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On the Stability with Respect to H-Manifolds for Cohen–Grossberg-Type Bidirectional Associative Memory Neural Networks with Variable Impulsive Perturbations and Time-Varying Delays

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Abstract: The present paper is devoted to Bidirectional Associative Memory (BAM) Cohen–Grossberg-type impulsive neural networks with time-varying delays. Instead of impulsive discontinuities at fixed moments of time, we consider variable impulsive perturbations. The stability with respect to manifolds notion is introduced for the neural network model under consideration. By means of the Lyapunov function method sufficient conditions that guarantee the stability properties of solutions are established. Two examples are presented to show the validity of the proposed stability criteria.

Keywords: global exponential stability; h-manifolds; BAM Cohen–Grossberg neural networks; time-varying delays; variable impulsive perturbations

1. Introduction

The Cohen–Grossberg-type neural network models were first proposed by Cohen and Grossberg [1] in 1983, and since then an extensive work on this subject has been done by numerous researchers due to the opportunities for applications of such models in key fields of science and engineering such as parallel computing, associative memory, pattern recognition, signal and image processing, etc. [2–4].

On the other side, it is well known that BAM neural networks were first proposed by Kosko [5–7] and this type of models also has been investigated intensively due to its extension of the single-layer auto-associative Hebbian correlation to two-layer hetero-associative circuits [8].

It is also well known that time delays naturally exist in neural network models, due mainly to the limited speed of signal transmissions and amplifiers switching. Time delays, also known as synaptic transmission delays, may affect the dynamical behaviors and synchronization control of neural networks. That is why numerous researchers considered delay effects on both Cohen–Grossberg and BAM neural networks, and excellent results have been reported in the literature. We will direct the reader to see [9–11] for some results on delayed Cohen–Grossberg neural networks, and [8,12–14] for results on BAM neural networks with delays, including some very recent publications [15–18].

In addition, the hybrid Cohen–Grossberg-type BAM neural networks with delays are an important subject of research and hence, is very well studied by numerous researchers. See, for example, references [19–23] and the references therein. Most of the above cited authors considered time-varying delays in their investigations of such neural networks. Indeed, it is noted that in real-world applications, models with time-varying delays provide generally a more realistic description than these with constant delays, in particular, in the population dynamics neural network models [24].

Since impulsive phenomena exist in numerous fields of science and engineering [25–29] and the impulsive control methods are find to be more efficient than other control strategies [30–38], there has been an increasing research interest on the impulsive generalizations of Cohen–Grossberg-type BAM neural networks with delays. For example, in [39] the problem of existence of equilibrium states of Cohen–Grossberg BAM neural networks with delays and impulses and their global exponential stability behavior are investigated using topological degree theory, the Lyapunov functional method and some analysis techniques. Sufficient conditions for existence, uniqueness and global exponential stability of equilibrium states of a class of Cohen–Grossberg-type BAM impulsive neural networks with time-varying delays has been proposed in [40] by establishing a delay differential inequality and utilizing the homeomorphism and M-matrix theories. Efficient criteria for existence, uniqueness and exponential stability of equilibrium points on the base on the application of Lyapunov functionals, impulsive control and some analysis methods are established in [41]. The author demonstrates the importance of the impulsive control technique in stabilizing unstable Cohen–Grossberg-type BAM neural networks with time-varying delays. The paper [42] deals with the questions of the existence and global exponential stability of single periodic solutions for Cohen–Grossberg-type BAM neural networks with impulsive perturbations and both continuously distributed and finite distributed delays. A class of Markovian jumping Cohen–Grossberg BAM-type neural networks with impulsive effects and mixed time delays has been investigated in [43] and criteria for global exponential stability of its states have been proposed. The authors in [44] study the impulsive effects on the exponential stability behavior of delayed Cohen–Grossberg-type BAM neural networks.

However, all cited above papers considered impulsive perturbations and impulsive control at fixed instants. Such systems are particular cases of more general and realistic impulsive models with variable impulsive perturbations. It is now well recognized that systems with impulsive perturbations at variable time, including neural network systems, are more useful from the applied point of view [45–49]. However, their investigations are related to some difficulties such as bifurcation, "merging" of solutions, "beating" phenomena, loss of the property of autonomy, etc. [29,47]. Most of the challenges are due to the fact that the impulsive effects such as instantaneous perturbations and abrupt changes on distinct solutions are not, in general, the same for all solutions. A novelty in our research is that we will propose an impulsive delayed Cohen–Grossberg-type bidirectional associative memory neural network with variable impulsive perturbations in this paper.

Due to the importance for applications the investigations on impulsive Cohen–Grossberg-type neural networks with delays and variable impulsive perturbations just began. Recently, some results on such models has been proposed in [50]. However, the authors in the above cited paper do not consider BAM neural networks. Hence, the topic of impulsive Cohen–Grossberg-type neural network with delays and variable impulsive perturbations is far from completion.

It is not surprised that in the existing results on Cohen–Grossberg-type BAM impulsive neural networks with time-varying delays the most investigated problem is the problem of exponential stability of their solutions. The authors of [39–44] considered mainly the stability of single solutions of great importance such as equilibrium points and periodic solutions. However, in many applications the stability properties of manifolds or sets related to a system are very significant for the qualitative behavior of the correspondent system [47,51–54]. Therefore, it is important and interesting to further investigate and generalize the stability results to stability of manifolds case.

Stimulated by the above discussions, in this paper, we will generalize the concept of the stability of a single solution, and investigate the stability of Cohen–Grossberg-type BAM impulsive neural networks with time-varying delays and variable impulsive perturbations with respect to manifolds. These manifolds will be determined by particular functions [55]. To the best of our knowledge, such generalization of the stability notion is not already studied for the Cohen–Grossberg-type BAM neural networks.

The remaining part of the paper is arranged as follows. In Section 2, the class of Cohen–Grossberg-type BAM impulsive neural networks with time-varying delays and variable impulsive perturbations is introduced. Some notations, assumptions and definitions are also given. Section 3 is devoted to our main h-global exponential stability result. The proof is performed by using the Lyapunov function technique and differential inequalities. In Section 4, two examples are provided to show the efficacy of the obtained criteria. Finally, some conclusions and open problems are presented in Section 5.

2. Preliminary Notes

Let $\mathbb{R}^n$ denotes the n-dimensional Euclidean space endowed with the norm $||x|| = \sum_{i=1}^{n} |x_i|$ and $\mathbb{R}_+ = [0, \infty)$. In the case when $z = (x, y)^T \in \mathbb{R}^{n+m}$, $||z|| = \sum_{i=1}^{n} |x_i| + \sum_{j=1}^{m} |y_j|$.

The goal of this paper is to investigate the qualitative properties, in this case the global exponential stability of the solutions with respect to a manifold defined by a function, for Cohen–Grossberg-type BAM impulsive neural networks with time-varying delays and variable impulsive perturbations of the type:

$$\begin{cases} \dot{x}_i(t) = -a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{m} c_{ji} f_j(y_j(t)) \\ \qquad - \sum_{j=1}^{m} d_{ji} g_j(y_j(t - \sigma_j(t))) - I_i\Big], \quad t \neq \tau_k(x(t), y(t)), \\ \dot{y}_j(t) = -\hat{a}_j(y_j(t))\Big[\hat{b}_j(y_j(t)) - \sum_{i=1}^{n} p_{ij} \hat{f}_i(x_i(t)) \\ \qquad - \sum_{i=1}^{n} q_{ij} \hat{g}_i(x_i(t - \hat{\sigma}_i(t))) - J_j\Big], \quad t \neq \tau_k(x(t), y(t)), \\ (x_i(t^+), y_j(t^+))^T = (x_i(t) + P_{ik}(x_i(t)), y_j(t) + Q_{jk}(y_j(t)))^T, \ t = \tau_k(x(t), y(t)), \end{cases} \tag{1}$$

where the model parameters c_{ji}, d_{ji}, p_{ij}, q_{ij}, I_i, $J_j \in \mathbb{R}$, the functions a_i, $\hat{a}_j$, b_i, $\hat{b}_j$, f_j, $\hat{f}_i$, g_j, $\hat{g}_i$, σ_j, $\hat{\sigma}_i$, P_{ik}, $Q_{jk} \in C[\mathbb{R}, \mathbb{R}]$, $t > \sigma_j$, $t > \hat{\sigma}_i$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $\tau_k : \mathbb{R}^{n+m} \to \mathbb{R}$, $k = 1, 2, \ldots$.

System (1) is a generalization of the existing models of impulsive BAM Cohen–Grossberg neural networks with time-varying delays [39–44], where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $y(t) = (y_1(t), y_2(t), \ldots, y_m(t))^T$, $x_i(t)$, $i = 1, 2, \ldots, n$ and $y_j(t)$, $j = 1, 2, \ldots, m$, are the states of the ith unit and jth unit, respectively, at time t, the functions $\sigma_j(t)$ $(0 \leq \sigma_j(t) \leq \sigma_j)$, $\hat{\sigma}_i(t)$ $(0 \leq \hat{\sigma}_i(t) \leq \hat{\sigma}_i)$ correspond to the transmission time delays at time t, c_{ji}, p_{ij}, d_{ji} and q_{ij} represent the connection weights, f_j, $\hat{f}_i$, g_j and $\hat{g}_i$ are the signal functions, I_i and J_j are the external inputs, a_i and $\hat{a}_j$ represent the amplification functions, b_i and $\hat{b}_j$ represent the appropriately behaved functions, such that all solutions of (1) remain bounded.

Different from the existing models [39–44], we consider variable impulsive perturbations in (1) such that P_{ik} and Q_{jk} represent the abrupt changes of the states at the impulsive moments, where $(x_i(t), y_j(t))^T = (x_i(t^-), y_j(t^-))^T$ and $(x_i(t^+), y_j(t^+))^T = \lim_{h \to 0^+} (x_i(t_k + h), y_j(t_k + h))^T$, are, respectively, the states of the ith unit from the first neural field and the jth unit from the second neural field before and after an impulsive perturbation at the moment t. Note that the abrupt changes $P_{ik}(x_i(t)) = \Delta x_i(t) = x_i(t^+) - x_i(t)$ and $Q_{jk}(y_j(t)) = \Delta y_j(t) = y_j(t^+) - y_j(t)$ can be considered as impulsive controls [31–38].

Let J be an interval, $J \subset \mathbb{R}_+$, and define the following class of piecewise continuous functions $PC[J, \mathbb{R}^n] = \{s : J \to \mathbb{R}^n : \ s(t)$ is piecewise continuous on J with points of discontinuity $t_k \in J$ at which $s(t_k^-)$ and $s(t_k^+)$ exist and $s(t_k^-) = s(t_k)\}$.

Let $\sigma = \max_{1\leq j\leq m}\sigma_j$, $\hat{\sigma} = \max_{1\leq i\leq n}\hat{\sigma}_i$, $\varphi_0 \in PCB[[-\hat{\sigma},0],\mathbb{R}^n]$, $\phi_0 \in PCB[[-\sigma,0],\mathbb{R}^m]$, where $PCB[J,\mathbb{R}^n] = \{s \in PC[J,\mathbb{R}^n] : s(t)$ is bounded on J$\}$.

For a $t_0 \in \mathbb{R}_+$ we denote by $z(t) = (x(t),y(t))^T = (x(t;t_0,\varphi_0),y(t;t_0,\phi_0))^T$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ the solution of model (1) which satisfies initial conditions of the type:

$$\begin{cases} x(t;t_0,\varphi_0) = \varphi_0(t-t_0) = \varphi_0(\xi), \ -\hat{\sigma} \leq \xi = t-t_0 \leq 0, \\ y(t;t_0,\phi_0) = \phi_0(t-t_0) = \phi_0(\xi), \ -\sigma \leq \xi = t-t_0 \leq 0, \\ x(t_0^+;t_0,\varphi_0) = \varphi_0(0), \ y(t_0^+;t_0,\phi_0) = \phi_0(0). \end{cases} \tag{2}$$

As usual, the solution $z(t) = (x(t),y(t))^T$ of the problem (1), (2) is [29,45–49] a function from the class $PC[J,\mathbb{R}^{n+m}]$, i.e., at the moments t_{l_k} when the integral curve of the solution $(x(t),y(t))$ meets the hypersurfaces

$$\theta_k = \{(t,x,y) \in [t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m : \ t = \tau_k(x,y)\},$$

we have:

$$x_i(t_{l_k}^-) = x_i(t_{l_k}), \ \ x_i(t_{l_k}^+) = x_i(t_{l_k}) + P_{il_k}x_i(t_{l_k}),$$

$$y_j(t_{l_k}^-) = y_j(t_{l_k}), \ \ y_j(t_{l_k}^+) = y_j(t_{l_k}) + Q_{jl_k}(y_j(t_{l_k})).$$

The points $t_{l_1}, t_{l_2}, \ldots$ $(t_0 < t_{l_1} < t_{l_2} < \ldots)$ are called impulsive moments at which impulsive control techniques can be applied [34–38]. Note that, in general, the number k of the hypersurface θ_k may not be equal to the number l_k of the impulsive moment t_{l_k}. Furthermore, different solutions may have different impulsive moments.

Denote by $\nu = \max\{\sigma,\hat{\sigma}\}$, $\psi_0 = (\varphi_0,\phi_0)^T$, and let $\mathcal{PCB} = PCB[[-\nu,0],\mathbb{R}^{n+m}]$. To eliminate any opportunity of "beating" of solutions, and to assurance existence, uniqueness and continuability of the solution $z(t) = z(t;t_0,\psi_0)$ of the initial value problem (IVP) (1), (2), on the interval $[t_0,\infty)$ for $\psi_0 \in \mathcal{PCB}$ and $t_0 \in \mathbb{R}_+$ we assume that:

1. $\tau_0(x,y) \equiv t_0$ for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, the functions $\tau_k(x,y)$ are continuous, and the following relations hold:

   $$t_0 < \tau_1(x,y) < \tau_2(x,y) < \ldots, \ \tau_k(x,y) \to \infty \text{ as } k \to \infty,$$

   uniformly on $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.
2. The functions a_i, $\hat{a}_j$, b_i, $\hat{b}_j$, f_j, $\hat{f}_i$, g_j, $\hat{g}_i$, σ_j, $\hat{\sigma}_i$, P_{ik}, Q_{jk} are smooth enough on $[t_0,\infty)$.

Let $h = h(t,z)$, $h : [t_0-\nu,\infty) \times \mathbb{R}^{n+m} \to \mathbb{R}$, be a function in $[t_0-\nu,\infty) \times \mathbb{R}^{n+m}$. We will consider the h-manifolds

$$M_t(n+m) = \{z \in \mathbb{R}^{n+m} : \ h(t,z) = 0, \ t \in [t_0,\infty)\},$$
$$M_{t,\nu}(n+m) = \{z \in \mathbb{R}^{n+m} : \ h(t,z) = 0, \ t \in [t_0-\nu,t_0]\},$$
$$M_t(n+m)(\varepsilon) = \{z \in \mathbb{R}^{n+m} : \ ||h(t,z)|| < \varepsilon, \ t \in [t_0,\infty)\}, \ \varepsilon > 0,$$
$$M_{t,\nu}(n+m)(\varepsilon) = \Big\{\psi \in \mathcal{PCB} : \sup_{-\nu\leq\xi\leq 0} ||h(t,\psi(\xi))|| < \varepsilon; \ t \in [t_0-\nu,t_0]\Big\}.$$

The next hypotheses will be very important in the proofs of our main results:

Hypothesis 1. *The functions a_i, $\hat{a}_j$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$ are bounded and there exist positive constants $\underline{a}_i$, $\underline{\hat{a}}_j$ such that $\underline{a}_i \leq a_i(t) \leq \bar{a}_i$ and $\underline{\hat{a}}_j \leq a_j(t) \leq \bar{\hat{a}}_j$ for $t \in \mathbb{R}$.*

Hypothesis 2. *For the functions b_i and $\hat{b}_j$ there exist positive constants B_i, $\hat{B}_j$ respectively, such that*

$$\frac{b_i(\chi_1) - b_i(\chi_2)}{\chi_1 - \chi_2} \geq B_i, \ \ \frac{\hat{b}_j(\chi_1) - \hat{b}_j(\chi_2)}{\chi_1 - \chi_2} \geq \hat{B}_j,$$

for any $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$ *and* $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Hypothesis 3. *There exist constants* $L_j > 0$, $M_j > 0$, $\hat{L}_i > 0$, $\hat{M}_i > 0$ *such that*

$$|f_j(\chi_1) - f_j(\chi_2)| \leq L_j|\chi_1 - \chi_2|, \quad |g_j(\chi_1) - g_j(\chi_2)| \leq M_j|\chi_1 - \chi_2|$$

$$|\hat{f}_i(\chi_1) - \hat{f}_i(\chi_2)| \leq \hat{L}_i|\chi_1 - \chi_2|, \quad |\hat{g}_i(\chi_1) - \hat{g}_i(\chi_2)| \leq \hat{M}_i|\chi_1 - \chi_2|$$

for all $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Hypothesis 4. *The function h is continuous on* $[t_0 - \nu, \infty) \times \mathbb{R}^{n+m}$ *and the sets* $M_t(n+m)$, $M_{t,\nu}(n+m)$ *are* $(n+m-1)$*-dimensional manifolds in* $\mathbb{R}^{n+m}$.

Hypothesis 5. *Each solution* $z(t)$ *of the IVP (1), (2) satisfying*

$$||h(t, z(t, t_0, \psi_0))|| \leq \ H < \infty$$

is defined on $[t_0, \infty) \times \mathbb{R}^{n+m}$, $H > 0$.

Note that, a constant state $z^* = (x^*, y^*)^T \in R^{n+m}$,

$$z^* = (x^*, y^*)^T = (x_1^*, x_2^*, \ldots, x_n^*, y_1^*, y_2^*, \ldots, y_m^*)^T$$

is called an *equilibrium* state of (1), if and only if

$$b_i(x_i^*) = \sum_{j=1}^{m} c_{ji} f_j(y_j^*) + \sum_{j=1}^{m} d_{ji} g_j(y_j^*) + I_i, \ \ \hat{b}_j(y_j^*) = \sum_{i=1}^{n} p_{ij} \hat{f}_i(x_i^*) + \sum_{i=1}^{n} q_{ij} \hat{g}_i(x_i^*) + J_j,$$

$$P_{ik}(x_i^*) = 0, \ \ Q_{jk}(y_j^*) = 0, \ i = 1, 2, \ldots, n, \ \ j = 1, 2, \ldots, n, \ \ k = 1, 2, \ldots.$$

Suppose that z^* is an equilibrium of (1), and for a solution $z(t) = (x(t), y(t))^T$ of (1) with an initial function $\psi_0 \in \mathcal{PCB}$ consider $\tilde{z}(t) = (\tilde{x}(t), \tilde{y}(t))^T$, where $\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t))^T$, $\tilde{y}(t) = (\tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_m(t))^T$, $\tilde{x}_i(t) = x_i(t) - x_i^*$, $\tilde{y}_j(t) = y_j(t) - y_j^*$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Next, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots$, we will also use the notations:
$\alpha_i(\tilde{x}_i(t)) = a_i(\tilde{x}_i(t) + x_i^*)$, $\hat{\alpha}_j(\tilde{y}_j(t)) = \hat{a}_j(\tilde{y}_j(t) + y_j^*)$;
$\beta_i(\tilde{x}_i(t)) = b_i(\tilde{x}_i(t) + x_i^*) - b_i(x_i^*)$, $\hat{\beta}_j(\tilde{y}_j(t)) = \hat{b}_j(\tilde{y}_j(t) + y_j^*) - \hat{b}_j(y_j^*)$;
$F_j(\tilde{y}_j(t)) = f_j(\tilde{y}_j(t) + y_j^*) - f_j(y_j^*)$, $G_j(\tilde{y}_j(t)) = g_j(\tilde{y}_j(t) + y_j^*) - g_j(y_j^*)$;
$\hat{F}_i(\tilde{x}_i(t)) = \hat{f}_i(\tilde{x}_i(t) + x_i^*) - \hat{f}_i(x_i^*)$, $\hat{G}_i(\tilde{x}_i(t)) = \hat{g}_i(\tilde{x}_i(t) + x_i^*) - \hat{g}_i(x_i^*)$;
$\hat{P}_{ik}(\tilde{x}_i(t)) = P_{ik}(\tilde{x}_i(t) + x_i^*)$, $\hat{Q}_{jk}(\tilde{y}_j(t)) = Q_{jk}(\tilde{y}_j(t) + y_j^*)$, $t \in \theta_k$.

Then $\tilde{z}(t)$ satisfies

$$\begin{cases} \dot{\tilde{x}}_i(t) = -\alpha_i(\tilde{x}_i(t))\Bigg[\beta_i(\tilde{x}_i(t)) - \sum_{j=1}^{m} c_{ji}F_j(\tilde{y}_j(t)) \\ \qquad\qquad - \sum_{j=1}^{m} d_{ji}G_j(\tilde{y}_j(t-\sigma_j(t)))\Bigg], \quad t \neq \tau_k(x(t),y(t)), \\ \dot{\tilde{y}}_j(t) = -\hat{\alpha}_j(\tilde{y}_j(t))\Bigg[\hat{\beta}_j(\tilde{y}_j(t)) - \sum_{i=1}^{n} p_{ij}\hat{F}_i(\tilde{x}_i(t)) \\ \qquad\qquad - \sum_{i=1}^{n} q_{ij}\hat{G}_i(\tilde{x}_i(t-\hat{\sigma}_i(t)))\Bigg], \quad t \neq \tau_k(x(t),y(t)), \\ (\tilde{x}_i(t^+),\tilde{y}_j(t^+))^T = (\tilde{x}_i(t) + \hat{P}_{ik}(\tilde{x}_i(t)), \tilde{y}_j(t) + \hat{Q}_{jk}(\tilde{y}_j(t)))^T, \ t = \tau_k(x(t),y(t)), \end{cases} \tag{3}$$

In this research, we will introduce the following definition of global exponential stability of the equilibrium z^* of (1) with respect to the function h (or with respect to the manifold defined by this function), which generalizes the stability definitions given in [39–44].

Definition 1. *We will say that the equilibrium z^* of (1) is globally exponentially stable with respect to the function h if there exists a constant $\mu > 0$ such that*

$$\tilde{z}(t) \in M_t(n+m)\big(\mathcal{M}(\psi_0)\exp(-\mu(t-t_0))\big), \quad t \geq t_0, \ t_0 \in \mathbb{R}_+, \ \psi_0 \in \mathcal{PCB},$$

where $\mathcal{M}(0) = 0$, $\mathcal{M}(\psi)$ is Lipschitz continuous with respect to $\psi \in \mathcal{PCB}$, and $\mathcal{M}(\psi) \geq 0$.

Introduce the following sets

$$\mathcal{G}_k = \{(t,x,y) : \tau_{k-1}(x,y) < t < \tau_k(x,y), \ (x,y) \in \mathbb{R}^{n+m}\}, \ k = 1,2,\ldots, \ \ \mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k.$$

In our investigations, we will use the Lyapunov-Razumikhin approach, which for impulsive systems requires a definition of Lyapunov's like piecewise continuous functions [29].

Definition 2. *We will say that a function $V : \mathbb{R}_+ \times \mathbb{R}^{n+m} \to \mathbb{R}_+$, $V = V(t,x,y) = V(t,z)$, belongs to the class V_0 if:*

1. The function V is continuous in $\mathcal{G}$ and locally Lipschitz continuous with respect to (x,y) on each of the sets $\mathcal{G}_k$, $k = 1,2,\ldots$.

2. For $(t_0^*, x_0^*, y_0^*) \in \theta_k$ and each $k = 1,2,\ldots$ there exist the finite limits

$$V(t_0^{*-}, x_0^*, y_0^*) = \lim_{\substack{(t,x,y)\to(t_0^*,x_0^*,y_0^*) \\ (t,x,y)\in\mathcal{G}_k}} V(t,x,y), \ \ V(t_0^{*+}, x_0^*, y_0^*) = \lim_{\substack{(t,x,y)\to(t_0^*,x_0^*,y_0^*) \\ (t,x,y)\in\mathcal{G}_{k+1}}} V(t,x,y)$$

and $V(t_0^{*-}, x_0^*, y_0^*) = V(t_0^*, x_0^*, y_0^*)$.

For $(t,z) \in \mathcal{G}$, we will apply the following upper right-hand derivative of a function V from the class V_0 with respect to system (1), defined by [29]

$$D^+V(t,\psi(0)) = \lim_{h\to 0^+} \sup \frac{1}{h}\big[V(t+h, z(t+h;t_0,\psi)) - V(t,\psi(0))\big],$$

for $(t,\psi) \in \mathbb{R}_+ \times \mathcal{PCB}$.

Let each of the points t_k be a solution of some of the equations $t = \tau_k(z(t))$, $k = 1, 2, \ldots$, i.e., $t_1, t_2, \ldots$ are the impulsive control instants for the IVP (1), (2). In the proof of the main result, we will apply the following comparison result from [29].

Lemma 1. *Let the function $V \in V_0$ be such that for $t \in [t_0, \infty)$, $\psi \in \mathcal{PCB}$,*

$$V(t^+, \psi(0) + \Delta\psi) \leq V(t, \psi(0)), \;\; t = t_k, \; k = 1, 2, \ldots,$$

and the inequality

$$D^+V(t, \psi(0)) \leq -\mu V(t, \psi(0)), \; t \neq t_k, \; k = 1, 2, \ldots$$

is valid whenever $V(t + \xi, \psi(\xi)) \leq V(t, \psi(0))$, $-\nu \leq \xi \leq 0$, $\mu > 0$.

Then

$$V(t, z(t; t_0, \psi_0)) \leq sup_{-\nu \leq \xi \leq 0} V(t_0^+, \psi_0(\xi)) \exp(-\mu(t - t_0))$$

for $t \in [t_0, \infty)$.

3. *H*-Stability Results

We will now derive our main $h-$stability results for the equilibrium state of the model (1).

Theorem 1. *Assume that:*

1. *Hypotheses 1–5 are satisfied.*
2. *There exists a positive number μ such that*

$$\min_{1 \leq i \leq n} \Big(\underline{a}_i B_i - \overline{a}_j \sum_{j=1}^{m} |p_{ij}| \hat{L}_i| \Big) + \min_{1 \leq j \leq m} \Big(\underline{\hat{a}}_j \hat{B}_j - \overline{\hat{a}}_j \sum_{i=1}^{n} |c_{ji}| L_j| \Big)$$

$$-\Big(\max_{1 \leq j \leq m} \overline{a}_j \sum_{i=1}^{n} |d_{ji}| M_j + \max_{1 \leq i \leq n} \overline{\hat{a}}_i \sum_{j=1}^{m} |q_{ij}| \hat{M}_i \Big) \geq \mu.$$

3. *The functions P_{ik} and Q_{jk} are such that*

$$P_{ik}(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*), \; Q_{jk}(y_j(t_k)) = -\mu_{jk}(y_j(t_k) - y_j^*),$$

where $0 < \gamma_{ik} < 2$, $0 < \mu_{jk} < 2$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots$.

4. *There exists a function $h(t, z)$ such that the next inequalities hold*

$$||h(t, z)|| \leq ||z|| < \Lambda(H)||h(t, z)||, \quad (t, z) \in [t_0, \infty) \times \mathbb{R}^{n+m},$$

where $\Lambda(H) \geq 1$ exists for any $0 < H \leq \infty$.

Then the equilibrium z^* of the Cohen–Grosberg-type BAM impulsive delayed neural network system (1) is globally exponentially stable with respect to the function h.

Proof. We define a Lyapunov function

$$V(t, \tilde{z}(t)) = ||\tilde{z}|| = \sum_{i=1}^{n} |\tilde{x}_i(t)| + \sum_{j=1}^{m} |\tilde{y}_j(t)|.$$

In the case when $t = t_k,\ k = 1, 2, \ldots$ by condition 3 of Theorem 1 and Hypotheses 1–3 and 5, we get

$$\begin{aligned} V(t_k^+, \tilde{z}(t_k^+)) = ||\tilde{z}(t_k^+)|| &= \sum_{i=1}^{n} |(1-\gamma_{ik})\tilde{x}_i(t_k)| + \sum_{j=1}^{m} |(1-\mu_{jk})\tilde{y}_j(t_k)| \\ &< \sum_{i=1}^{n} |\tilde{x}_i(t_k)| + \sum_{j=1}^{m} |\tilde{y}_j(t_k)| = V(t_k, \tilde{z}(t_k)). \end{aligned} \tag{4}$$

Let now $t \geq t_0$ and $t \neq \tau_k(x,y)$, $k = 1, 2, \ldots$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Then, from the fact that z^* is an equilibrium of (1), using Hypotheses 1–5, for the upper right-hand derivative $D^+V(t, \tilde{\psi}(0))$, $\tilde{\psi} \in \mathcal{PCB}$, along the solutions of system (3), we get

$$\begin{aligned} D^+V(t,\tilde{\psi}(0)) \leq \sum_{i=1}^{n} \Bigg(&- \underline{a}_i B_i |\tilde{\varphi}(0)| + \overline{a}_i \sum_{j=1}^{m} |c_{ji}| L_j |\tilde{\phi}(0)| + \overline{a}_i \sum_{j=1}^{m} |d_{ji}| M_j |\tilde{\phi}(-\sigma_j(0))| \Bigg) \\ &+ \sum_{j=1}^{m} \Bigg(- \underline{\hat{a}}_j \hat{B}_j |\tilde{\phi}(0)| + \hat{\overline{a}}_j \sum_{i=1}^{n} |p_{ij}| \hat{L}_i |\tilde{\varphi}(0)| + \hat{\overline{a}}_j \sum_{i=1}^{n} |q_{ij}| \hat{M}_i |\tilde{\varphi}(-\hat{\sigma}_i(0))| \Bigg) \\ &\leq - \sum_{i=1}^{n} \Bigg(\underline{a}_i B_i - \hat{\overline{a}}_i \sum_{j=1}^{m} |p_{ij}| \hat{L}_i \Bigg) |\tilde{\varphi}(0)| - \sum_{j=1}^{m} \Bigg(\underline{\hat{a}}_j \hat{B}_j - \overline{a}_j \sum_{i=1}^{n} |c_{ji}| L_j \Bigg) |\tilde{\phi}(0)| \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{a}_i |d_{ji}| M_j |\tilde{\phi}(-\sigma_j(0))| + \sum_{j=1}^{m} \sum_{i=1}^{n} \hat{\overline{a}}_j |q_{ij}| \hat{M}_i |\tilde{\varphi}(-\hat{\sigma}_i(0))| \\ &\leq - \Bigg[\min_{1 \leq i \leq n} \Big(\underline{a}_i B_i - \overline{a}_j \sum_{j=1}^{m} |p_{ij}| \hat{L}_i \Big) + \min_{1 \leq j \leq m} \Big(\underline{\hat{a}}_j \hat{B}_j - \hat{\overline{a}}_j \sum_{i=1}^{n} |c_{ji}| L_j \Big) \Bigg] V(t, \tilde{\psi}(0)) \\ &+ \Big(\max_{1 \leq j \leq m} \overline{a}_j \sum_{i=1}^{n} |d_{ji}| M_j + \max_{1 \leq i \leq n} \hat{\overline{a}}_i \sum_{j=1}^{m} |q_{ij}| \hat{M}_i \Big) \sup_{-\nu \leq \xi \leq 0} V(s, \tilde{\psi}(\xi)), \end{aligned}$$

where $\tilde{\psi} = (\tilde{\varphi}, \tilde{\phi})^T = (\varphi - x^*, \phi - y^*)^T$.

The last inequality and condition 2 of Theorem 1 imply

$$D^+V(t, \tilde{\psi}(0)) \leq -\mu V(t, \tilde{\psi}(0)),\ t \neq t_k,\ k = 1, 2, \ldots \tag{5}$$

whenever $V(t+\xi, \tilde{\psi}(\xi)) \leq V(t, \tilde{\psi}(0))$, $-\nu \leq \xi \leq 0$.

Then using (4) and (5), by Lemma 1, we obtain

$$V(t, \tilde{z}(t)) \leq \sup_{-\nu \leq \xi \leq 0} V(0, \tilde{\psi}_0(\xi)) \exp(-\mu(t-t_0)),\ t > t_0.$$

Hence, from condition 4 of Theorem 1 we have

$$\begin{aligned} ||h(t, \tilde{z}(t, t_0, \tilde{\psi}_0)|| \leq V(t, \tilde{z}(t)) &\leq \sup_{-\nu \leq \xi \leq 0} V(0, \tilde{\psi}_0(\xi)) \exp(-\mu(t-t_0)) \\ &< \Lambda(H) \sup_{-\nu \leq \xi \leq 0} ||h(t_0^+, \psi_0(\xi) - z^*)|| \exp(-\mu(t-t_0)), \quad t \geq t_0. \end{aligned}$$

Let $\mathcal{M} = \mathcal{M}(\psi_0) = \Lambda(H) \sup_{-\nu \leq \xi \leq 0} ||h(t_0^+, \psi_0(\xi) - z^*)||$.

Then

$$||h(t, z(t, t_0, \psi_0) - z^*|| < \mathcal{M} \exp(-\mu(t-t_0)), \quad t \geq t_0,$$

where we have that $\mathcal{M} \geq 0$ and $\mathcal{M} = 0$ only for $h(t_0^+, \psi_0(\xi) - z^*) = 0$, $\xi \in [-\nu, 0]$.

The last estimate implies the global exponential stability of the equilibrium state z^* of (1) with respect to the function h. $\square$

Remark 1. *The concept of stability with respect to manifolds defined by a particular function h generalizes numerous stability notions. Hence, Theorem 1 can be applied to a number of concrete situations depending on the choice of the norm $||z||$ and the Lyapunov function $V(t,z)$. Some of the most applicable cases are when the function h is*

$$h(t,z) = z - z^*,$$

where z^* is an arbitrary nontrivial solution of (1) (an equilibrium, periodic solution, almost periodic solution, etc.);

$$h(t,z) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2 + y_1^2 + y_2^2 + \cdots + y_m^2};$$

$$h(t,z) = d(z, A),$$

where $A \subset \mathbb{R}^{n+m}$ and d is the distance function. Therefore, the proposed stability result extends and generalizes the existing stability results for Cohen–Grossberg-type BAM impulsive neural networks with time-varying delays.

Remark 2. *The stability criteria provided by Theorem 1 also generalize the results in [39–44] considering variable impulsive perturbations and $h-$manifolds. It is worth noting that, in the case when $h(t,z) = z - z^*$, the impulsive moments of both solutions $z(t)$ and $z^*(t)$ can be different which is not considered in [39–44]. However, considering impulsive perturbations at variable time in impulsive neural network models is more natural and realistic, and, therefore, the new results offer an extended horizon for applications. Observe also that, if the impulsive events are realized at fixed times or when $\tau_k(x,y) = t_k$, $k = 1,2,\ldots$, and the function $h(t,z) = z$, then the exponential stability criteria in [39–44] can be obtained as corollaries from our result.*

4. Illustrative Examples

In this section, we will demonstrate the validity of the obtained in Theorem 1 criteria for global exponential stability with respect to manifolds.

Example 1. *Consider the following Cohen–Grossberg-type BAM impulsive neural networks with time-varying delays*

$$\begin{cases} \dot{x}_i(t) = -a_i(x_i(t))\Bigg[b_i(x_i(t)) - \sum\limits_{j=1}^{2} c_{ji} f_j(y_j(t)) \\ \qquad\qquad - \sum\limits_{j=1}^{2} d_{ji} g_j(y_j(t-\sigma_j(t))) - I_i\Bigg], \quad t \neq \tau_k(x(t), y(t)), \\ \dot{y}_j(t) = -\hat{a}_j(y_j(t))\Bigg[\hat{b}_j(y_j(t)) - \sum\limits_{i=1}^{2} p_{ij} \hat{f}_i(x_i(t)) \\ \qquad\qquad - \sum\limits_{i=1}^{2} q_{ij} \hat{g}_i(x_i(t-\hat{\sigma}_i(t))) - J_j\Bigg], \quad t \neq \tau_k(x(t), y(t)), \end{cases} \tag{6}$$

with impulsive perturbations of the type

$$x(t^+) - x(t) = \begin{pmatrix} -1+\frac{1}{2k} & 0 \\ 0 & -1+\frac{1}{2k} \end{pmatrix} x(t), \quad t = \tau_k(x(t), y(t)), \quad k = 1, 2, \ldots,$$
$$y(t^+) - y(t) = \begin{pmatrix} -1+\frac{1}{3k} & 0 \\ 0 & -1+\frac{1}{3k} \end{pmatrix} y(t), \quad t = \tau_k(x(t), y(t)), \quad k = 1, 2, \ldots, \tag{7}$$

where $t > 0$,

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad I_1 = I_2 = J_1 = J_2 = 1,$$

$$f_j(y_j) = g_j(y_j) = \frac{|y_j+1| - |y_j-1|}{2}, \quad \hat{f}_i(x_i) = \hat{g}_i(x_i) = \frac{|x_i+1| - |x_i-1|}{2}, \quad i = 1, 2, \; j = 1, 2,$$
$$0 \le \sigma_j(t) \le 1, \; 0 \le \hat{\sigma}_i(t) \le 1, \quad a_i(x_i) = \hat{a}_j(y_j) = 1, \quad b_1(x_i) = 2x_i, \quad b_2(x_i) = 3x_i,$$
$$\hat{b}_1(y_j) = \hat{b}_2(y_j) = 2y_j, \quad i = 1, 2, \; j = 1, 2,$$
$$(c_{ij})_{2\times 2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ 0.6 & -0.5 \end{pmatrix}, \; (d_{ij})_{2\times 2} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 \\ -0.4 & 0.2 \end{pmatrix},$$
$$(p_{ij})_{2\times 2} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.7 & -0.6 \\ 0.9 & 0.8 \end{pmatrix}, \; (q_{ij})_{2\times 2} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} 0.2 & -0.1 \\ 0.1 & -0.2 \end{pmatrix},$$
$$\tau_k(x, y) = |x| + |y| + k, \; k = 1, 2, \ldots.$$

We have that all assumptions of Theorem 1 are satisfied for

$$L_1 = L_2 = 1, \quad M_1 = M_2 = 1, \quad \hat{L}_1 = \hat{L}_2 = 1, \quad \hat{M}_1 = \hat{M}_2 = 1,$$
$$\underline{a}_i = \overline{a}_i = 1, \quad \underline{\hat{a}}_i = \overline{\hat{a}}_i = 1, \quad B_1 = 2, \quad B_2 = 3, \quad \hat{B}_1 = \hat{B}_2 = 2.$$

We can verify that condition 2 of Theorem 1 is satisfied for $0 < \mu \le 0.2$.

In addition, the given functions τ_k *are continuous on their domains,* $\tau_k(x, y) \to \infty$ *as* $k \to \infty$ *uniformly on* $(x, y) \in R^4$*, and also*

$$0 < \tau_1(x, y) < \tau_2(x, y) < \ldots, \; (x, y) \in \mathbb{R}^4.$$

Condition 3 of Theorem 1 is also true since $\gamma_{ik} = 1 - \frac{1}{2k}$, $\mu_{jk} = 1 - \frac{1}{3k}$ *for* $i, j = 1, 2$, $k = 1, 2, \ldots$.

Therefore, according to Theorem 1, we conclude that the zero equilibrium $(x^*, y^*) = (0, 0)$ *of the model (6), (7) is globally exponentially stable with respect to the function* $h = \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}$*. The stable behavior of each neural state is shown in Figure 1.*

Example 2. *Let again consider the Cohen–Grossberg impulsive BAM neural network with time-varying delays (6), and replace the impulsive condition (7) with the following equations*

$$x(t^+) - x(t) = \begin{pmatrix} -1+\frac{1}{2k} & 0 \\ 0 & \frac{1}{2k} \end{pmatrix} x(t), \quad t = \tau_k(x(t), y(t)), \quad k = 1, 2, \ldots,$$
$$y(t^+) - y(t) = \begin{pmatrix} -1+\frac{1}{3k} & 0 \\ 0 & -1+\frac{1}{3k} \end{pmatrix} y(t), \quad t = \tau_k(x(t), y(t)), \quad k = 1, 2, \ldots. \tag{8}$$

Since we have that

$$\gamma_{2k} = -\frac{1}{2k} < 0, \quad k = 1, 2, \ldots,$$

then the condition 3 of Theorem 1 is not satisfied. Hence, we can not make any conclusion on the stability behavior of the equilibrium state of (6). For example, in this case when $\tau_k(x, y) = |x| + |y| + k$, the unstable behavior of the node $x_2(t)$ with respect to the function $h = \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}$ is shown in Figure 2.

Remark 3. *The presented examples not only show the effectiveness of the proposed theoretical results, but also illustrated how the stability behavior of a class of Cohen–Grossberg BAM delayed neural networks can be controlled via appropriate impulsive perturbations.*

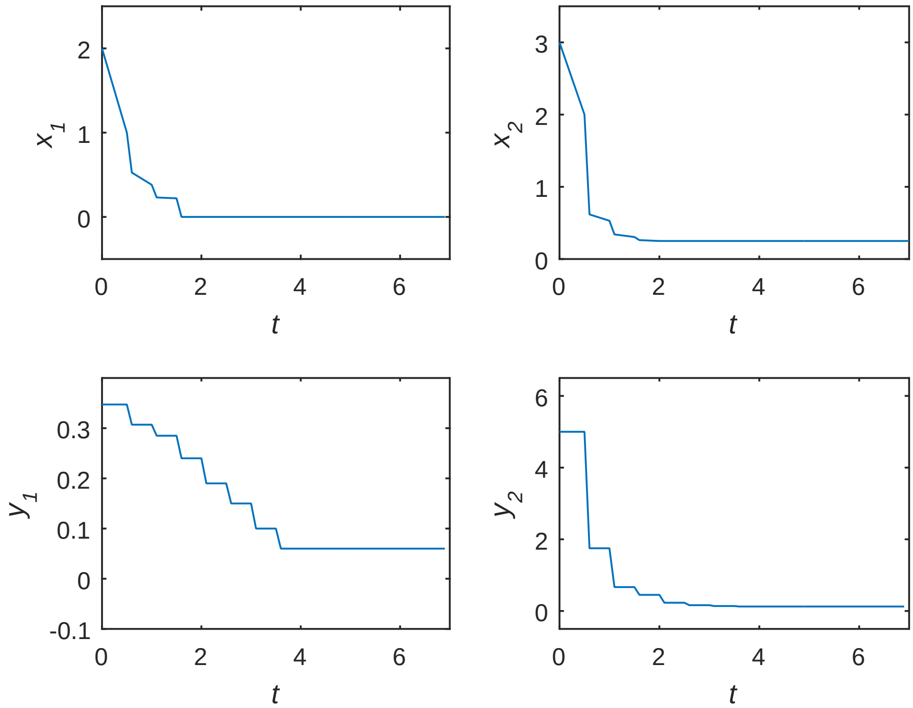

Figure 1. The global exponentially stable behavior of model (6), (7) with respect to the function $h = \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}$. Behavior of the state variables $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$.

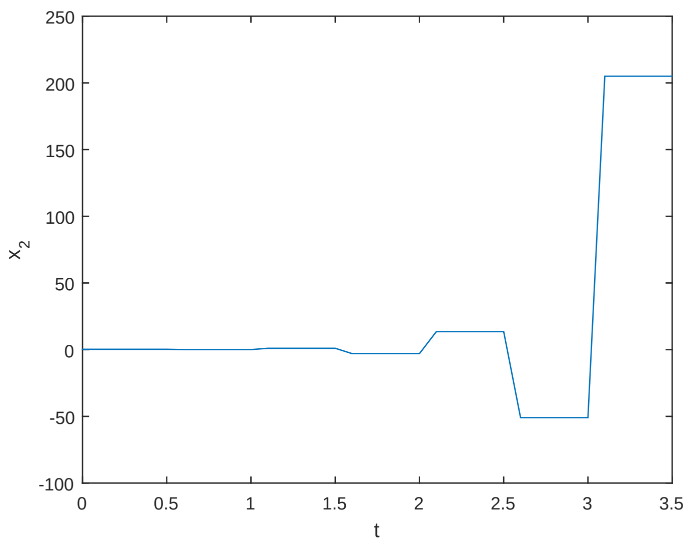

Figure 2. The unstable behavior of the state variable $x_2(t)$ of the model (6) with impulsive perturbations (8).

5. Conclusions

In this paper, the important notion of global exponential stability of single solutions of Cohen–Grossberg BAM impulsive neural networks with time-varying delays is extended and generalized. We introduce the concept of stability with respect to a manifold defined by a function h with specific properties. Thus, our research generalize some existing results in the literature on global exponential stability of solutions of impulsive BAM Cohen–Grossberg neural networks. In addition, instead of impulsive effects at fixed moments of time, we consider variable impulsive perturbations. The proposed notion and the results obtained in the paper can be extended to various other types of impulsive neural network models.

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