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On Two Bivariate Kinds of Poly-Bernoulli and Poly-Genocchi Polynomials

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Abstract: In this paper, we introduce two bivariate kinds of poly-Bernoulli and poly-Genocchi polynomials and study their basic properties. Finally, we consider some relationships for Stirling numbers of the second kind related to bivariate kinds of poly-Bernoulli and poly-Genocchi polynomials.

Keywords: poly-Bernoulli polynomials; poly-Genocchi polynomials; Appell polynomials; generating functions; Stirling numbers of the second kind

MSC: 11B68; 11C08; 11Y35

1. Introduction

Numerous mathematicians including Kim and Ryoo [1], Kim and Kim [2], Kim et al. [3–5], Khan [6,7] have concentrated their study on polynomials and its combination with Bernoulli, Genocchi, Euler, and tangent numbers. One of the essential classes of these sequences is the class of Appell polynomials. Various numerical problem of functional equations associated with pure and applied mathematics in the theory of approximation, differential equations, summation techniques, interpolation problems, quadrature rules, and their multidimensional extensions (see [8,9]). The Appell polynomials $A_n(z)$ are defined by means of the following generating function

$$A(t)e^{zt} = A_0(z) + A_1(z)\frac{t}{1!} + A_2(z)\frac{t^2}{2!} + \cdots + A_n(z)\frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} A_n(z)\frac{t^n}{n!}, \tag{1}$$

where

$$A(t) = A_0 + A_1\frac{t}{1!} + A_2\frac{t^2}{2!} + \cdots + A_n\frac{t^n}{n!} + \cdots, \quad A_0 \neq 0.$$

Differentiating generating function (1) with respect z and equating the coefficients of $\frac{t^n}{n!}$, we have

$$\frac{d}{dz}A_n(z) = nA_{n-1}(z), \quad A_0(z) \neq 0, z = x + iy \in \mathbb{C}, \quad n \in \mathbb{N}.$$

The special cases of Appell polynomials are the poly-Bernoulli and poly-Genocchi polynomials, (see [4,10]).

The poly-Bernoulli polynomials are defined by, (see [2–7,11])

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \tag{2}$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, \quad (k \in \mathbb{Z})$$

is called the classical polylogarithm function, (see [1–7,10,11]).

For $k = 1$ in (2), we have

$$\frac{\mathrm{Li}_1(1-e^{-t})}{e^t-1}e^{xt} = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{3}$$

where $B_n(x)$ are called the Bernoulli polynomials, (see [1–16]).

In (2015), Kim et al. [10] introduced the poly-Genocchi polynomials are defined by means of the following generating function

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}. \tag{4}$$

For $k = 1$, we have

$$\frac{2\mathrm{Li}_1(1-e^{-t})}{e^t+1}e^{xt} = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \tag{5}$$

where $G_n(x)$ are called the Genocchi polynomials, (see [3,14]).

The Stirling numbers of the first kind are defined by the coefficients in the expansion of $(x)_n$ in terms of power of x as follows, (see [1,2,7])

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 0). \tag{6}$$

Subsequently, the Stirling numbers of the second kind are defined by, (see [2,4,5])

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, (n \geq 0). \tag{7}$$

Recently, Jamei et al. [13,14] introduced and investigated the new type of Bernoulli and Genocchi polynomials defined by means of the following generating function

$$\frac{t}{e^t-1}e^{xt}\cos yt = \sum_{n=0}^{\infty} B_n^{(c)}(x,y)\frac{t^n}{n!}, \tag{8}$$

$$\frac{t}{e^t-1}e^{xt}\sin yt = \sum_{n=0}^{\infty} B_n^{(s)}(x,y)\frac{t^n}{n!}, \tag{9}$$

and

$$\frac{2t}{e^t+1}e^{xt}\cos yt = \sum_{n=0}^{\infty} G_n^{(c)}(x,y)\frac{t^n}{n!}, \tag{10}$$

$$\frac{2t}{e^t+1}e^{xt}\sin yt = \sum_{n=0}^{\infty} G_n^{(s)}(x,y)\frac{t^n}{n!}, \tag{11}$$

respectively.

They have also considered the two functions $e^{xt}\cos yt$ and $e^{xt}\sin yt$ as follows (see [12–16]):

$$e^{xt}\cos yt = \sum_{k=0}^{\infty} C_k(x,y)\frac{t^k}{k!}, \tag{12}$$

and

$$e^{xt}\sin yt = \sum_{k=0}^{\infty} S_k(x,y)\frac{t^k}{k!}, \tag{13}$$

where

$$C_k(x,y) = \sum_{j=0}^{[\frac{k}{2}]}(-1)^j \binom{k}{2j} x^{k-2j}y^{2j}, \tag{14}$$

and

$$S_k(x,y) = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k}{2j+1} (-1)^j x^{k-2j-1}y^{2j+1}. \tag{15}$$

In (2018), Kim and Ryoo [1] introduced the cosine Bernoulli polynomials of a complex variable, the sine Bernoulli polynomials of a complex variable and the cosine Euler polynomials of a complex variable, the sine Euler polynomials of a complex variable, respectively are defined as follows

$$\frac{t}{e^t-1}e^{(x+iy)t} = \sum_{n=0}^{\infty} B_n(x+iy)\frac{t^n}{n!}, \tag{16}$$

and

$$\frac{2}{e^t+1}e^{(x+iy)t} = \sum_{n=0}^{\infty} E_n(x+iy)\frac{t^n}{n!}. \tag{17}$$

From (16) and (17), we get

$$\frac{t}{e^t-1}e^{xt}\cos yt = \sum_{n=0}^{\infty}\frac{B_n(x+iy)+B_n(x-iy)}{2}\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{(c)}(x,y)\frac{t^n}{n!},$$

and

$$\frac{t}{e^t-1}e^{xt}\sin yt = \sum_{n=0}^{\infty}\frac{B_n(x+iy)-B_n(x-iy)}{2i}\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{(s)}(x,y)\frac{t^n}{n!},$$

$$\frac{2}{e^t+1}e^{xt}\cos yt = \sum_{n=0}^{\infty}\frac{E_n(x+iy)+E_n(x-iy)}{2}\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{(c)}(x,y)\frac{t^n}{n!},$$

and

$$\frac{2}{e^t+1}e^{xt}\sin yt = \sum_{n=0}^{\infty}\frac{E_n(x+iy)-E_n(x-iy)}{2i}\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{(s)}(x,y)\frac{t^n}{n!}.$$

This article is organized as follows. In Section 2, we introduce the cosine poly-Bernoulli and sine poly-Bernoulli polynomials and derive some identities of these polynomials. In Section 3, we establish the cosine poly-Genocchi and sine poly-Genocchi polynomials and derive some identities of these polynomials. Finally Section 4, we investigated some relationships for Stirling numbers of the second kind related to poly-Bernoulli and poly-Genocchi polynomials.

2. Poly-Bernoulli Polynomials of Complex Variable

This section presents sine and cosine variant of poly-Bernoulli polynomials. These variants are processed by separating the real $\Re$ and imaginary $\Im$ parts of the complex poly-Bernoulli polynomials and study on their basic properties are expressed. Now, we consider the poly-Bernoulli polynomials that are given by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{(x+iy)t}=\sum_{n=0}^{\infty}B_n^{(k)}(x+iy)\frac{t^n}{n!}. \tag{18}$$

On the other hand, we observe that, (see [1])

$$e^{(x+iy)t}=e^{xt}e^{iyt}=e^{xt}(\cos yt+i\sin yt), \tag{19}$$

Thus, by (18) and (19), we have

$$\sum_{n=0}^{\infty}B_n^{(k)}(x+iy)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{(x+iy)t}=\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}(\cos yt+i\sin yt), \tag{20}$$

and

$$\sum_{n=0}^{\infty}B_n^{(k)}(x-iy)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{(x-iy)t}=\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}(\cos yt-i\sin yt). \tag{21}$$

From (20) and (21), we get

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}\cos yt=\sum_{n=0}^{\infty}\left(\frac{B_n^{(k)}(x+iy)+B_n^{(k)}(x-iy)}{2}\right)\frac{t^n}{n!}, \tag{22}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}\sin yt=\sum_{n=0}^{\infty}\left(\frac{B_n^{(k)}(x+iy)-B_n^{(k)}(x-iy)}{2i}\right)\frac{t^n}{n!}. \tag{23}$$

Definition 1. *The two bivariate kinds of cosine poly-Bernoulli polynomials $B_n^{(k,c)}(x,y)$ and sine poly-Bernoulli polynomials $B_n^{(k,s)}(x,y)$, for non negative integer n are defined by*

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}\cos yt=\sum_{n=0}^{\infty}B_n^{(k,c)}(x,y)\frac{t^n}{n!},(k\in\mathbb{Z}) \tag{24}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}\sin yt=\sum_{n=0}^{\infty}B_n^{(k,s)}(x,y)\frac{t^n}{n!},(k\in\mathbb{Z}) \tag{25}$$

respectively.

Note that $B_n^{(k,c)}(x,0)=B_n^{(k,c)}(x), B_n^{(k,s)}(x,0)=0,(n\geq 0)$.
For instance, we have

$$B_0^{(2,c)}(x,y)=1,\quad B_1^{(2,c)}(x,y)=-\frac{3}{4}+x,\quad B_2^{(2,c)}(x,y)=\frac{17}{36}-\frac{3x}{2}+x^2-y^2,$$

$$B_3^{(2,c)}(x,y)=-\frac{5}{24}+\frac{17x}{12}-\frac{9x^2}{4}+x^3+\frac{9y^2}{4}-3xy^2,$$

$$B_4^{(2,c)}(x,y)=\frac{7}{450}-\frac{5x}{6}+\frac{17x^2}{6}-3x^3+x^4-\frac{17y^2}{6}+9xy^2-6x^2y^2+y^4,$$

$$B_5^{(2,c)}(x,y)=\frac{7}{120}+\frac{7x}{90}-\frac{25x^2}{12}+\frac{85x^3}{18}-\frac{15x^4}{4}+x^5+\frac{25y^2}{12}-\frac{85xy^2}{6}+\frac{45x^2y^2}{2}-10x^3y^2-\frac{15y^4}{4}+5xy^4,$$

$$B_6^{(2,c)}(x,y)=-\frac{38}{2205}+\frac{7x}{20}+\frac{7x^2}{30}-\frac{25x^3}{6}+\frac{85x^4}{12}-\frac{9x^5}{2}+x^6-\frac{7y^2}{30}+\frac{25xy^2}{2}-\frac{85x^2y^2}{2}$$
$$+45x^3y^2-15x^4y^2+\frac{85y^4}{12}-\frac{45xy^4}{2}+15x^2y^4-y^6,$$

and

$$B_0^{(2,s)}(x,y)=0,\quad B_1^{(2,s)}(x,y)=y,$$

$$B_2^{(2,s)}(x,y)=-\frac{3y}{2}+2xy,$$

$$B_3^{(2,s)}(x,y)=\frac{17y}{12}-\frac{9xy}{2}+3x^2y-y^3,$$

$$B_4^{(2,s)}(x,y)=-\frac{5y}{6}+\frac{17xy}{3}-9x^2y+4x^3y+3y^3-4xy^3,$$

$$B_5^{(2,s)}(x,y)=\frac{7y}{90}-\frac{25xy}{6}+\frac{85x^2y}{6}-15x^3y+5x^4y-\frac{85y^3}{18}+15xy^3-10x^2y^3+y^5,$$

$$B_6^{(2,s)}(x,y)=\frac{7y}{20}+\frac{7xy}{15}-\frac{25x^2y}{2}+\frac{85x^3y}{3}-\frac{45x^4y}{2}+6x^5y+\frac{25y^3}{6}-\frac{85xy^3}{3}$$
$$+45x^2y^3-20x^3y^3-\frac{9y^5}{2}+6xy^5.$$

From (22)–(25), we have

$$B_n^{(k,c)}(x,y)=\frac{B_n^{(k)}(x+iy)+B_n^{(k)}(x-iy)}{2},\tag{26}$$

$$B_n^{(k,s)}(x,y)=\frac{B_n^{(k)}(x+iy)-B_n^{(k)}(x-iy)}{2i}.\tag{27}$$

Remark 1. *For $x=0$ in (24) and (25), we get new type polynomials as follows*

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\cos yt=\sum_{n=0}^{\infty}B_n^{(k,c)}(y)\frac{t^n}{n!},(k\in\mathbb{Z})\tag{28}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\sin yt=\sum_{n=0}^{\infty}B_n^{(k,s)}(y)\frac{t^n}{n!},(k\in\mathbb{Z})\tag{29}$$

respectively.

It is clear that

$$B_n^{(k,c)}(0)=B_n^{(k,c)},B_n^{(k,s)}(y)=0,(n\geq 0).$$

From (28) and (29), we can derive the following equations

$$\sum_{n=0}^{\infty}B_n^{(k,c)}(y)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\cos yt$$

$$=\sum_{n=0}^{\infty}B_n^{(k,c)}\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^m y^{2m}\frac{t^n}{2m!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}(-1)^m y^{2m}B_{n-2m}^{(k)}\right)\frac{t^n}{n!},\tag{30}$$

and

$$\sum_{n=0}^{\infty}B_n^{(k,s)}(y)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\sin yt$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\left[\frac{n-1}{2}\right]} \begin{pmatrix} n \\ 2m+1 \end{pmatrix} (-1)^m y^{2m+1} B_{n-2m-1}^{(k)} \right) \frac{t^n}{n!}. \tag{31}$$

Therefore, by (30) and (31), we get

$$B_n^{(k,c)}(y) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \begin{pmatrix} n \\ 2m \end{pmatrix} (-1)^m y^{2m} B_{n-2m}^{(k)}, \tag{32}$$

and

$$B_n^{(k,s)}(y) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \begin{pmatrix} n \\ 2m+1 \end{pmatrix} (-1)^m y^{2m+1} B_{n-2m-1}^{(k)}. \tag{33}$$

Now, we start some basic properties of these polynomials.

Theorem 1. *For $n \geq 0$, we have*

$$B_n^{(k)}(x+iy) = \sum_{l=0}^{n} \binom{n}{l} (x+iy)^{n-l} B_l^{(k)}$$

$$= \sum_{l=0}^{n} \binom{n}{l} (iy)^{n-l} B_l^{(k)}(x), \tag{34}$$

and

$$B_n^{(k)}(x-iy) = \sum_{l=0}^{n} \binom{n}{l} (x-iy)^{n-l} B_l^{(k)}$$

$$= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} (iy)^{n-l} B_l^{(k)}(x). \tag{35}$$

Proof. By using (20) and (21), we can easily get. So we omit the proof. □

Theorem 2. *$B_n^{(k,c)}(x,y)$ and $B_n^{(k,s)}(x,y)$ can be represented in terms of poly-Bernoulli numbers as follows*

$$B_n^{(k,c)}(x,y) = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} B_{n-m}^{(k)} C_m(x,y), \tag{36}$$

and

$$B_n^{(k,s)}(x,y) = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} B_{n-m}^{(k)} S_m(x,y). \tag{37}$$

Proof. By noting the general identity, we have

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} b_n \frac{t^m}{m!} \right) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_{n-m} b_m \right) \frac{t^n}{n!}.$$

Now

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(x,y) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} \left(e^{xt} \cos yt\right) = \left(\sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} C_m(x,y) \frac{t^m}{m!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} B_{n-m}^{(k)} C_m(x,y) \right) \frac{t^n}{n!},$$

which proves (36). The proof of (37) is similar. □

Theorem 3. *For every $n \in \mathbb{Z}^+$, the following formula holds true*

$$B_n^{(k,c)}(1-x,y) = (-1)^n B_n^{(k,c)}(x,y), \tag{38}$$

and

$$B_n^{(k,s)}(1-x,y) = (-1)^{n+1} B_n^{(k,s)}(x,y). \tag{39}$$

Proof. From (24), we have

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(1-x,y)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})e^{(1-x)t}}{e^t-1}\cos yt,$$

as well as

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n B_n^{(k,c)}(x,y)\frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-e^{t})e^{-xt}}{e^{-t}-1}\cos(-yt)\\ &= \frac{\mathrm{Li}_k(1-e^{-t})e^{(1-x)t}}{e^t-1}\cos(yt).\end{aligned}$$

Similarly Equation (39) can be proved. □

Corollary 1. *For every $n \in \mathbb{Z}^+$, we have*

$$B_{2n+1}^{(k,c)}\left(\frac{1}{2},y\right) = 0,$$

and

$$B_{2n}^{(k,s)}\left(\frac{1}{2},y\right) = 0.$$

Theorem 4. *For every $n \in \mathbb{Z}^+$, the following formula holds true*

$$B_n^{(k,c)}(x+r,y) = \sum_{m=0}^{n}\binom{n}{m} B_m^{(k,c)}(x,y)r^{n-m}, \tag{40}$$

and

$$B_n^{(k,s)}(x+r,y) = \sum_{m=0}^{n}\binom{n}{m} B_m^{(k,s)}(x,y)r^{n-m}. \tag{41}$$

Proof. Replacing x by $x+r$ in (24), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(k,c)}(x+r,y)\frac{t^n}{n!} &= \left(\frac{\mathrm{Li}_k(1-e^{-t})e^{xt}}{e^t-1}\cos yt\right)e^{rt}\\ &= \left(\sum_{n=0}^{\infty} B_n^{(k,c)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} r^n\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} B_m^{(c,k)}(x,y)r^{n-m}\right)\frac{t^n}{n!},\end{aligned}$$

which proves (40). The result (41) can be similarly proved. □

Theorem 5. *For every $n \in \mathbb{N}$, the following formula holds true*

$$\frac{\partial B_n^{(k,c)}(x,y)}{\partial x} = nB_{n-1}^{(k,c)}(x,y), \tag{42}$$

$$\frac{\partial B_n^{(k,c)}(x,y)}{\partial y} = -nB_{n-1}^{(k,s)}(x,y), \tag{43}$$

and

$$\frac{\partial B_n^{(k,s)}(x,y)}{\partial x} = nB_{n-1}^{(k,s)}(x,y), \tag{44}$$

$$\frac{\partial B_n^{(k,c)}(x,y)}{\partial y} = nB_{n-1}^{(k,c)}(x,y). \tag{45}$$

Proof. Equation (24) yields

$$\sum_{n=1}^{\infty} \frac{\partial B_n^{(k,c)}(x,y)}{\partial x} \frac{t^n}{n!} = \frac{t\mathrm{Li}_k(1-e^{-t})e^{xt}}{e^t-1}\cos yt = \sum_{n=0}^{\infty} B_n^{(k,c)}(x,y)\frac{t^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} B_{n-1}^{(k,c)}(x,y)\frac{t^n}{(n-1)!} = \sum_{n=1}^{\infty} nB_{n-1}^{(k,c)}(x,y)\frac{t^n}{n!},$$

proving (42). Other (43), (44) and (45) can be similarly derived. □

Theorem 6. *For $n \geq 0$, the following formula holds true*

$$B_n^{(2,c)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m m!}{m+1} B_{n-m}^{(c)}(x,y), \tag{46}$$

and

$$B_n^{(2,s)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m m!}{m+1} B_{n-m}^{(s)}(x,y). \tag{47}$$

Proof. From Equation (24), we have

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(x,y)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}\cos yt$$

$$= \frac{e^{xt}\cos yt}{e^t-1}\underbrace{\int_0^t \frac{1}{e^z-1}\int_0^t \frac{1}{e^z-1}\cdots\frac{1}{e^z-1}}_{(k-1)-\text{times}}\int_0^t \frac{z}{e^z-1}dz\cdots dz.$$

In particular for $k = 2$, we have

$$\sum_{n=0}^{\infty} B_n^{(2,c)}(x,y)\frac{t^n}{n!} = \frac{e^{xt}\cos yt}{e^t-1}\int_0^t \frac{z}{e^z-1}dz = \left(\sum_{m=0}^{\infty}\frac{t^m B_m}{m+1}\right)\frac{t}{e^t-1}e^{xt}\cos yt$$

$$= \left(\sum_{m=0}^{\infty}\frac{B_m m!}{m+1}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} B_n^{(c)}(x,y)\frac{t^n}{n!}\right).$$

Replacing n by $n-m$ in R.H.S. of above equation, we have

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m}\frac{B_m m!}{m+1}B_{n-m}^{(c)}(x,y)\frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the result (46). The proof of (47) is similar. □

3. Poly-Genocchi Polynomials of Complex Variable

This section presents sine and cosine variant of poly-Genocchi polynomials. These variants are processed by separating the real $\Re$ and imaginary $\Im$ parts of the complex poly-Genocchi polynomials and study on their basic properties are expressed. Now, we consider the poly-Genocchi polynomials that are given by the generating function

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{(x+iy)t}=\sum_{n=0}^{\infty}G_n^{(k)}(x+iy)\frac{t^n}{n!}. \tag{48}$$

By using (48) and (19), we have

$$\sum_{n=0}^{\infty}G_n^{(k)}(x+iy)\frac{t^n}{n!}=\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{(x+iy)t}=\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}(\cos yt+i\sin yt), \tag{49}$$

and

$$\sum_{n=0}^{\infty}G_n^{(k)}(x-iy)\frac{t^n}{n!}=\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{(x-iy)t}=\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}(\cos yt-i\sin yt). \tag{50}$$

From (49) and (50), we get

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt}\cos yt=\sum_{n=0}^{\infty}\left(\frac{G_n^{(k)}(x+iy)+G_n^{(k)}(x-iy)}{2}\right)\frac{t^n}{n!}, \tag{51}$$

and

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt}\sin yt=\sum_{n=0}^{\infty}\left(\frac{G_n^{(k)}(x+iy)-G_n^{(k)}(x-iy)}{2i}\right)\frac{t^n}{n!}. \tag{52}$$

Definition 2. *The two bivariate kinds of cosine poly-Genocchi polynomials $G_n^{(k,c)}(x,y)$ and sine poly-Genocchi polynomials $G_n^{(k,s)}(x,y)$, for non negative integer n are defined by*

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt}\cos yt=\sum_{n=0}^{\infty}G_n^{(k,c)}(x,y)\frac{t^n}{n!},(k\in\mathbb{Z}) \tag{53}$$

and

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt}\sin yt=\sum_{n=0}^{\infty}G_n^{(k,s)}(x,y)\frac{t^n}{n!},(k\in\mathbb{Z}) \tag{54}$$

respectively.

From (51)–(54), we have

$$G_n^{(k,c)}(x,y)=\frac{G_n^{(k)}(x+iy)+G_n^{(k)}(x-iy)}{2},$$

$$G_n^{(k,s)}(x,y)=\frac{G_n^{(k)}(x+iy)-G_n^{(k)}(x-iy)}{2i}.$$

Note that

$$G_n^{(k,c)}(x,0)=G_n^{(k)}(x),G_n^{(k,s)}(x,0)=0,(n\geq 0).$$

The cosine poly-Genocchi and sine poly-Genocchi polynomials can be determined explicitly. A few of them are

$$G_2^{(2,c)}(x,y)=-\frac{3}{2}+2x,$$

$$G_3^{(2,c)}(x,y) = \frac{11}{12} - \frac{9x}{2} + 3x^2 - 3y^2,$$

$$G_4^{(2,c)}(x,y) = \frac{2}{3} + \frac{11x}{3} - 9x^2 + 4x^3 + 9y^2 - 12xy^2,$$

$$G_5^{(2,c)}(x,y) = -\frac{77}{60} + \frac{10x}{3} + \frac{55x^2}{6} - 15x^3 + 5x^4 - \frac{55y^2}{6} + 45xy^2 - 30x^2y^2 + 5y^4,$$

$$G_6^{(2,c)}(x,y) = -\frac{31}{15} - \frac{77x}{10} + 10x^2 + \frac{55x^3}{3} - \frac{45x^4}{2} + 6x^5 - 10y^2 - 55xy^2 + 135x^2y^2 - 60x^3y^2 - \frac{45y^4}{2} + 30xy^4,$$

and

$$G_2^{(2,s)}(x,y) = 2y,$$

$$G_3^{(2,s)}(x,y) = -\frac{9y}{2} + 6xy,$$

$$G_4^{(2,s)}(x,y) = \frac{11y}{3} - 18xy + 12x^2y - 4y^3,$$

$$G_5^{(2,s)}(x,y) = \frac{10y}{3} + \frac{55xy}{3} - 45x^2y + 20x^3y + 15y^3 - 20xy^3,$$

$$G_6^{(2,s)}(x,y) = -\frac{77y}{10} + 20xy + 55x^2y - 90x^3y + 30x^4y - \frac{55y^3}{3} + 90xy^3 - 60x^2y^3 + 6y^5.$$

Remark 2. *For $x = 0$ in (53) and (54), we get new type polynomials as follows*

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\cos yt = \sum_{n=0}^{\infty} G_n^{(k,c)}(y)\frac{t^n}{n!}, (k \in \mathbb{Z}) \tag{55}$$

and

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\sin yt = \sum_{n=0}^{\infty} G_n^{(k,s)}(y)\frac{t^n}{n!}, (k \in \mathbb{Z}) \tag{56}$$

respectively.

It is clear that

$$G_n^{(k,c)}(0) = G_n^{(k,c)}, G_n^{(k,s)}(y) = 0, (n \geq 0).$$

From (55) and (56), we can derive the following equations

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(k,c)}(y)\frac{t^n}{n!} &= \frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\cos yt \\ &= \sum_{n=0}^{\infty} G_n^{(k)}\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^m y^{2m}\frac{t^n}{2m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n}{2m}(-1)^m y^{2m} G_{n-2m}^{(k)}\right)\frac{t^n}{n!},\end{aligned} \tag{57}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(k,s)}(y)\frac{t^n}{n!} &= \frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\sin yt \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2m+1}(-1)^m y^{2m+1} G_{n-2m-1}^{(k)}\right)\frac{t^n}{n!}.\end{aligned} \tag{58}$$

Therefore, by (57) and (58), we get

$$G_n^{(k,c)}(y) = \sum_{m=0}^{[\frac{n}{2}]} \left(\begin{array}{c} n \\ 2m \end{array} \right) (-1)^m y^{2m} G_{n-2m}^{(k)}, \tag{59}$$

and

$$G_n^{(k,s)}(y) = \sum_{m=0}^{[\frac{n-1}{2}]} \left(\begin{array}{c} n \\ 2m+1 \end{array} \right) (-1)^m y^{2m+1} G_{n-2m-1}^{(k)}. \tag{60}$$

Theorem 7. *For $n \geq 0$, we have*

$$G_n^{(k)}(x+iy) = \sum_{l=0}^{n} \binom{n}{l} (x+iy)^{n-l} G_l^{(k)}$$

$$= \sum_{l=0}^{n} \binom{n}{l} (iy)^{n-l} G_l^{(k)}(x), \tag{61}$$

and

$$G_n^{(k)}(x-iy) = \sum_{l=0}^{n} \binom{n}{l} (x-iy)^{n-l} G_l^{(k)}$$

$$= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} (iy)^{n-l} G_l^{(k)}(x). \tag{62}$$

Proof. By using (50) and (51), we can easily get. So we omit the proof. □

Theorem 8. *$G_n^{(k,c)}(x,y)$ and $G_n^{(k,s)}(x,y)$ can be represented in terms of poly-Genocchi numbers as follows*

$$G_n^{(k,c)}(x,y) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) G_{n-m}^{(k)} C_m(x,y). \tag{63}$$

and

$$G_n^{(k,s)}(x,y) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) G_{n-m}^{(k)} S_m(x,y). \tag{64}$$

Proof. By noting the general identity, we have

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} b_n \frac{t^m}{m!} \right) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_{n-m} b_m \right) \frac{t^n}{n!}.$$

Now

$$\sum_{n=0}^{\infty} G_n^{(k,c)}(x,y) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} \left(e^{xt} \cos yt \right) = \left(\sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} C_m(x,y) \frac{t^m}{m!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) G_{n-m}^{(k)} C_m(x,y) \right) \frac{t^n}{n!},$$

which proves (63). The proof of (64) is similar. □

Theorem 9. *For every $n \in \mathbb{Z}^+$, the following formula holds true*

$$G_n^{(k,c)}(1-x,y) = (-1)^n G_n^{(k,c)}(x,y), \tag{65}$$

and

$$G_n^{(k,s)}(1-x,y) = (-1)^{n+1} G_n^{(k,s)}(x,y). \tag{66}$$

Proof. From (53), we have

$$\sum_{n=0}^{\infty} G_n^{(k,c)}(1-x,y)\frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-e^{-t})e^{(1-x)t}}{e^t+1}\cos yt,$$

as well as

$$\sum_{n=0}^{\infty}(-1)^n G_n^{(k,c)}(x,y)\frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-e^{t})e^{-xt}}{e^{-t}+1}\cos(-yt)$$

$$= \frac{\mathrm{Li}_k(1-e^{-t})e^{(1-x)t}}{e^t-1}\cos(yt).$$

Similarly Equation (66) can be proved. □

Theorem 10. *For every $n \in \mathbb{Z}^+$, the following formula holds true*

$$G_n^{(k,c)}(x+r,y) = \sum_{m=0}^{n}\binom{n}{m} G_m^{(k,c)}(x,y)r^{n-m}, \tag{67}$$

and

$$G_n^{(k,s)}(x+r,y) = \sum_{m=0}^{n}\binom{n}{m} G_m^{(k,s)}(x,y)r^{n-m}. \tag{68}$$

Proof. Replacing x by $x+r$ in (53). we have

$$\sum_{n=0}^{\infty} G_n^{(k,c)}(x+r,y)\frac{t^n}{n!} = \left(\frac{2\mathrm{Li}_k(1-e^{-t})e^{xt}}{e^t+1}\cos yt\right)e^{rt}$$

$$= \left(\sum_{n=0}^{\infty} G_n^{(k,c)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} r^n\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} G_m^{(k,c)}(x,y)r^{n-m}\right)\frac{t^n}{n!},$$

which proves (67). The result (68) can be similarly proved. □

Theorem 11. *For every $n \in \mathbb{N}$, the following formula holds true*

$$\frac{\partial G_n^{(k,c)}(x,y)}{\partial x} = nG_{n-1}^{(k,c)}(x,y), \tag{69}$$

$$\frac{\partial G_n^{(k,c)}(x,y)}{\partial y} = -nG_{n-1}^{(k,s)}(x,y), \tag{70}$$

and

$$\frac{\partial G_n^{(k,s)}(x,y)}{\partial x} = nG_{n-1}^{(k,s)}(x,y), \tag{71}$$

$$\frac{\partial G_n^{(k,s)}(x,y)}{\partial y} = nG_{n-1}^{(k,c)}(x,y). \tag{72}$$

Proof. Equation (53) yields

$$\sum_{n=1}^{\infty}\frac{\partial G_n^{(k,c)}(x,y)}{\partial x}\frac{t^n}{n!} = \frac{2t\mathrm{Li}_k(1-e^{-t})e^{xt}}{e^t+1}\cos yt = \sum_{n=0}^{\infty} G_n^{(k,c)}(x,y)\frac{t^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} G_{n-1}^{(k,c)}(x,y)\frac{t^n}{(n-1)!} = \sum_{n=1}^{\infty} nG_{n-1}^{(k,c)}(x,y)\frac{t^n}{n!},$$

proving (69). Other (70), (71) and (72) can be similarly derived. □

Theorem 12. *For $n \geq 0$, the following formula holds true*

$$G_n^{(2,c)}(x,y) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \frac{B_m m!}{m+1} G_{n-m}^{(c)}(x,y), \tag{73}$$

and

$$G_n^{(2,s)}(x,y) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \frac{B_m m!}{m+1} G_{n-m}^{(s)}(x,y). \tag{74}$$

Proof. From Equation (53), we have

$$\sum_{n=0}^{\infty} G_n^{(k,c)}(x,y)\frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt}\cos yt$$

$$= \frac{2e^{xt}\cos yt}{e^t+1}\underbrace{\int_0^t \frac{1}{e^z-1}\int_0^t \frac{1}{e^z-1}\cdots\frac{1}{e^z-1}}_{(k-1)-\text{times}}\int_0^t \frac{z}{e^z-1}dz\cdots dz.$$

In particular for $k = 2$, we have

$$\sum_{n=0}^{\infty} G_n^{(2,c)}(x,y)\frac{t^n}{n!} = \frac{2e^{xt}\cos yt}{e^t+1}\int_0^t \frac{z}{e^z-1}dz = \left(\sum_{m=0}^{\infty}\frac{t^m B_m}{m+1}\right)\frac{2t}{e^t+1}e^{xt}\cos yt$$

$$= \left(\sum_{m=0}^{\infty}\frac{B_m m!}{m+1}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} G_n^{(c)}(x,y)\frac{t^n}{n!}\right).$$

Replacing n by $n-m$ in R.H.S. of above equation, we have

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array} \right)\frac{B_m m!}{m+1}G_{n-m}^{(c)}(x,y)\frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the result (73). The proof of (74) is similar. □

4. Relationship between Stirling Numbers of the Second Kind

In this section, we prove some relationships for Stirling numbers of the second kind related to poly-Bernoulli polynomials of complex variable and poly-Genocchi polynomials of complex variable. We start a following theorem.

Theorem 13. *For every $n \in \mathbb{Z}^+$, the following formula holds true*

$$B_n^{(k,c)}(1+x,y) - B_n^{(k,c)}(x,y) = \sum_{p=1}^{n}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\left(\begin{array}{c} n \\ p \end{array} \right)C_{n-p}(x,y), \tag{75}$$

and

$$B_n^{(k,s)}(1+x,y) - B_n^{(k,s)}(x,y) = \sum_{p=1}^{n}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\left(\begin{array}{c} n \\ p \end{array} \right)S_{n-p}(x,y). \tag{76}$$

Proof. Using (24), we have

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(1+x,y)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})e^{xt}(e^t-1+1)}{e^t-1}\cos yt$$

$$= \mathrm{Li}_k(1-e^{-t})e^{xt}\cos yt + \frac{\mathrm{Li}_k(1-e^{-t})e^{xt}}{e^t-1}\cos yt$$

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(1+x,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n^{(k,c)}(x,y)\frac{t^n}{n!}$$

$$= \sum_{p=1}^{\infty}\left(\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\right)\frac{t^p}{p!}e^{xt}\cos yt,$$

$$= \left(\sum_{p=1}^{\infty}\left(\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}C_n(x,y)\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on either side, we get the result (75). The proof of (76) is similar. □

Corollary 2. *For $k=1$ in Theorem 4.1, we get*

$$B_n^{(c)}(1+x,y) - B_n^{(c)}(x,y) = nC_{n-1}(x,y),$$

and

$$B_n^{(s)}(1+x,y) - B_n^{(s)}(x,y) = nS_{n-1}(x,y).$$

Theorem 14. *For $n \geq 0$, the following formula holds true*

$$B_n^{(k,c)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)B_{n-i}^{(k,c)}(y), \tag{77}$$

and

$$B_n^{(k,s)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)B_{n-i}^{(k,s)}. \tag{78}$$

Proof. From (24), we have

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(x,y)\frac{t^n}{n!} = \left(\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\right)((e^t-)+1)^x\cos yt$$

$$= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\right)\sum_{n=0}^{\infty}\binom{x}{l}(e^t-1)^l$$

$$= \sum_{l=0}^{\infty}(x)_l\frac{(e^t-1)^l}{l!}\left(\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\right)\cos yt$$

$$= \sum_{l=0}^{\infty}(x)_l\sum_{n=0}^{\infty}S_2(,l)\frac{t^n}{n!}\sum_{n=0}^{\infty}B_n^{(k,c)}(y)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)B_{n-i}^{(k,c)}(y)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of t^n on both sides , we get (77). The proof of (78) is similar. □

Theorem 15. *For $n \geq 0$, the following formula holds true*

$$B_n^{(k,c)}(x,y) = \sum_{p=0}^{n} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k (p+1)} \binom{n}{p} B_{n-p}^{(c)}(x,y), \tag{79}$$

and

$$B_n^{(k,s)}(x,y) = \sum_{p=0}^{n} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k (p+1)} \binom{n}{p} B_{n-p}^{(s)}(x,y). \tag{80}$$

Proof. From Equation (24), we have

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(x,y) \frac{t^n}{n!} = \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t} \right) \left(\frac{t}{e^t - 1} e^{xt} \cos yt \right). \tag{81}$$

Now

$$\begin{aligned} \frac{1}{t}\mathrm{Li}_k(1-e^{-t}) &= \frac{1}{t} \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{l^k} = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} (1-e^{-t})^l, \\ &= \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} l! \sum_{p=l}^{\infty} (-1)^p S_2(p,l) \frac{t^p}{p!}, \\ &= \frac{1}{t} \sum_{p=1}^{\infty} \sum_{l=1}^{p} \frac{(-1)^{l+p}}{l^k} l! S_2(p,l) \frac{t^p}{p!}, \\ &= \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!}. \end{aligned} \tag{82}$$

Thus, by (81) and (82), we obtain

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(x,y) \frac{t^n}{n!} = \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!} \left(\sum_{n=0}^{\infty} B_n^{(c)}(x,y) \frac{t^n}{n!} \right).$$

Now replacing n by $n-p$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on either side, we get the result (79). The proof of (80) is similar. □

Theorem 16. *For $d \in \mathbb{N}$ with $d \equiv 1 (\mod 2)$, we have*

$$B_n^{(k,c)}(x,y) = \sum_{p=0}^{n} \binom{n}{p} d^{n-p-1} \sum_{l=0}^{p+1} \sum_{a=0}^{d-1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k} (-1)^a B_{n-p}^{(c)} \left(\frac{a+x}{d}, y \right), \tag{83}$$

and

$$B_n^{(k,s)}(x,y) = \sum_{p=0}^{n} \binom{n}{p} d^{n-p-1} \sum_{l=0}^{p+1} \sum_{a=0}^{d-1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k} (-1)^a B_{n-p}^{(s)} \left(\frac{a+x}{d}, y \right). \tag{84}$$

Proof. From Equation (24) can be written as

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(k,c)}(x,y) \frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} e^{xt} \cos yt \\ &= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t} \right) \left(\frac{t}{e^{dt} - 1} \sum_{a=0}^{d-1} (-1)^a e^{(a+x)t} \cos yt \right), \end{aligned}$$

$$= \left(\sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!} \right) \left(\sum_{m=0}^{\infty} d^{m-1} \sum_{a=0}^{d-1} (-1)^a B_n^{(c)} \left(\frac{a+x}{d}, y \right) \frac{t^n}{n!} \right).$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on either side, we get the result (83). The proof of (84) is similar. □

Theorem 17. *For every $n \in \mathbb{Z}^+$, the following formula holds true*

$$G_n^{(k,c)}(1+x,y) + G_n^{(k,c)}(x,y) = \sum_{p=1}^{n} \sum_{l=1}^{p} \frac{(-1)^{l+p}}{l^k} l! S_2(p,l) \left(\begin{array}{c} n \\ p \end{array} \right) C_{n-p}(x,y), \tag{85}$$

and

$$G_n^{(k,s)}(1+x,y) + G_n^{(k,s)}(x,y) = \sum_{p=1}^{n} \sum_{l=1}^{p} \frac{(-1)^{l+p}}{l^k} l! S_2(p,l) \left(\begin{array}{c} n \\ p \end{array} \right) S_{n-p}(x,y). \tag{86}$$

Proof. Using (53), we have

$$\sum_{n=0}^{\infty} G_n^{(k,s)}(1+x,y) \frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-e^{-t}) e^{xt}(e^t - 1 + 1)}{e^t + 1} \cos yt$$

$$= 2\mathrm{Li}_k(1-e^{-t}) e^{xt} \cos yt - \frac{2\mathrm{Li}_k(1-e^{-t}) e^{xt}}{e^t+1} \cos yt$$

$$\sum_{n=0}^{\infty} B_n^{(c,k)}(1+x,y) \frac{t^n}{n!} + \sum_{n=0}^{\infty} B_n^{(k,c)}(x,y) \frac{t^n}{n!}$$

$$= 2 \sum_{p=1}^{\infty} \left(\sum_{l=1}^{p} \frac{(-1)^{l+p}}{l^k} l! S_2(p,l) \right) \frac{t^p}{p!} e^{xt} \cos yt,$$

$$= 2 \left(\sum_{p=1}^{\infty} \left(\sum_{l=1}^{p} \frac{(-1)^{l+p}}{l^k} l! S_2(p,l) \right) \frac{t^p}{p!} \right) \left(\sum_{n=0}^{\infty} C_n(x,y) \frac{t^n}{n!} \right).$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on either side, we get the result (85). The proof of (86) is similar. □

Theorem 18. *For $n \geq 0$, the following formula holds true*

$$B_n^{(k,c)}(x,y) = \sum_{p=0}^{n} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k (p+1)} \left(\begin{array}{c} n \\ p \end{array} \right) G_{n-p}^{(c)}(x,y), \tag{87}$$

and

$$G_n^{(k,s)}(x,y) = \sum_{p=0}^{n} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k (p+1)} \left(\begin{array}{c} n \\ p \end{array} \right) G_{n-p}^{(s)}(x,y). \tag{88}$$

Proof. From Equation (53), we have

$$\sum_{n=0}^{\infty} B_n^{(k,c)}(x,y) \frac{t^n}{n!} = \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t} \right) \left(\frac{2t}{e^t+1} e^{xt} \cos yt \right). \tag{89}$$

Now

$$\frac{1}{t}\mathrm{Li}_k(1-e^{-t}) = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{l^k} = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} (1-e^{-t})^l,$$

$$= \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k}l!\sum_{p=l}^{\infty}(-1)^p S_2(p,l)\frac{t^p}{p!},$$

$$= \frac{1}{t}\sum_{p=1}^{\infty}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\frac{t^p}{p!},$$

$$= \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}. \tag{90}$$

Thus (89) and (90), we obtain

$$\sum_{n=0}^{\infty}G_n^{(k,c)}(x,y)\frac{t^n}{n!} = \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\left(\sum_{n=0}^{\infty}G_n^{(c)}(x,y)\frac{t^n}{n!}\right).$$

Now replacing n by $n-p$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on either side, we get the result (87). The proof of (88) is similar. □

Theorem 19. *For $d \in \mathbb{N}$ with $d \equiv 1(\mod 2)$, we have*

$$G_n^{(k,c)}(x,y) = \sum_{p=0}^{n}\binom{n}{p}d^{n-p-1}\sum_{l=0}^{p+1}\sum_{a=0}^{d-1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k}(-1)^a G_{n-p}^{(c)}\left(\frac{a+x}{d},y\right), \tag{91}$$

and

$$G_n^{(k,s)}(x,y) = \sum_{p=0}^{n}\binom{n}{p}d^{n-p-1}\sum_{l=0}^{p+1}\sum_{a=0}^{d-1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k}(-1)^a G_{n-p}^{(s)}\left(\frac{a+x}{d},y\right). \tag{92}$$

Proof. Now Equation (53) can be written as

$$\sum_{n=0}^{\infty}G_n^{(k,c)}(x,y)\frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt}\cos yt$$

$$= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t}\right)\left(\frac{2t}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a e^{(a+x)t}\cos yt\right),$$

$$= \left(\sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}d^{n-1}\sum_{a=0}^{d-1}(-1)^a G_n^{(c)}\left(\frac{a+x}{d},y\right)\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on either side, we get the result (91). The proof of (92) is similar. □

5. Conclusions

In this paper, we introduced the bivariate kind of poly-Bernoulli and poly-Genocchi polynomials by defining the two specific generating functions. We also investigate some analytical properties (for example, summation formulae, differential formulae and relations with other well-known polynomials and numbers) for our introduced polynomials in a systematic way. We also derived new identities and relations involving the Stirling numbers of the second kind. The results of this article may potentially be used in mathematics, in mathematical physics, and engineering.

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