon t(K_i) = 1\}$, where two vertices K_i, K_j are joined if $d(K_i, K_j) = 1$, we get Figure 1.

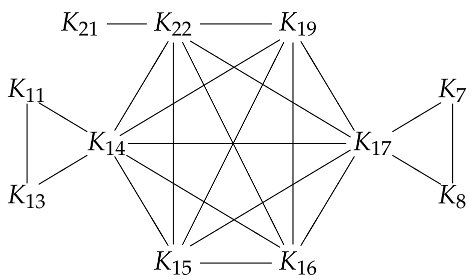

Figure 1. Graph of $(29, 5)$-arcs with distance 1.

4. Construction of Some Generalized Arcs in PG$(2, q)$

Recall from Lemma 1 that the complement of a $(b, 3)$-blocking set B is a $(q^2 + q + 1 - b, q - 2)$ -arc in PG$(2, q)$. When q is odd, for a $(b, 3)$-blocking set B is known that $b = |B| \geq 4q$ if B contains a line [26]. The set B_0 for odd q in Lemma 3 is such a $(4q, 3)$-blocking set.

In this section, we generalize some construction results in Section 3 by constructing some $(4q, 3)$-blocking sets in PG$(2, q)$ for odd q. Obviously, we have $a_i(K) = a_{q+1-i}(K^c)$, where K^c is the complement of K in PG$(2, q)$.

Theorem 3. *For odd $q \geq 5$, let C be a conic in* PG$(2, q)$*. For any three points P_1, P_2, P_3 in C, let ℓ_i be the tangent of C through P_i and ℓ_{ij} be the secant of C through P_i and P_j, and let $P_{ij} = \ell_i \cap \ell_j$ for $1 \leq i \leq j \leq 3$. Take any two points P and Q from the three points P_{12}, P_{23}, P_{13}, and let $B = C \cup \ell_{12} \cup \ell_{23} \cup \ell_{13} \cup \{P, Q\}$. Then $K = B^c$ is a $(q^2 - 3q + 1, q - 2)$-arc with spectrum*
$(a_0, a_1, a_2, a_3) = (5, 1, 10, 15)$ for $q = 5$ and
$(a_0, a_{q-5}, a_{q-4}, a_{q-3}, a_{q-2}) = (3, q - 3, \frac{(q-3)(q-4)}{2}, 2q, \frac{(q+5)(q-2)}{2})$ for $q \geq 7$.

Proof. Let $C = \{P_1, P_2, \ldots, P_{q+1}\}$ be a conic in PG$(2, q)$ and let ℓ be a line. If ℓ contains none of P_1, P_2, P_3, then ℓ meets $\ell_{12} \cup \ell_{23} \cup \ell_{13}$ at three points. Thus, $|\ell \cap B| \geq 3$. If ℓ contains exactly one of P_1, P_2, P_3, say P', ℓ meets $\ell_{12} \cup \ell_{23} \cup \ell_{13}$ at two points. Then, ℓ is a secant or a tangent of C. If ℓ is a secant of C, ℓ meets C at P' and another point. So, $|\ell \cap B| \geq 3$. If ℓ is a tangent of C, ℓ is ℓ_1, ℓ_2 or ℓ_3, and ℓ contains at least one of the points P and Q. So, $|\ell \cap B| \geq 3$. If ℓ contains two of P_1, P_2 and P_3, then ℓ is ℓ_{12}, ℓ_{23} or ℓ_{13}. Thus, B is a $(4q, 3)$-blocking set. Without loss of generality, we may take $P = P_{13}$ and $Q = P_{12}$. Assume $q \geq 7$. The $(q+1)$-lines for B are ℓ_{12}, ℓ_{23}, ℓ_{13}. So, $a_{q+1}(B) = 3$. The 6-lines are the secants through P or Q except $\langle P, P_2 \rangle$ and $\langle Q, P_3 \rangle$. Hence $a_6(B) = 2\left(\frac{q-1}{2} - 1\right) = q - 3$. For $q = 5$, the above $(q+1)$-lines are also 6-lines for B, and $a_6(B) = 5$. Now, assume $q \geq 5$. The 5-lines are the secants of C passing through none of P_1, P_2, P_3 except the 6-lines. So, $a_5(B) = \binom{q+1-3}{2} - a_6(B) =$

$(q-3)(q-4)/2$. The 4-lines are the external lines of C through P or Q, the secants $\langle P, P_2\rangle$, $\langle Q, P_3\rangle$, the tangents at $P_4, P_5, \ldots, P_{q+1}$ and $\langle P, Q\rangle$. Hence, $a_4(B) = q-1+2+(q+1-3)+1 = 2q$. Finally, $a_3(B) = q^2+q+1-a_4(B)-a_5(B)-a_6(B)-a_{q+1}(B) = (q+5)(q-2)/2$. $\square$

By some transitions from B in Theorem 3, we get the following.

Theorem 4. *Under the conditions of Theorem 3 with $q \geq 7$, take $P = P_{13}$, $Q = P_{12}$ and a point Q' in ℓ_2 with $Q' \notin \{Q, P_2, l_{13} \cap \ell_2\}$. Let $B' = (B \setminus \{Q\}) \cup \{Q'\}$ and $\ell = \langle P, Q'\rangle$. Then $K = (B')^c$ forms a $(q^2-3q+1, q-2)$-arc with spectrum*

(1) $(a_0, a_{q-5}, a_{q-4}, a_{q-3}, a_{q-2}) = (3, q-3, \frac{(q-3)(q-4)}{2}, 2q, \frac{(q+5)(q-2)}{2})$ *if ℓ is a tangent,*

(2) $(a_0, a_{q-6}, a_{q-5}, a_{q-4}, a_{q-3}, a_{q-2}) = (3, 1, q-6, \frac{q^2-7q+18}{2}, 2q-1, \frac{(q+5)(q-2)}{2})$ *if ℓ is a secant,*

(3) $(a_0, a_{q-5}, a_{q-4}, a_{q-3}, a_{q-2}) = (3, q-4, \frac{q^2-7q+18}{2}, 2q-3, \frac{q^2+3q-8}{2})$ *if ℓ is an external line.*

Proof. Since ℓ is a tangent of C if and only if $Q' = P_{23}$, we get the spectrum (1) from Theorem 3 if ℓ is a tangent. As we have already seen in the proof of Theorem 3, the tangent $\langle Q, P\rangle$ and the secant $\langle Q, P_3\rangle$ are 4-lines, the other $(q-3)/2$ secants through Q are 6-lines and the $(q-1)/2$ external lines through Q are 4-lines for B. Note that $a_{q+1}(B') = a_{q+1}(B)$, for $Q' \in \ell_2 \setminus \{P_2, \ell_{13} \cap \ell_2\}$.

If ℓ is a secant, then for B, the tangent ($\neq \ell_2$) through Q' is a 4-line, the secant ℓ is a 6-line, the secants $\langle Q', P_1\rangle$, $\langle Q', P_3\rangle$ are 3-lines, other $(q-7)/2$ secants on Q' are 5-lines and the $(q-1)/2$ external lines on Q' are 3-lines. Hence, $a_3(B') = a_3(B)+2+(q-1)/2-2-(q-1)/2 = a_3(B)$, $a_4(B') = a_4(B)-2-(q-1)/2-1+2+(q-1)/2 = a_4(B)-1$, $a_5(B') = a_5(B)+(q-3)/2+1-(q-7)/2 = a_5(B)+3$, $a_6(B') = a_6(B)-(q-3)/2-1+(q-7)/2 = a_6(B)-3$, $b'_7 = 1$.

If ℓ is an external line, then for B, the tangent ($\neq \ell_2$) through Q' is a 4-line, the secants $\langle Q', P_1\rangle$, $\langle Q', P_3\rangle$ are 3-lines, other $(q-5)/2$ secants on Q' are 5-lines, the external line ℓ is a 4-line and the $(q-3)/2$ external lines on Q' are 3-lines. Hence, $a_3(B') = a_3(B)+2+(q-1)/2-2-(q-3)/2 = a_3(B)+1$, $a_4(B') = a_4(B)-2-(q-1)/2-1+2-1+(q-3)/2 = a_4(B)-3$, $a_5(B') = a_5(B)+(q-3)/2+1-(q-5)/2+1 = a_5(B)+3$, $a_6(B') = a_6(B)-(q-3)/2+(q-5)/2 = a_6(B)-1$. $\square$

We note that the construction of a $(4q, 3)$-blocking set with spectrum (1) or (3) in Theorem 4 is also valid for $q = 5$, but not for the spectrum (2) since ℓ is a secant if and only if $Q' = \ell_{13} \cap \ell_2$ when $q = 5$.

For $q = 7$, the $(q^2-3q+1, q-2)$-arcs of Theorem 4 (1), (2) and (3) are equivalent to K_{13}, K_{14} and K_{11}, respectively. The next lemma is given in [3, Corollary 7.5].

Lemma 20 ([3]). *In $\mathrm{PG}(2, q)$ with $q \geq 4$, there is a unique conic through a 5-arc.*

We can get one more $(4q, 3)$-blocking set in $\mathrm{PG}(2, q)$ from the set B in Theorem 3 by exchanging two points.

Theorem 5. *Let $q = p^h \geq 7$ for an odd prime $p \neq 3$. Under the conditions of Theorem 3, let C be the conic $\{\mathbf{P}(1, a, a^2) \colon a \in \mathbb{F}_q\} \cup \{\mathbf{P}(0, 0, 1)\}$ and take $P_1 = \mathbf{P}(1, 1, 1)$, $P_2 = \mathbf{P}(0, 0, 1)$, $P_3 = \mathbf{P}(1, 0, 0)$, $P_4 = \mathbf{P}(1, 2^{-1}, 2^{-2})$, $P_5 = \mathbf{P}(1, 2, 2^2)$, $S = \langle P_1, P_4\rangle \cap \langle P_2, P_5\rangle$ and $T = \langle P_1, P_5\rangle \cap \langle P_3, P_4\rangle$. Let $B_1 = (B \setminus \{P_4, P_5\}) \cup \{S, T\}$. Then $K' = (B_1)^c$ is a $(q^2-3q+1, q-2)$-arc, which is not projectively equivalent to any arc in Theorems 3 and 4.*

Proof. Note that $P_4 \neq P_5$ if $p \neq 3$ and that $S = \mathbf{P}(1, 2, 2+2^{-1})$, $T = \mathbf{P}(2+2^{-1}, 2, 1)$. Since $P = \ell_1 \cap \ell_3 = \mathbf{P}(1, 2^{-1}, 0)$ and $Q = \ell_1 \cap \ell_2 = \mathbf{P}(0, 1, 2)$, the lines $\langle P, P_2\rangle$ and $\langle Q, P_3\rangle$ are passing through P_4 and P_5, respectively. Let $B_1^- = B \setminus \{P_4, P_5\}$. Then, the 2-lines for B_1^- are $\langle P_1, P_4\rangle$, $\langle P_1, P_5\rangle$, $\langle P_2, P_5\rangle$ and $\langle P_3, P_4\rangle$. Hence, adding $S = \langle P_1, P_4\rangle \cap \langle P_2, P_5\rangle$ and $T = \langle P_1, P_5\rangle \cap \langle P_3, P_4\rangle$ to B_1^-, $B_1 = B_1^- \cup \{S, T\}$ forms a $(4q, 3)$-blocking set. It can be checked using computer that B_1 has spectrum $(a_3(B_1), a_4(B_1), a_5(B_1), a_7(B_1), a_8(B_1)) = (28, 18, 6, 2, 3)$ for $q = 7$, $(a_3(B_1), a_4(B_1), a_5(B_1), a_6(B_1), a_7(B_1), a_{12}(B_1)) = (66, 38, 16, 8, 2, 3)$ for $q = 11$ and

$(a_3(B_1), a_4(B_1), a_5(B_1), a_6(B_1), a_{14}(B_1)) = (93, 44, 27, 16, 3)$ for $q = 13$. Hence, B_1 is not projectively equivalent to any blocking set in Theorems 3 and 4. Assume $q \geq 17$ and suppose B_1 contains a conic C'. Since $C \neq C'$, it follows from Lemma 20 that C' could contain at most 4 points from C, 6 points from $\ell_{12} \cup \ell_{13} \cup \ell_{23}$ and the other 4 points, in total at most 14 points from B_1, a contradiction. Thus, B_1 contains no conic for $q \geq 17$. On the other hand, the blocking sets in Theorem 3 and 4 contain a conic. Hence, the arc $(B_1)^c$ is not projectively equivalent to any of the arcs in the previous theorems. $\square$

For $q = 7$, the $(q^2 - 3q + 1, q - 2)$-arc of Theorem 5 is equivalent to K_{20}.

Remark 5. *(1) Assume $q = 5$ in Theorem 5. From Table 2 in Section 1, there exist two inequivalent $(11, 3)$-arcs (equivalently, $(20, 3)$-blocking sets) in $\mathrm{PG}(2, 5)$, see also ([3], Table 12.5). The $(11, 3)$-arcs have spectrum*

(a) $(a_0, a_1, a_2, a_3) = (5, 1, 10, 15)$ *or*
(b) $(a_0, a_1, a_2, a_3) = (4, 4, 7, 16)$.

There are four 6-lines ℓ_{12}, ℓ_{13}, ℓ_{23} and $\langle S, T\rangle$ for the arc $(B_1)^c$ in Theorem 5 when $q = 5$. So, $(B_1)^c$ has spectrum (b) and hence $(B_1)^c$ is projectively equivalent to the arc in Theorem 4 (3).

(2) When $q = 7$, the line $\langle P, S\rangle$ in the proof of Theorem 5 is a secant of C. On the other hand, when $q = 13$, $\langle P, S\rangle$ is an external line of C. Thus, depending on the value of q, the line $\langle P, S\rangle$ can form a tangent, a secant or an external line of C. That is why we could not determine the spectrum of the $(q^2 - 3q + 1, q - 2)$-arc in Theorem 5.

Next, we determine the spectrum of the arc B_0 in Lemma 3 for odd q to find one more inequivalent arc.

Theorem 6. *For odd $q \geq 5$, let $B = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4 \cup \{P_1, P_2\}$, consisting of the lines $\ell_1 = [100]$, $\ell_2 = [010]$, $\ell_3 = [001]$, $\ell_4 = [111]$ and the points $P_1 = \mathbf{P}(-1, 1, 1)$, $P_2 = \mathbf{P}(1, -1, 1)$. Then, B^c forms a $(q^2 - 3q + 1, q - 2)$-arc with spectrum $(a_0, a_{q-4}, a_{q-3}, a_{q-2}) = (4, 2q - 6, q^2 - 7q + 17, 6q - 14)$.*

Proof. Note that no three of the lines $\ell_1, \ell_2, \ell_3, \ell_4$ are concurrent. Let $\mathcal{Q} = \{Q_{ij} = l_i \cap \ell_j : 1 \leq i < j \leq 4\}$, $r_1 = \langle Q_{14}, Q_{23}\rangle$, $r_2 = \langle Q_{13}, Q_{24}\rangle$ and $r_3 = \langle Q_{12}, Q_{34}\rangle$. Then, P_1 and P_2 are equal to $r_2 \cap r_3$ and $r_1 \cap r_3$, respectively. Hence, $r_3 = \langle P_1, P_2\rangle$ is a 4-line. Let ℓ be a line. Then ℓ meets $\bigcup_{i=1}^4 \ell_i$ at two, three or four points. When $|\ell \cap (\bigcup_{i=1}^4 \ell_i)| = 2$, ℓ is r_1, r_2 or r_3. So, ℓ contains P_1 or P_2. Thus, B^c is a $(q^2 - 3q + 1, q - 2)$-arc. Now, the $(q + 1)$-lines for B are $\ell_1, \ldots, \ell_4$, and $a_{q+1}(B) = 4$. The 5-lines for B are the lines containing one of P_1, P_2 but none of $\mathcal{Q}$. Hence, $a_5(B) = 2(q + 1 - 4)$. The 3-lines for B are the lines through one of two points Q_{12}, Q_{34} containing no other point of $\mathcal{Q}$, the lines through one point $(\neq Q_{12}, Q_{34})$ of $\mathcal{Q}$ containing none of $\{P_1, P_2\}$, and two more lines r_1, r_2. Thus, $a_3(B) = 2(q + 1 - 3) + 4(q + 1 - 4) + 2 = 6q - 14$. Finally, $a_4(B) = q^2 + q + 1 - a_{q+1}(B) - a_5(B) - a_3(B) = q^2 - 7q + 17$. $\square$

Theorem 7. *Under the conditions of Theorem 6, let $P_3 = r_1 \cap r_2$. Take $P_2' \in r_1 \setminus \{P_2, P_3, Q_{14}, Q_{23}\}$ and let $B' = (B \setminus \{P_2\}) \cup \{P_2'\}$. Then, $K = (B')^c$ is a $(q^2 - 3q + 1, q - 2)$-arc with spectrum $(a_0, a_1, a_2, a_3) = (5, 1, 10, 15)$ for $q = 5$ and $(a_0, a_{q-5}, a_{q-4}, a_{q-3}, a_{q-2}) = (4, 1, 2q - 9, q^2 - 7q + 20, 6q - 15)$ for $q \geq 7$.*

Proof. Since the 3-line for B through P_2 is r_1 only, B' forms a $(4q, 3)$-blocking set. The lines through P_2 for K except $r_1 = \langle P_2, P_2'\rangle$ are three 4-lines $\langle P_2, Q_{13}\rangle$, $\langle P_2, Q_{24}\rangle$, $\langle P_1, P_2\rangle$ and $(q - 3)$ 5-lines. On the other hand, the lines through P_2' for K other than r_1 are four 3-lines $\langle P_2', Q_{ij}\rangle$ with $Q_{ij} \in \mathcal{Q} \setminus r_1$, one 5-line $\langle P_2', P_1\rangle$ and $(q - 5)$ 4-lines. Hence, $a_3(B') = a_3(B) + 3 - 4$, $a_4(B') = a_4(B) - 3 + (q - 3) + 4 - (q - 5)$, $a_5(B') = a_5(B) - (q - 3) - 1 + (q - 5)$, $a_6(B') = 1$ (or $a_6(B') = 1 + 4 = 5$ for $q = 5$). Now, our assertion follows from Theorem 6. $\square$

From the above theorems we get the following.

Corollary 1. *There exist at least six projectively inequivalent* $(q^2-3q+1, q-2)$*-arcs in* $\mathrm{PG}(2,q)$ *for* $q = p^h \geq 7$ *with odd prime* $p \neq 3$.

Finally, we consider the case q is even. Assume $q \geq 4$. Then, it is known that a $(b,3)$-blocking set B containing a line satisfies $b \geq 4q-1$ [27]. The set B_0 for even q in Lemma 3 is such a $(4q-1,3)$-blocking set with spectrum

$$(a_3(B_0), a_4(B_0), a_5(B_0), a_{q+1}(B_0)) = (6q-9, q^2-6q+8, q-2, 4).$$

When $q = 4$, the complement of a $(4q-1,3)$-blocking set is a 6-arc (a hyperoval). So, assume $q \geq 8$. We can construct two more $(4q-1,3)$-blocking sets as follows.

Theorem 8. *For even* $q \geq 8$*, let* C *be a conic in* $\mathrm{PG}(2,q)$ *with nucleus* N*. For any three points* P_1*,* P_2*,* P_3 *in* $C \cup \{N\}$ *with* $P_1, P_2 \in C$*, let* $\ell_{ij} = \langle P_i, P_j \rangle$ *for* $1 \leq i < j \leq 3$*. Then,*

(1) $B = C \cup \ell_{12} \cup \ell_{23} \cup \ell_{13}$ *is a* $(4q-1,3)$*-blocking set with spectrum*

$$(a_3(B), a_5(B), a_{q+1}(B)) = \left(\frac{(q+6)(q-1)}{2}, \frac{(q-1)(q-2)}{2}, 3\right)$$

with $|\mathrm{Aut}(B)| = 2(q-1)$ *if* $P_3 = N$*,*

(2) $B = C \cup \ell_{12} \cup \ell_{23} \cup \ell_{13} \cup \{N\}$ *is a* $(4q-1,3)$*-blocking set with spectrum*

$$(a_3(B), a_5(B), a_{q+1}(B)) = \left(\frac{(q+6)(q-1)}{2}, \frac{(q-1)(q-2)}{2}, 3\right)$$

with $|\mathrm{Aut}(B)| = 6$ *if* $P_3 \neq N$*.*

The $(4q-1,3)$-blocking sets in Theorem 8 were first found for $q = 8$, see [7].

Corollary 2. *There exist at least three projectively inequivalent* $(q^2-3q+2, q-2)$*-arcs (equivalently,* $(4q-1,3)$*-blocking sets) in* $\mathrm{PG}(2,q)$ *for every even* $q \geq 8$.

Remark 6. *From Table 1,* $(q^2-3q+2, q-2)$*-arcs are optimal for* $q = 4, 8$ *and give the known lower bound on* $m_{14}(2,16)$ *for* $q = 16$.

Author Contributions: Writing–original draft, I.B., E.J.C., T.M. and T.O. All authors have read and agreed to the published version of the manuscript.

Funding: The first author was partially supported by the Bulgarian Science Fund under Contract DN-02-2/13.12.2016. The second author was supported by the National Research Foundation of Korea funded by the Korean Government(NRF-2018R1D1A1B07044992). The third author was partially supported by JSPS KAKENHI Grant Number JP16K05256.

Acknowledgments: The second author thanks to M. Grassl for his helpful discussion and computer search with Magma in the earlier version of this paper. The authors are very grateful to the reviewers for their comments.

Conflicts of Interest: The authors declare no conflict of interest.

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