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Starlikeness Condition for a New Differential-Integral Operator

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Abstract: A new differential-integral operator of the form $I^n f(z) = (1-\lambda)S^n f(z) + \lambda L^n f(z)$, $z \in U$, $f \in A$, $0 \leq \lambda \leq 1$, $n \in \mathbb{N}$ is introduced in this paper, where S^n is the Sălăgean differential operator and L^n is the Alexander integral operator. Using this operator, a new integral operator is defined as: $F(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z I^n f(z) \cdot t^{\beta+\gamma-2} dt\right]^{\frac{1}{\beta}}$, where $I^n f(z)$ is the differential-integral operator given above. Using a differential subordination, we prove that the integral operator $F(z)$ is starlike.

Keywords: differential subordination; analytic function; univalent function; convex function; starlike function; dominant; best dominant

1. Introduction and Preliminaries

The introduction and study of operators has been a topic that emerged at the very beginning of the theory of functions of a complex variable. The first operators were introduced during the first years of the twentieth century by mathematicians like J.W. Alexander, R. Libera, S. Bernardi, P. T. Mocanu and many more. The Alexander integral operator is such an example, defined by J. W. Alexander in 1915 [1]. This paper is cited in nearly 500 papers. The use of operators has facilitated the introduction of special classes of univalent functions and studying properties of the functions in those classes, such as convexity, starlikeness, coefficient estimates, and distortion properties. The Sălăgean differential operator was introduced in 1983 [2] and is cited by over 1300 papers. It has been used in obtaining new classes of functions and proving many interesting results related to them. The operator we introduce in this paper gives a new perspective in the theory related to operators by combining the integral Alexander operator and the differential Sălăgean operator. The results were obtained also using the means of the theory of differential subordinations introduced by Professors Miller and Mocanu in two papers in 1978 and 1980 and condensed in the monograph published by them in 2000 [3]. This theory has remarkable applications allowing easier proofs of already known results and facilitating the emergence of new ones. The idea of combining integral and differential operators is illustrated in the very recent paper [4] where a differential-integral operator was defined and using the method of the subordination chains, differential subordinations in their special Briot-Bouquet form were studied obtaining their best dominant and, as a consequence, criteria containing sufficient conditions for univalence were formulated. Similar work containing subordination results related to a class of univalent functions obtained by the use of an operator introduced by using a differential operator and an integral one can be seen in [5].

We use the well-known notations:

- $\mathcal{H}(U)$ is the class of functions analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$,
- For $a \in \mathbb{C}$, $n \in \mathbb{N}$, $\mathcal{H}_{[a,n]} = \{f \in \mathcal{H}(U) : f(z) = a + b_n z^n + \ldots, z \in U\}$,
- $A_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots, z \in U\}$, with $A_1 = A$,
- $S^* = \left\{ f \in A, \operatorname{Re} \dfrac{zf'(z)}{f(z)} > 0, z \in U \right\}$ is the class of starlike functions in U,
- $K = \left\{ f \in A, \operatorname{Re} \dfrac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}$ is the class of normalized convex functions in U.

The definitions of subordination, solution of the differential subordination and best dominant of the solutions of the differential subordination are recalled next as they can be found in the monograph published by Professors Miller and Mocanu in 2000 [3], which gives the core of the theory of differential subordination:

If f and g are analytic in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$, for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbf{C}^3 \times U \to \mathbf{C}$ and h be a univalent function in U. If p is a analytic function in U which satisfies the following (second-order) differential subordination:

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z),$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination or more simply a dominant, if $p \prec q$, for all p satisfying the differential subordination. A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for every dominant q is said to be the best dominant.

A well-known lemma from the theory of differential subordinations that is used in proving the new results is shown as follows:

Lemma 1. *[3] Let g be univalent in U and let θ and ϕ be analytic in a domain D containing $g(U)$, with $\phi(w) \neq 0$, when $w \in g(U)$. Set*

$$Q(z) = zq'(z) \cdot \phi[q(z)], \quad h(z) = \theta[q(z)] + Q(z),$$

and suppose that

(i) Q is starlike and

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re} \left[\dfrac{\theta'[q(z)]}{\phi[q(z)]} + \dfrac{zQ'(z)}{Q(z)} \right] > 0$, $z \in U$.

If p is analytic in U, with $p(0) = q(0)$, $p'(0) = \ldots = p^{(n-1)}(0) = 0$, $p(U) \subset D$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z) \cdot \phi[q(z)] = h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

In order to define the new differential-integral operator, we need the following definitions:

Definition 1. *[2] For $f \in A$, $n \in \mathbb{N} = \mathbb{N}^* \cup \{0\}$, let S^n be the differential operator given by $S^n : A \to A$ with*

$$\begin{aligned} &S^0 f(z) = f(z) \\ &\vdots \\ &S^{n+1} f(z) = z[S^n f(z)]', \ z \in U. \end{aligned}$$

Remark 1. *If $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*

$$S^n f(z) = z + \sum_{j=2}^{\infty} j^n \cdot a_j z^j. \tag{1}$$

Definition 2. *[6] For $f \in A$, $n \in \mathbb{N} = \mathbb{N}^* \cup \{0\}$, let L^n be the integral operator given by $L^n : A \to A$ with*

$$\begin{aligned} &L^0 f(z) = f(z) \\ &L^1 f(z) = \int_0^z \frac{L^0 f(t)}{t} dt \\ &\vdots \\ &L^n f(z) = \int_0^z \frac{L^{n-1} f(t)}{t} dt, \ z \in U. \end{aligned}$$

Remark 2. *(a) For $n = 1$, $L^1 f(z) = \int_0^z \frac{f(t)}{t} dt$ becomes Alexander integral operator [1].*

(b) For $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, we obtain:

$$L^n f(z) = z + \sum_{j=2}^{\infty} \frac{1}{j^n} \cdot a_j z^j. \tag{2}$$

2. Main Results

Using Definition 1 and Definition 2, we introduce a new operator, as follows:

Definition 3. *Let $0 \le \lambda \le 1$, $n \in \mathbb{N} = \mathbb{N}^* \cup \{0\}$. Denote by I^n the differential-integral operator $I^n : A \to A$ given by*

$$I^n f(z) = (1 - \lambda) S^n f(z) + \lambda L^n f(z), \ z \in U, \tag{3}$$

where S^n is Sălăgean differential operator, and L^n is Alexander integral operator.

Remark 3. *(a) For $\lambda = 0$, $I^n f(z) = S^n f(z)$, the differential-integral operator is equivalent to Sălăgean differential operator.*

(b) For $\lambda = 1$, $I^n f(z) = L^n f(z)$, the differential-integral operator becomes Alexander integral operator.

(c) For $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *we obtain*

$$I^n f(z) = z + \sum_{j=2}^{\infty} \left[(1-\lambda) j^n + \lambda \cdot \frac{1}{j^n} \right] a_j z^j, \ z \in U. \tag{4}$$

Using the differential-integral operator introduced in Definition 3, we define a new integral operator, which can be seen as generalization of some well-known integral operators.

Definition 4. *Let* $\gamma \geq 0$, $0 < \beta \leq 1$, $n \in \mathbb{N} = \mathbb{N}^* \cup \{0\}$, *and* $f \in A$, $I^n f \in A$, *where* I^n *is given by Equation (3). The integral operator* $F : A \to \mathcal{H}_{[a,n]}$ *is defined as:*

$$F(z) = \left[\frac{\beta+\gamma}{z^\gamma} \int_0^z I^n f(t) \cdot t^{\beta+\gamma-2} dt \right]^{\frac{1}{\beta}}. \tag{5}$$

Remark 4. *(a) For* $n = 0$, $\beta = 1$, $\gamma > 0$, *we have*

$$F(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t) \cdot t^{\gamma-1} dt,$$

which is the Bernardi integral operator [7].

(b) For $n = 0$, $\beta = 1$, $\gamma = 1$, *we have*

$$F(z) = \frac{2}{z} \int_0^z f(t) dt,$$

which is the Libera integral operator [8].

(c) For $n = 0$, $\beta = 1$, $\gamma = 0$, *we have*

$$F(z) = \int_0^z \frac{f(t)}{t} dt,$$

which is Alexander integral operator [1].

(d) For $\beta = 1$, $n \in \mathbb{N}^*$, $\gamma > 0$, *we have*

$$F(z) = \frac{1+\gamma}{z^\gamma} \int_0^z I^n f(t) \cdot t^{\gamma-1} dt$$

which was studied in [6].

(e) For $n = 0$, *we have*

$$F(z) = [\frac{\beta+\gamma}{z^\gamma} \int_0^z f(t) \cdot t^{\beta+\gamma-2} dt]^{\frac{1}{\beta}},$$

$\beta > 0$, $\beta + \gamma > 0$ *and* $\beta \geq 2\gamma(1-\beta)$ *studied in [9] where the authors have proved that* $F \in S^*$.

Using a differential subordination, we prove that the operator given by Equation (5) is starlike.

Theorem 1. *Let* $0 < \beta \leq 1$, $\gamma \geq 0$, $n \in \mathbb{N} = \mathbb{N}^* \cup \{0\}$, *and let*

$$h(z) = \frac{1+z}{1-z} + \frac{2z}{(1-z)[1+\beta+\gamma+(1-\beta-\gamma)z]}, \ z \in U.$$

If $f \in A$ and

$$\frac{1}{\beta} \cdot \frac{z(I^n f(z))'}{I^n f(z)} + \frac{\beta - 1}{\beta} \prec h(z),\ z \in U, \tag{6}$$

then

$$F(z) = \left[\frac{\beta+\gamma}{z^\gamma} \int_0^z I^n f(z) \cdot t^{\beta+\gamma-2} dt\right]^{\frac{1}{\beta}}$$

is starlike, i.e., $F \in S^*$, where $I^n f$ is given by Equation (3).

Proof. From Equation (5) we have

$$F^\beta(z) = \frac{\beta+\gamma}{z^\gamma} \int_0^z I^n f(z) \cdot t^{\beta+\gamma-2} dt,$$

and

$$z^\gamma F^\beta(z) = (\beta+\gamma) \int_0^z I^n f(z) \cdot t^{\beta+\gamma-2} dt. \tag{7}$$

Differentiating Equation (7), we obtain

$$F^\beta(z) \left[\gamma + \beta z \cdot \frac{F'(z)}{F(z)}\right] = (\beta+\gamma) I^n f(z) \cdot z^{\beta-1}. \tag{8}$$

We let

$$p(z) = z \cdot \frac{F'(z)}{F(z)},\ z \in U. \tag{9}$$

Using Equations (9) in (8), we have

$$F^\beta(z)[\gamma + \beta p(z)] = (\beta+\gamma) I^n f(z) \cdot z^{\beta-1}. \tag{10}$$

Differentiating (10), we get

$$\beta \cdot \frac{zF'(z)}{F(z)} + \beta \cdot \frac{zp'(z)}{\gamma + \beta p(z)} = \frac{z(I^n f(z))'}{I^n f(z)} + \beta - 1. \tag{11}$$

Using (9) in (11), we have

$$p(z) + \frac{zp'(z)}{\gamma + \beta p(z)} = \frac{1}{\beta} \cdot \frac{z(I^n f(z))'}{I^n f(z)} + \frac{\beta - 1}{\beta}. \tag{12}$$

Using Relation (12), the differential subordination of Equation (6) becomes:

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z) = \frac{1+z}{1-z} + \frac{2z}{(1-z)[1+\beta+\gamma+(1-\beta-\gamma)z]}. \tag{13}$$

In order to prove the theorem, we shall use Lemma 1.
If we let $\theta : D \subset \mathbb{C} \to \mathbb{C}$ and $\phi : D \subset \mathbb{C} \to \mathbb{C}$ be analytic,

$$\theta(w) = q,\ \phi(w) = \frac{1}{w + \beta + \gamma} \text{ in a domain } D.$$

For $w = q(z) = \dfrac{1+z}{1-z}$, we obtain

$$\phi[q(z)] = \frac{1}{\dfrac{1+z}{1-z} + \beta + \gamma} = \frac{1-z}{1+\beta+\gamma+z(1-\beta-\gamma)} \tag{14}$$

$$Q(z) = z \cdot q'(z) \cdot \phi[q(z)] = \frac{2z}{(1-z)[1+\beta+\gamma+(1-\beta-\gamma)z]} \tag{15}$$

and

$$h(z) = \theta[q(z)] + Q(z) = \frac{1+z}{1-z} + \frac{2z}{(1-z)[1+\beta+\gamma+(1-\beta-\gamma)z]}. \tag{16}$$

Next we show that conditions in Lemma 1 are satisfied. We prove that the function Q is starlike. Differentiating Equation (15), we have

$$\frac{zQ'(z)}{Q(z)} = \frac{z}{1-z} + \frac{1+\beta+\gamma}{1+\beta+\gamma+(1-\beta-\gamma)z}.$$

We take

$$\begin{aligned}
\operatorname{Re}\frac{zQ'(z)}{Q(z)} &= \operatorname{Re}\frac{z}{1-z} + (1+\beta+\gamma)\operatorname{Re}\frac{1}{1+\beta+\gamma+(1-\beta-\gamma)z} \\
&= -\frac{1}{2} + \frac{(1+\beta+\gamma)^2 + (1+\beta+\gamma)(1-\beta-\gamma)\cos\alpha}{[1+\beta+\gamma+(1-\beta-\gamma)\cos\alpha]^2 + (1-\beta-\gamma)^2\sin^2\alpha} \\
&= \frac{2(\beta+\gamma)}{[1+\beta+\gamma+(1-\beta-\gamma)\cos\alpha]^2 + (1-\beta-\gamma)^2\sin^2\alpha} > 0.
\end{aligned}$$

We have shown that $\operatorname{Re}\dfrac{zQ'(z)}{Q(z)} > 0$, $z \in U$, i.e., $Q \in S^*$, hence (i) from Lemma 1 is satisfied.

We evaluate now:

$$\operatorname{Re}\phi[q(z)] = \operatorname{Re}\frac{1-z}{1+\beta+\gamma+(1-\beta-\gamma)z}$$

$$= \frac{2\beta + 2\gamma(1-\cos\alpha)}{[1+\beta+\gamma+(1-\beta-\gamma)\cos\alpha]^2 + (1-\beta-\gamma)^2\sin^2\alpha} > 0,\ 0 < \beta \le 1,\ \gamma \ge 0.$$

Since Q is starlike and $\operatorname{Re}\phi[q(z)] > 0$, we have

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} > 0,\ z \in U.$$

Next we prove that $p(0) = q(0)$, $p \in \mathcal{H}_{[1,1]}$ and p is analytic in U, where

$$p(z) = \frac{zF'(z)}{F(z)},\ z \in U.$$

From Equation (4), we have

$$I^n f(z) = z + \sum_{j=2}^{\infty} \left[(1-\lambda)j^n + \lambda \cdot \frac{1}{j^n}\right] a_j z^j = z + \sum_{j=2}^{\infty} b_j z^j,$$

where
$$b_j = \left[(1-\lambda)j^n + \lambda \cdot \frac{1}{j^n}\right] a_j.$$
From Equation (5), we can write
$$\begin{aligned} F(z) &= \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z \left(t + \sum_{j=2}^{\infty} b_j z^j\right) t^{\beta+\gamma-2} dt\right]^{\frac{1}{\beta}} \\ &= \left[z^\beta + \sum_{j=2}^{\infty} c_j z^{j+\beta-1}\right]^{\frac{1}{\beta}} \end{aligned}$$
and we obtain
$$F^\beta(z) = z^\beta + \sum_{j=2}^{\infty} c_j z^{j+\beta-1},\ z \in U. \tag{17}$$
Differentiating Equation (17), we have
$$\beta F^{\beta-1}(z) \cdot F'(z) = \beta z^{\beta-1} + \sum_{j=2}^{\infty} c_j (j+\beta-1) \cdot z^{j+\beta-2}.$$
Further, we deduce
$$p(z) = \frac{zF'(z)}{F(z)} = \frac{z^\beta + \sum_{j=2}^{\infty} d_j z^{j+\beta-1}}{z^\beta + \sum_{j=2}^{\infty} c_j z^{j+\beta-1}} = 1 + p_1 z + p_2 z^2 + \ldots. \tag{18}$$
For $z = 0$, we obtain $p(0) = 1$ and $p \in \mathcal{H}_{[1,1]}$, hence F it is analytic in U.
Since $q(z) = \dfrac{1+z}{1-z}$, we have $q(0) = 1$, $p(0) = q(0) = 1$ and
$$\theta[p(z)] + zp'(z) \cdot \phi[p(z)] \prec \theta[q(z)] + zq'(z) \cdot \phi[q(z)] = h(z). \tag{19}$$
We have proved that we can use Lemma 1. By applying it, we have $p(z) \prec q(z)$, i.e.,
$$\frac{zF'(z)}{F(z)} \prec q(z) = \frac{1+z}{1-z},\ z \in U. \tag{20}$$
Since $q(z) = \dfrac{1+z}{1-z}$ is a convex function and
$$\mathrm{Re}\,\frac{1+z}{1-z} > 0,\ z \in U,$$
the differential subordination in Equation (20) implies
$$\mathrm{Re}\,\frac{zF'(z)}{F(z)} > \mathrm{Re}\, q(z) > 0,\ \text{hence } F \in S^*.$$
□

Example 1. *Let* $\lambda = \frac{1}{2}$, $n = 1$, $\beta = \frac{1}{2}$, $\gamma = 2$,

$$h(z) = \frac{1+z}{1-z} + \frac{4z}{(1-z)(7-3z)},\ f(z) = z + \frac{1}{4}z^2,\ z \in U,$$

$$S^1 f(z) = zf'(z) = z + \frac{1}{2}z^2,$$

$$L^1 f(z) = \int_0^z \frac{t + \frac{1}{4}t^2}{t} dt = z + \frac{1}{8}z^2,$$

$$I^1 f(z) = \frac{1}{2}S^1 f(z) + \frac{1}{2}L^1 f(z) = z + \frac{5}{16}z^2,$$

$$F(z) = \left(z^{\frac{1}{2}} + \frac{25}{112}z^{\frac{3}{2}}\right)^2 = z + \frac{25}{56}z^2 + \frac{625}{12544}z^3$$

and

$$\frac{zF'(z)}{F(z)} = \frac{1 + \frac{25}{28}z + \frac{1875}{12544}z^2}{1 + \frac{25}{56}z + \frac{625}{12544}z^2},\ q(z) = \frac{1+z}{1-z}$$

$$\frac{1}{\beta} \cdot \frac{z(I^1 f(z))'}{I^1 f(z)} + \frac{\beta - 1}{\beta} = \frac{4(8+5z)}{16+5z} - 1\,.$$

From Theorem 1, we have:
If $f \in A$, *and*

$$\frac{4(8+5z)}{16+5z} - 1 \prec \frac{1+z}{1-z} + \frac{4z}{(1-z)(7-3z)},\ z \in U$$

then

$$p(z) = \frac{zF'(z)}{F(z)} = \frac{1 + \frac{25}{28}z + \frac{1875}{12544}z^2}{1 + \frac{25}{56}z + \frac{625}{12544}z^2} \prec \frac{1+z}{1-z},$$

meaning that $F(z) = z + \frac{25}{56}z^2 + \frac{625}{12544}z^3$ *is a starlike function.*

3. Conclusions

A new differential-integral operator is introduced proving that this operator is starlike. An example is given to show how the result can be applied in finding such operators. As it is the case for most operators, special classes of univalent functions could be introduced using it and this is subject to further studies. Another problem that can be studied is related to the parameters β and γ used in the definition of the operator. In this paper they are positive but the case of β and γ being complex numbers could be subject of further investigation. Starlikeness of certain order $0 \leq \alpha < 1$ can also be further studied both for the case of β and γ positive and for β and γ complex numbers.

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