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Certain Hadamard Proportional Fractional Integral Inequalities

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Abstract: In this present paper we study the non-local Hadamard proportional integrals recently proposed by Rahman et al. (Advances in Difference Equations, (2019) 2019:454) which containing exponential functions in their kernels. Then we establish certain new weighted fractional integral inequalities involving a family of n ($n \in \mathbb{N}$) positive functions by utilizing Hadamard proportional fractional integral operator. The inequalities presented in this paper are more general than the inequalities existing in the literature.

Keywords: fractional integrals; hadamard proportional fractional integrals; fractional integral inequalities

MSC: 26A33; 26D10; 26D53

1. Introduction

The field of fractional integral inequalities play an important role in the field of differential equations and applied mathematics. These inequalities have many applications in applied sciences such as probability, statistical problems, numerical quadrature and transform theory. In the last few decades, many mathematicians have paid their valuable considerations to this field and obtained a bulk of various fractional integral inequalities and their applications. The interested readers are referred to the work of [1–5] and the references cited therein. A variety of different kinds of certain classical integral inequalities and their extensions have been investigated by considering the classical Riemann–Liouville (RL) fractional integrals, fractional derivatives and their various extensions. In [6], the authors presented integral inequalities via generalized (k, s) -fractional integrals. Dahmani and Tabharit [7] proposed weighted Grüss-type inequalities by utilizing Riemann–Liouville fractional integrals. Dahmani [8] presented several new inequalities in fractional integrals. Nisar et al. [9] investigated several inequalities for extended gamma function and confluent hypergeometric k -function. Gronwall type inequalities associated with Riemann–Liouville k and Hadamard k -fractional derivative with applications can be found in the work of Nisar et al. [10]. Rahman et al. [11] presented certain inequalities for generalized fractional integrals. Sarikaya and Budak [12] investigated Ostrowski type inequalities by employing local fractional integrals. The generalized (k, s) -fractional integrals and their applications can be found in the work of Sarikaya et al. [13]. In [14], Set et al. proposed the generalized Grüss-type inequalities by employing generalized k -fractional integrals. Set et al. [15] have introduced the generalized version of Hermite–Hadamard type inequalities via fractional

integrals. Agarwal et al. [16] studied the Hermite–Hadamard type inequalities by considering the Riemann–Liouville k -fractional integrals. Fractional integral inequalities via Hadamard fractional integral, Saigo fractional integral and fractional q -integral operators are found in [17–19]. In [20], Huang et al. investigated Hermite–Hadamard type inequalities for k -fractional conformable integrals. Mubeen et al. [21] proposed the Minkowski’s inequalities involving the generalized k -fractional conformable integrals. The chebyshev type inequalities involving generalized k -fractional conformable integrals can be found in the work of Qi et al. [22]. Rahman et al. investigated Chebyshev type inequalities by utilizing fractional conformable integrals [23,24]. In [25,26], Nisar et al. proposed generalized Chebyshev-type inequalities and certain Minkowski’s type inequalities by employing generalized conformable integrals. Recently, Tassaddiq et al. [27] investigated certain inequalities for the weighted and the extended Chebyshev functionals by using fractional conformable integrals. Nisar et al. [28] established certain new inequalities for a class of n ($n \in \mathbb{N}$) positive continuous and decreasing functions by employing generalized conformable fractional integrals. Certain generalized fractional integral inequalities for Marichev–Saigo–Maeda (MSM) fractional integral operators were recently established by Nisar et al. [29]. Rahman et al. [30] recently investigated Grüss type inequalities for generalized k -fractional conformable integrals. In [31–33], Rahman et al. recently investigated Minkowski’s inequalities, fractional proportional integral inequalities and fractional proportional inequalities for convex functions by employing fractional proportional integrals.

Moreover, the recent research focuses to the development of the theory of fractional calculus and its applications in multiple disciplines of sciences. In last three centuries, the field of fractional calculus has earned more recognition due to its wide applications in diverse domains. Recently, several kinds of various fractional integral and derivative operators have been investigated. The idea of fractional conformable derivative operators with some drawbacks was proposed by Khalil et al. [34]. The properties of the fractional conformable derivative operators was investigated by Abdeljawad [35]. Abdeljawad and Baleanu [36] proposed several monotonicity results for fractional difference operators with discrete exponential kernels. Abdeljawad and Baleanu [37] have presented fractional derivative operator with exponential kernel and their discrete version. A new fractional derivative operator with the non-local and non-singular kernel was proposed by Atangana and Baleanu [38]. Jarad et al. [39] proposed the fractional conformable integral and derivative operators. The idea of conformable derivative by employing local proportional derivatives was proposed by Anderson and Unles [40]. In [41], Caputo and Fabrizio proposed fractional derivative without a singular kernel. Later on, Losada and Nieto [42] gave certain properties of fractional derivative without a singular kernel.

In [43], Liu et al. investigated several integral inequalities. Later on, Dahmani [44] proposed certain classes of weighted fractional integral inequalities for a family of n positive increasing and decreasing functions by utilizing Riemann–Liouville fractional integrals. Houas [45] utilized Hadamard fractional integrals and established several weighted type integral inequalities. Recently, Jarad et al. [46] and Rahman et al. [47] proposed the idea of non-local fractional proportional and Hadamard proportional integrals which concerning exponential functions in their kernels. The aim of this paper is to establish weighted type inequalities by using the non-local Hdamard proportional integrals. The paper is organized as follows. In Section 2, we present some basic definitions and mathematical formulas. In Section 3, we establish certain weighted Hadamard proportional fractional integral inequalities. Section 4 containing concluding remarks.

2. Preliminaries

This section is devoted to some well-known definitions and mathematical preliminaries of fractional calculus which will be used in this article. Jarad et al. [46] presented the left and right proportional integral operators.

Definition 1. The left and right sided proportional fractional integrals are respectively defined by

$$({}_a\mathcal{J}^{\tau,\nu}\mathcal{U})(\theta) = \frac{1}{\nu^\tau\Gamma(\tau)} \int_a^\theta \exp\left[\frac{\nu-1}{\nu}(\theta-t)\right](\theta-t)^{\tau-1}\mathcal{U}(t)dt, a < \theta \quad (1)$$

and

$$(\mathcal{J}_b^{\tau,\nu}\mathcal{U})(x) = \frac{1}{\nu^\tau\Gamma(\tau)} \int_\theta^b \exp\left[\frac{\nu-1}{\nu}(t-\theta)\right](t-\theta)^{\tau-1}\mathcal{U}(t)dt, \theta < b. \quad (2)$$

where the proportionality index $\nu \in (0, 1]$ and $\tau \in \mathbb{C}$ with $\Re(\tau) > 0$.

Remark 1. If we consider $\nu = 1$ in (1) and (2), then we get the well-known left and right Riemann–Liouville integrals which are respectively defined by

$$({}_a\mathcal{J}^\tau\mathcal{U})(x) = \frac{1}{\Gamma(\tau)} \int_a^\theta (\theta-t)^{\tau-1}\mathcal{U}(t)dt, a < \theta \quad (3)$$

and

$$(\mathcal{J}_b^\tau\mathcal{U})(\theta) = \frac{1}{\Gamma(\tau)} \int_\theta^b (t-\theta)^{\tau-1}\mathcal{U}(t)dt, \theta < b \quad (4)$$

where $\tau \in \mathbb{C}$ with $\Re(\tau) > 0$.

Recently, Rahman et al. [47] proposed the following generalized Hadamard proportional fractional integrals.

Definition 2. The left sided generalized Hadamard proportional fractional integral of order $\tau > 0$ and proportional index $\nu \in (0, 1]$ is defined by

$$({}_a\mathcal{H}^{\tau,\nu}\mathcal{U})(\theta) = \frac{1}{\nu^\tau\Gamma(\tau)} \int_a^\theta \exp\left[\frac{\nu-1}{\nu}(\ln\theta - \ln t)\right](\ln\theta - \ln t)^{\tau-1}\frac{\mathcal{U}(t)}{t}dt, a < \theta. \quad (5)$$

Definition 3. The right sided generalized Hadamard proportional fractional integral of order $\tau > 0$ and proportional index $\nu \in (0, 1]$ is defined by

$$(\mathcal{H}_b^{\tau,\nu}\mathcal{U})(\theta) = \frac{1}{\nu^\tau\Gamma(\tau)} \int_\theta^b \exp\left[\frac{\nu-1}{\nu}(\ln t - \ln\theta)\right](\ln t - \ln\theta)^{\tau-1}\frac{\mathcal{U}(t)}{t}dt, \theta < b. \quad (6)$$

Definition 4. The one sided generalized Hadamard proportional fractional integral of order $\tau > 0$ and proportional index $\nu \in (0, 1]$ is defined by

$$(\mathcal{H}_{1,\theta}^{\tau,\nu}\mathcal{U})(\theta) = \frac{1}{\nu^\tau\Gamma(\tau)} \int_1^\theta \exp\left[\frac{\nu-1}{\nu}(\ln\theta - \ln t)\right](\ln\theta - \ln t)^{\tau-1}\frac{\mathcal{U}(t)}{t}dt, \theta > 1, \quad (7)$$

where $\Gamma(\tau)$ is the classical well-known gamma function.

Remark 2. If we consider $\nu = 1$, then (5)–(7) will led to the following well-known Hadamard fractional integrals

$$({}_a\mathcal{H}^\tau\mathcal{U})(\theta) = \frac{1}{\Gamma(\tau)} \int_a^\theta (\ln\theta - \ln t)^{\tau-1}\frac{\mathcal{U}(t)}{t}dt, a < \theta, \quad (8)$$

$$(\mathcal{H}_b^\tau\mathcal{U})(\theta) = \frac{1}{\Gamma(\tau)} \int_\theta^b (\ln t - \ln\theta)^{\tau-1}\frac{\mathcal{U}(t)}{t}dt, \theta < b \quad (9)$$

and

$$(\mathcal{H}_{1,\theta}^\tau \mathcal{U})(\theta) = \frac{1}{\Gamma(\tau)} \int_1^\theta (\ln \theta - \ln t)^{\tau-1} \frac{\mathcal{U}(t)}{t} dt, \theta > 1. \quad (10)$$

One can easily prove the following results

Lemma 1.

$$\left(\mathcal{H}_{1,\theta}^{\tau,\nu} \exp\left[\frac{\nu-1}{\nu}(\ln \theta)\right] (\ln \theta)^{\lambda-1} \right) (\theta) = \frac{\Gamma(\lambda)}{\nu^\tau \Gamma(\tau + \lambda)} \exp\left[\frac{\nu-1}{\nu}(\ln \theta)\right] (\ln \theta)^{\tau+\lambda-1} \quad (11)$$

and the semi group property

$$\left(\mathcal{H}_{1,\theta}^{\tau,\nu} \right) \left(\mathcal{H}_{1,\theta}^{\lambda,\nu} \right) \mathcal{U}(\theta) = \left(\mathcal{H}_{1,\theta}^{\tau+\lambda,\nu} \right) \mathcal{U}(\theta). \quad (12)$$

Remark 3. If $\nu = 1$, then (11) will reduce to the result of [48] as defined by

$$\left(\mathcal{H}_{1,\theta}^\tau (\ln \theta)^{\lambda-1} \right) (\theta) = \frac{\Gamma(\lambda)}{\Gamma(\tau + \lambda)} (\ln \theta)^{\tau+\lambda-1}. \quad (13)$$

3. Main Results

In this section, we present certain new proportional fractional integral inequalities by utilizing Hadamard proportional fractional integral. Employ the left generalized proportional fractional integral operator to establish the generalization of some classical inequalities.

Theorem 1. Let the two functions \mathcal{U} and \mathcal{V} be positive and continuous on the interval $[1, \infty)$ and satisfy

$$(\mathcal{V}^\sigma(\zeta) \mathcal{U}^\sigma(\rho) - \mathcal{V}^\sigma(\rho) \mathcal{U}^\sigma(\zeta)) (\mathcal{U}^{\delta-\zeta}(\rho) - \mathcal{U}^{\delta-\zeta}(\zeta)) \geq 0, \quad (14)$$

where $\rho, \zeta \in (1, \theta), \theta > 1$ and for any $\sigma > 0, \delta \geq \zeta > 0$. Assume that the function $\mathcal{W} : [1, \infty) \rightarrow \mathbb{R}^+$ is positive and continuous. Then for all $\theta > 1$, the following inequality for Hadamard proportional fractional integral operator (7) holds;

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\delta+\sigma}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\zeta(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\zeta}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right], \end{aligned} \quad (15)$$

where $\delta \geq \zeta > 0, \tau, \nu, \sigma > 0$.

Proof. Consider the function

$$\mathcal{F}(\theta, \rho) = \frac{1}{\nu^\tau \Gamma(\tau)} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^\zeta(\rho)}{\rho}, \quad (16)$$

where $\tau > 0, \zeta > 0$ and $\rho \in (1, \theta), \theta > 1$.

Since the functions \mathcal{U} and \mathcal{V} satisfy (14) for all $\rho, \zeta \in (1, \theta), \theta > 1$ and for any $\sigma > 0, \delta \geq \zeta > 0$. Therefore, from (14), we have

$$\mathcal{V}^\sigma(\zeta) \mathcal{U}^{\sigma+\delta-\zeta}(\rho) + \mathcal{V}^\sigma(\rho) \mathcal{U}^{\sigma+\delta-\zeta}(\zeta) \geq \mathcal{V}^\sigma(\zeta) \mathcal{U}^\sigma(\rho) \mathcal{U}^{\delta-\zeta}(\zeta) + \mathcal{V}^\sigma(\rho) \mathcal{U}^\sigma(\zeta) \mathcal{U}^{\delta-\zeta}(\rho). \quad (17)$$

We observe that the function $\mathcal{F}(\theta, \rho)$ remains positive for all $\rho \in (1, \theta), \theta > 1$. Therefore multiplying (17) by $\mathcal{F}(\theta, \rho)$ (where $\mathcal{F}(\theta, \rho)$ is defined in (16)) and integrating the resultant estimates with respect to ρ over $(1, \theta)$, we get

$$\begin{aligned} & \mathcal{V}^\sigma(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^\xi(\rho)}{\rho} \mathcal{U}^{\sigma+\delta-\xi}(\rho) d\rho \\ & + \mathcal{U}^{\sigma+\delta-\xi}(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^\xi(\rho)}{\rho} \mathcal{V}^\sigma(\rho) d\rho \\ & \geq \mathcal{V}^\sigma(\zeta) \mathcal{U}^{\delta-\xi}(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^\xi(\rho)}{\rho} \mathcal{U}^\sigma(\rho) d\rho \\ & + \mathcal{U}^\sigma(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^\xi(\rho)}{\rho} \mathcal{V}^\sigma(\rho) \mathcal{U}^{\delta-\xi}(\rho) d\rho, \end{aligned}$$

which in view of (7) becomes,

$$\begin{aligned} & \mathcal{V}^\sigma(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\delta}(\theta) \right] + \mathcal{U}^{\sigma+\delta-\xi}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \\ & \geq \mathcal{V}^\sigma(\zeta) \mathcal{U}^{\delta-\xi}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\xi}(\rho) \right] + \mathcal{U}^\sigma(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right]. \end{aligned} \quad (18)$$

Now, multiplying (18) by $\mathcal{F}(\theta, \zeta)$ (where $\mathcal{F}(\theta, \zeta)$ can be obtain from (16) by replacing ρ by ζ) and integrating the resultant estimates with respect to ζ over $(1, \theta)$ and then by using (7), we obtain

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\delta}(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\delta}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\xi}(\rho) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\xi}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right], \end{aligned}$$

which gives the desired assertion (15). \square

Remark 4. If the inequality (14) reverses, then the inequality (15) will also reverse.

Theorem 2. Let the two functions \mathcal{U} and \mathcal{V} be positive and continuous on the interval $[1, \infty)$ and satisfy (14). Assume that the function $\mathcal{W} : [1, \infty) \rightarrow \mathbb{R}^+$ is positive and continuous. Then for all $\theta > 1$, the following inequality for Hadamard proportional fractional integral operator (7) holds;

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\delta}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\delta}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\xi}(\rho) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\xi}(\theta) \right], \end{aligned} \quad (19)$$

where $\delta \geq \xi > 0$, $\tau, \lambda, \nu, \sigma > 0$.

Proof. Taking product on both sides of (18) by $\mathcal{G}(\theta, \zeta) = \frac{1}{\nu^\lambda \Gamma(\lambda)} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta}$, where $\lambda > 0, \xi > 0$ and $\zeta \in (1, \theta), \theta > 1$. and integrating the resultant estimates with respect to ζ over $(1, \theta)$, we have

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\delta}(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta} \mathcal{V}^\sigma(\zeta) d\zeta \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta} \mathcal{U}^{\sigma+\delta-\xi}(\zeta) d\zeta \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\xi}(\rho) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta} \mathcal{V}^\sigma(\zeta) \mathcal{U}^{\delta-\xi}(\zeta) d\zeta \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta} \mathcal{U}^\sigma(\zeta) d\zeta, \end{aligned}$$

which in view of (7) yields,

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\delta}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^{\sigma}(\theta) \mathcal{U}^{\xi}(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^{\sigma}(\theta) \mathcal{U}^{\xi}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\delta}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\xi}(\rho) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^{\sigma}(\theta) \mathcal{U}^{\delta}(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^{\sigma}(\theta) \mathcal{U}^{\delta}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\sigma+\xi}(\theta) \right], \end{aligned}$$

which completes the proof of (19). \square

Remark 5. Applying Theorem 2 for $\tau = \lambda$, we get Theorem 1.

Theorem 3. Let the two functions \mathcal{U} and \mathcal{V} be positive and continuous on the interval $[1, \infty)$ such that the function \mathcal{U} is decreasing and the function \mathcal{V} is increasing on $[1, \infty)$ and assume that the function $\mathcal{W} : [1, \infty) \rightarrow \mathbb{R}^+$ is positive and continuous. Then for all $\theta > 1$, the following inequality for Hadamard proportional fractional integral operator (7) holds;

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\delta}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^{\sigma}(\theta) \mathcal{U}^{\xi}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\xi}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^{\sigma}(\theta) \mathcal{U}^{\delta}(\theta) \right], \end{aligned} \quad (20)$$

where $\delta \geq \xi > 0$, $\tau, \nu, \sigma > 0$.

Proof. Since the two functions \mathcal{U} and \mathcal{V} are positive and continuous on $[1, \infty)$ such that \mathcal{U} is decreasing and \mathcal{V} is increasing on $[1, \infty)$, then for all $\sigma > 0$, $\delta \geq \xi > 0$, $\rho, \zeta \in (1, \theta)$, $\theta > 1$, we have

$$(\mathcal{V}^{\sigma}(\zeta) - \mathcal{V}^{\sigma}(\rho)) (\mathcal{U}^{\delta-\xi}(\rho) - \mathcal{U}^{\delta-\xi}(\zeta)) \geq 0,$$

which follows that

$$\mathcal{V}^{\sigma}(\zeta) \mathcal{U}^{\delta-\xi}(\rho) + \mathcal{V}^{\sigma}(\rho) \mathcal{U}^{\delta-\xi}(\zeta) \geq \mathcal{V}^{\sigma}(\zeta) \mathcal{U}^{\delta-\xi}(\zeta) + \mathcal{V}^{\sigma}(\rho) \mathcal{U}^{\delta-\xi}(\rho). \quad (21)$$

Multiplying (21) by $\mathcal{F}(\theta, \rho)$ (where $\mathcal{F}(\theta, \rho)$ is defined in (16)) and integrating the resultant estimates with respect to ρ over $(1, \theta)$, we get

$$\begin{aligned} & \mathcal{V}^{\sigma}(\zeta) \frac{1}{\nu^{\tau} \Gamma(\tau)} \int_1^{\theta} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^{\xi}(\rho)}{\rho} \mathcal{U}^{\delta-\xi}(\rho) d\rho \\ & + \mathcal{U}^{\delta-\xi}(\zeta) \frac{1}{\nu^{\tau} \Gamma(\tau)} \int_1^{\theta} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^{\xi}(\rho)}{\rho} \mathcal{V}^{\sigma}(\rho) d\rho \\ & \geq \mathcal{V}^{\sigma}(\zeta) \mathcal{U}^{\delta-\xi}(\zeta) \frac{1}{\nu^{\tau} \Gamma(\tau)} \int_1^{\theta} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^{\xi}(\rho)}{\rho} d\rho \\ & + \frac{1}{\nu^{\tau} \Gamma(\tau)} \int_1^{\theta} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \mathcal{U}^{\xi}(\rho)}{\rho} \mathcal{V}^{\sigma}(\rho) \mathcal{U}^{\delta-\xi}(\rho) d\rho, \end{aligned}$$

which in view of (7) becomes,

$$\begin{aligned} & \mathcal{V}^{\sigma}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\delta}(\theta) \right] + \mathcal{U}^{\delta-\xi}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^{\sigma}(\theta) \mathcal{U}^{\xi}(\theta) \right] \\ & \geq \mathcal{V}^{\sigma}(\zeta) \mathcal{U}^{\delta-\xi}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^{\xi}(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^{\sigma}(\theta) \mathcal{U}^{\delta}(\theta) \right]. \end{aligned} \quad (22)$$

Again, multiplying (22) by $\mathcal{F}(\theta, \zeta)$ (where $\mathcal{F}(\theta, \zeta)$ can be obtained from (16)) and integrating the resultant estimates with respect to ζ over $(1, \theta)$ and then employing (7), we get

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\delta(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\delta(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\xi(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\xi(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right], \end{aligned}$$

which gives the desired assertion (20). \square

Theorem 4. Let the two functions \mathcal{U} and \mathcal{V} be positive and continuous on the interval $[1, \infty)$ such that the function \mathcal{U} is decreasing and the function \mathcal{V} is increasing on $[1, \infty)$ and assume that the function $\mathcal{W} : [1, \infty) \rightarrow \mathbb{R}^+$ is positive and continuous. Then for all $\theta > 1$, the following inequality for Hadamard proportional fractional integral operator (7) holds;

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\delta(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] + \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\delta(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\xi(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] + \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\xi(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right], \quad (23) \end{aligned}$$

where $\delta \geq \xi > 0$, $\tau, \lambda, \nu, \sigma > 0$.

Proof. Taking product on both sides of (22) by $\mathcal{G}(\theta, \zeta) = \frac{1}{\nu^\lambda \Gamma(\lambda)} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta}$, where $\lambda > 0, \xi > 0$ and $\zeta \in (1, \theta), \theta > 1$. and integrating the resultant estimates with respect to ζ over $(1, \theta)$, we have

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\delta(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta} \mathcal{V}^\sigma(\zeta) d\zeta \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta} \mathcal{U}^{\delta-\xi}(\zeta) d\zeta \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\xi(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta} \mathcal{V}^\sigma(\zeta) \mathcal{U}^{\delta-\xi}(\zeta) d\zeta \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \mathcal{U}^\xi(\zeta)}{\zeta} d\zeta. \end{aligned}$$

Consequently, in view of (7) it can be written as,

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\delta(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\xi(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\delta(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\xi(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}^\delta(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}^\xi(\theta) \right], \end{aligned}$$

which gives the desired inequality (23). \square

Remark 6. Applying Theorem 4 for $\tau = \lambda$, we get Theorem 3.

Next, we present some fractional proportional inequalities for a family of n positive functions defined on $[1, \infty)$ by utilizing Hadamard proportional fractional integral (7).

Theorem 5. Let the functions \mathcal{U}_j , ($j = 1, 2, \dots, n$) and \mathcal{V} be positive and continuous on the interval $[1, \infty)$. Suppose that for any fixed $k = 1, 2, \dots, n$,

$$(\mathcal{V}^\sigma(\zeta) \mathcal{U}_k^\sigma(\rho) - \mathcal{V}^\sigma(\rho) \mathcal{U}_k^\sigma(\zeta)) \left(\mathcal{U}_k^{\delta-\xi_k}(\rho) - \mathcal{U}_k^{\delta-\xi_k}(\zeta) \right) \geq 0, \quad (24)$$

where $\rho, \zeta \in (1, \theta)$, $\theta > 1$, $\sigma > 0$, $\delta > \xi_k > 0$. Assume that the function $\mathcal{W} : [1, \infty) \rightarrow \mathbb{R}^+$ is positive and continuous. Then for all $\theta > 1$, the following inequality for Hadamard proportional fractional integral operator (7) holds;

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right], \end{aligned} \quad (25)$$

where $\delta \geq \xi_k > 0$, $\tau, \nu, \sigma > 0$ and $k = 1, 2, \dots, n$.

Proof. Consider the function

$$\begin{aligned} \mathcal{F}_1(\theta, \rho) &= \frac{1}{\nu^\tau \Gamma(\tau)} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho}, \\ & \xi_j > 0, j = 1, 2, \dots, n, \rho \in (1, \theta), \theta > 1. \end{aligned} \quad (26)$$

Since the functions \mathcal{U}_j , ($j = 1, 2, \dots, n$) and \mathcal{V} satisfy (24) for any fixed $k = 1, 2, \dots, n$. Therefore, we can write

$$\mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta+\sigma-\xi_k}(\rho) + \mathcal{V}^\sigma(\rho) \mathcal{U}_k^{\delta+\sigma-\xi_k}(\zeta) \geq \mathcal{V}^\sigma(\zeta) \mathcal{U}_k^\sigma(\rho) \mathcal{U}_k^{\delta-\xi_k}(\zeta) + \mathcal{V}^\sigma(\rho) \mathcal{U}_k^\sigma(\zeta) \mathcal{U}_k^{\delta-\xi_k}(\rho). \quad (27)$$

Multiplying (27) by $\mathcal{F}_1(\theta, \rho)$ (where $\mathcal{F}_1(\theta, \rho)$ is defined in (26)) and integrating the resultant estimates with respect to ρ over $(1, \theta)$, $\theta > 1$, we have

$$\begin{aligned} & \mathcal{V}^\sigma(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho} \mathcal{U}_k^{\delta+\sigma-\xi_k}(\rho) d\rho \\ & + \mathcal{U}_k^{\delta+\sigma-\xi_k}(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho} \mathcal{V}^\sigma(\rho) d\rho \\ & \geq \mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta-\xi_k}(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho} \mathcal{U}_k^\sigma(\rho) d\rho \\ & + \mathcal{U}_k^\sigma(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho} \mathcal{V}^\sigma(\rho) \mathcal{U}_k^{\delta-\xi_k}(\rho) d\rho, \end{aligned}$$

which with aid of (7) gives

$$\mathcal{V}^\sigma(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] + \mathcal{U}_k^{\delta+\sigma-\xi_k}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \quad (28)$$

$$\geq \mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta-\xi_k}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] + \mathcal{U}_k^\sigma(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right]. \quad (29)$$

Now, multiplying (28) by $\mathcal{F}_1(\theta, \zeta)$ (where $\mathcal{F}_1(\theta, \zeta)$ can be obtained from (26) by replacing ρ by ζ) and integrating the resultant estimates with respect to ζ over $(1, \theta)$, $\theta > 1$ and then by applying (7), we obtain

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right], \end{aligned}$$

which completes the desired assertion (25). \square

Theorem 6. Let the functions \mathcal{U}_j , ($j = 1, 2, \dots, n$) and \mathcal{V} be positive and continuous on the interval $[1, \infty)$ and satisfy (24) for any fixed $k = 1, 2, \dots, n$. Assume that the function $\mathcal{W} : [1, \infty) \rightarrow \mathbb{R}^+$ is positive and continuous. Then for all $\theta > 1$, the following inequality for Hadamard proportional fractional integral operator (7) holds;

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & + \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & + \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right], \end{aligned} \quad (30)$$

where $\delta \geq \xi_k > 0$, $\tau, \lambda, \nu, \sigma > 0$ and $k = 1, 2, \dots, n$.

Proof. Taking product on both sides of (28) by $\mathcal{G}_1(\theta, \zeta) = \frac{1}{\nu^\lambda \Gamma(\lambda)} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta}$, where $\lambda > 0$, $\xi_j > 0$ and $\zeta \in (1, \theta)$, $\theta > 1$, $j = 1, 2, \dots, n$ and integrating the resultant estimates with respect to ζ over $(1, \theta)$, we have

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta} \mathcal{V}^\sigma(\zeta) d\zeta \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta} \mathcal{U}_k^{\delta+\sigma-\xi_k}(\zeta) d\zeta \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta} \mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta-\xi_k}(\zeta) d\zeta \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta} \mathcal{U}_k^\sigma(\zeta) d\zeta, \end{aligned}$$

which in view of (7) yields the desired assertion

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & + \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^{\delta+\sigma}(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & + \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right]. \end{aligned}$$

□

Remark 7. Applying Theorem 6 for $\tau = \lambda$, we get Theorem 5.

Theorem 7. Let the functions \mathcal{U}_j , ($j = 1, 2, \dots, n$) and \mathcal{V} be positive and continuous on the interval $[1, \infty)$ such that the function \mathcal{V} is increasing and the function \mathcal{U}_j for $j = 1, 2, \dots, n$ are decreasing on $[1, \infty)$. Assume that the function $\mathcal{W} : [1, \infty) \rightarrow \mathbb{R}^+$ is positive and continuous. Then for all $\theta > 1$, the following inequality for Hadamard proportional fractional integral operator (7) holds;

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right], \end{aligned} \quad (31)$$

where $\delta \geq \xi_k > 0$, $\tau, \nu, \sigma > 0$ and $k = 1, 2, \dots, n$.

Proof. Under the conditions stated in Theorem 7, we can write

$$(\mathcal{V}^\sigma(\zeta) - \mathcal{V}^\sigma(\rho)) \left(\mathcal{U}_k^{\delta-\xi_k}(\rho) - \mathcal{U}_k^{\delta-\xi_k}(\zeta) \right) \geq 0 \quad (32)$$

for any $\rho, \zeta \in (1, \theta)$, $\theta > 1$, $\sigma > 0$, $\delta > \xi_k > 0$, $k = 1, 2, 3, \dots, n$.

From (32), we have

$$\mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta-\xi_k}(\rho) + \mathcal{V}^\sigma(\rho) \mathcal{U}_k^{\delta-\xi_k}(\zeta) \geq \mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta-\xi_k}(\zeta) + \mathcal{V}^\sigma(\rho) \mathcal{U}_k^{\delta-\xi_k}(\rho). \quad (33)$$

Multiplying (27) by $\mathcal{F}_1(\theta, \rho)$ (where $\mathcal{F}_1(\theta, \rho)$ is defined in (26)) and integrating the resultant estimates with respect to ρ over $(1, \theta)$, $\theta > 1$, we have

$$\begin{aligned} & \mathcal{V}^\sigma(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho} \mathcal{U}_k^{\delta-\xi_k}(\rho) d\rho \\ & + \mathcal{U}_k^{\delta-\xi_k}(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho} \mathcal{V}^\sigma(\rho) d\rho \\ & \geq \mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta-\xi_k}(\zeta) \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho} d\rho \\ & + \frac{1}{\nu^\tau \Gamma(\tau)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \rho)} (\ln \theta - \ln \rho)^{\tau-1} \frac{\mathcal{W}(\rho) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\rho)}{\rho} \mathcal{V}^\sigma(\rho) \mathcal{U}_k^{\delta-\xi_k}(\rho) d\rho, \end{aligned}$$

which in view of Hadamard proportional fractional integral (7) becomes

$$\begin{aligned} & \mathcal{V}^\sigma(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] + \mathcal{U}_k^{\delta - \xi_k}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & \geq \mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta - \xi_k}(\zeta) \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right]. \end{aligned} \quad (34)$$

Now, multiplying (34) by $\mathcal{F}_1(\theta, \zeta)$ (where $\mathcal{F}_1(\theta, \zeta)$ can be obtained from (26) by replacing ρ by ζ) and integrating the resultant estimates with respect to ζ over $(1, \theta)$, $\theta > 1$ and then by applying (7), we obtain

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right], \end{aligned}$$

which is the desired assertion (31). \square

Theorem 8. Let the functions \mathcal{U}_j , ($j = 1, 2, \dots, n$) and \mathcal{V} be positive and continuous on the interval $[1, \infty)$ such that the function \mathcal{V} is increasing and the function \mathcal{U}_j for $j = 1, 2, \dots, n$ are decreasing on $[1, \infty)$. Assume that the function $\mathcal{W} : [1, \infty) \rightarrow \mathbb{R}^+$ is positive and continuous. Then for all $\theta > 1$, the following inequality for Hadamard proportional fractional integral operator (7) holds;

$$\begin{aligned} & \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & + \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\ & + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right], \end{aligned} \quad (35)$$

where $\delta \geq \xi_k > 0$, $\tau, \lambda, \nu, \sigma > 0$ and $k = 1, 2, \dots, n$.

Proof. To obtain the desire assertion (35), we multiply (34) by $\mathcal{G}_1(\theta, \zeta) = \frac{1}{\nu^\lambda \Gamma(\lambda)} e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta}$, where $\lambda > 0$, $\xi_j > 0$ and $\zeta \in (1, \theta)$, $\theta > 1$, $j = 1, 2, \dots, n$ and integrating the resultant estimates with respect to ζ over $(1, \theta)$, we have

$$\begin{aligned}
& \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta} \mathcal{V}^\sigma(\zeta) d\zeta \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\
& + \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta} \mathcal{U}_k^{\delta-\xi_k}(\zeta) d\zeta \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\
& \geq \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta} \mathcal{V}^\sigma(\zeta) \mathcal{U}_k^{\delta-\xi_k}(\zeta) d\zeta \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\
& + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \frac{1}{\nu^\lambda \Gamma(\lambda)} \int_1^\theta e^{\frac{\nu-1}{\nu}(\ln \theta - \ln \zeta)} (\ln \theta - \ln \zeta)^{\lambda-1} \frac{\mathcal{W}(\zeta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\zeta)}{\zeta} d\zeta.
\end{aligned}$$

which in view of Hadamard proportional fractional integral (7) gives

$$\begin{aligned}
& \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\
& + \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\
& \geq \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \\
& + \mathcal{H}_{1,\theta}^{\tau,\nu} \left[\mathcal{W}(\theta) \mathcal{V}^\sigma(\theta) \mathcal{U}_k^\delta(\theta) \prod_{j \neq k}^n \mathcal{U}_j^{\xi_j}(\theta) \right] \mathcal{H}_{1,\theta}^{\lambda,\nu} \left[\mathcal{W}(\theta) \prod_{j=1}^n \mathcal{U}_j^{\xi_j}(\theta) \right],
\end{aligned}$$

which completes the proof of (35). \square

Remark 8. Applying Theorem 8 for $\tau = \lambda$, we get Theorem 7.

4. Concluding Remarks

Recently Jarad et al. [46] introduced the idea of generalized proportional fractional integral operators which comprises exponential in their kernels. Later on, Rahman et al. [47] studied these operators and defined Hadamard proportional fractional integrals. They established certain inequalities for convex functions by employing Hadamard proportional fractional integrals. In [49], the authors defined bounds of generalized proportional fractional integral operators for convex functions and their applications. Motivated by the above, here we presented certain inequalities by employing Hadamard proportional fractional integrals. The inequalities obtained in this paper generalized the inequalities presented earlier by Houas [45].

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