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The Mean Minkowski Content of Homogeneous Random Fractals

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Received: 26 April 2020; Accepted: 26 May 2020; Published: 1 June 2020



Abstract: Homogeneous random fractals form a probabilistic generalisation of self-similar sets with more dependencies than in random recursive constructions. Under the Uniform Strong Open Set Condition we show that the mean D -dimensional (average) Minkowski content is positive and finite, where the mean Minkowski dimension D is, in general, greater than its almost sure variant. Moreover, an integral representation extending that from the special deterministic case is derived.

Keywords: fractals; random homogeneous constructions; code trees; mean Minkowski content

1. Introduction

In fractal geometry and analysis, the D -dimensional Minkowski content, and its average version, of a nonempty compact set $K \subset \mathbb{R}^d$ are respectively defined as

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{D-d} \mathcal{L}^d(K_\varepsilon), \text{ and } \lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{D-d} \mathcal{L}^d(K_\varepsilon) \frac{1}{\varepsilon} d\varepsilon,$$

provided the corresponding limit exists, where K_ε denotes the parallel set of K of distance ε . K is said to be Minkowski measurable if the first limit is positive and finite. In this case, D is a determined number called the Minkowski dimension of K . These notions have been considered in the literature for several classes of fractal sets. In [1] and the references therein relationships to spectral analysis, certain Zeta functions and fractal drums are established. Another approach is based on the application of renewal theorems from probability theory in order to determine related geometric quantities.

For self-similar sets satisfying the Open Set Condition, as introduced in [2], this idea goes back to [3]. The Minkowski content of such sets was determined in [4] under the stronger separation condition and in [5] for the general case. (For local versions see also [6], Chapter 10.) Note that here the Hausdorff dimension of the sets coincides with their Minkowski dimension, which was shown before (for references see, e.g., [7]). In [8], where these results are extended to fractal versions of higher order mean curvatures, explicit numerical values are calculated for the examples of the Sierpinski gasket and variants of the Sierpinski carpet.

For stochastically self-similar sets in the sense of [9–11], the almost sure (average) Minkowski content was determined in [5], where a renewal theorem for branching random walks was used as a main tool. Again, for this model, the a.s. Hausdorff dimension coincides with the a.s. Minkowski dimension. Moreover, the mean Minkowski content agrees with the almost sure variant, i.e., the latter is constant. Extensions to the geometric higher order mean curvatures can be found in [12].

The above random sets are also called random recursive fractals. Homogeneous random fractals have much more dependencies in their construction. A first special case was studied [13], in particular, the a.s. Hausdorff dimension was determined and shown to be equal to the a.s. Minkowski dimension. From the results in [14,15], respectively, this follows for the general case.

V -variable random fractals in the sense of [16] and preceding parts provide a certain interpolation between homogeneous random fractals as special case $V = 1$ and random recursive constructions

($V = \infty$). In [16] the corresponding a.s. Hausdorff dimension was determined. However, recently it has been shown in [17] that—in distinction to the random recursive case—no related gauge function provides a positive and finite Hausdorff measure. We conjecture that an a.s. positive and finite Minkowski content also does not exist.

In order to find geometric parameters for such sets, too, it makes sense to consider mean values in the probabilistic sense. In the present paper, we show under some general conditions, that the (average) mean Minkowski content of a homogeneous random fractal exists. Moreover, we derive a formula in terms of expectations, which is the same as that for the included deterministic self-similar sets. Extensions to fractal curvatures in the sense of [8,12] are possible. Some basic techniques of proof are close to those from the former variants. In particular, we use the classical Renewal theorem in the sense of [18]. It turns out that the a.s. Minkowski dimension of such sets is, in general, less than the associated mean Minkowski dimension.

2. Construction of Homogeneous Random Fractals and Statement of the Results

For fixed $0 < r_{\min} < r_{\max} < 1$ let Sim be the set of contractive similarities $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with contraction ratios $r_{\min} \leq r \leq r_{\max}$ equipped with the topology given by uniform convergence on compact sets. \mathcal{B} denotes the associated Borel σ -algebra. The space $\Omega_0 := \bigcup_{k=1}^{\infty} \text{Sim}^k$ together with the σ -algebra $\mathcal{F}_0 := \{A \subset \Omega_0 : A \cap \text{Sim}^k \in \otimes_{i=1}^k \mathcal{B} \text{ for all } k \in \mathbb{N}\}$, and with a distribution \mathbb{P}_0 on it provide the primary probability space $[\Omega_0, \mathcal{F}_0, \mathbb{P}_0]$. (Here and in the following the symbol \subset is used for \subseteq .) The basic probability space for the random construction model is the product space

$$[\Omega, \mathcal{F}, \mathbb{P}] := \bigotimes_{n=1}^{\infty} [\Omega_0, \mathcal{F}_0, \mathbb{P}_0] \quad (1)$$

and the expectation symbol \mathbb{E} will be used for integration with respect to \mathbb{P} .

The elements of Ω are denoted by

$$\omega = \omega_1, \omega_2, \dots := (f_1(1), \dots, f_{N(1)}(1)), (f_1(2), \dots, f_{N(2)}(2)), \dots,$$

and $r_i(n)$ are the contraction ratios of the similarities $f_i(n)$. For $f_i(1)$, $r_i(1)$ and $N(1)$ we will often write f_i , r_i and N , resp. Below we will use the measurable mapping

$$\theta : \Omega \rightarrow \Omega \text{ with } \theta(\omega_1, \omega_2, \omega_3, \dots) := \omega_2, \omega_3, \dots,$$

and for a random element $\xi(\omega)$ we will write

$$\xi^n(\omega) := \xi(\theta^n(\omega)).$$

(If it is clear from the context, the argument ω will be omitted.)

The above random similarities $(f_1(n), \dots, f_{N(n)}(n))$ with distribution \mathbb{P}_0 play the role of the random *iterated function system* (IFS) of random length $N(n)$ in the n -th construction step. For different n they are independent of each other.

Throughout the paper, we suppose the *Uniform Open Set Condition* (UOSC), i.e., there exists a nonempty bounded open set $O \subset \mathbb{R}^d$ such that a.s. with respect to \mathbb{P}_0 we have

$$\bigcup_{i=1}^N f_i(O) \subset O \quad \text{and} \quad f_i(O) \cap f_j(O) = \emptyset, i \neq j. \quad (2)$$

Then with \mathbb{P} -probability 1 all IFS in the product space fulfil this (UOSC).

As usual, the corresponding random fractal set is introduced by means of a random *coding tree*: $\Sigma_n = \Sigma_n(\omega) := \{\sigma_1 \dots \sigma_n : 1 \leq \sigma_i \leq N(i), i = 1, \dots, n\}$ is the set of all *nodes* at level n and $\Sigma_* := \bigcup_{n=0}^{\infty} \Sigma_n$ is the set of all nodes of the tree, where Σ_0 denotes the empty code at level 0.

Recall that $\Sigma_l^k(\omega) = \Sigma_l(\theta^k \omega)$. For $\sigma = \sigma_1 \dots \sigma_k \in \Sigma_k$ and $\tau = \tau_1 \dots \tau_l \in \Sigma_l^k$ we write $\sigma\tau := \sigma_1 \dots \sigma_k \tau_1 \dots \tau_l \in \Sigma_{k+l}$ for the *concatenation* of these codes. If $\sigma = \sigma_1 \dots \sigma_n \in \Sigma_n$ and $0 \leq k \leq n$, then $\sigma|k := \sigma_1 \dots \sigma_k$ denotes the *restriction* to the first k components of σ , and $|\sigma| := n$ is the *length* of σ . With each such σ we associate the same random IFS $(f_1(n+1), \dots, f_{N(n+1)}(n+1))$, where $n = 0, 1, 2, \dots$. This leads to the homogeneous structure. (In the V -variable case these random IFS are chosen by means of V different types. Here we have $V = 1$, and in the case of random recursive constructions, where $V = \infty$, for different $\sigma \in \Sigma_n$ the IFS are i.i.d.) Furthermore we define the random mappings

$$f_\sigma = f_\sigma(\omega) := f_{\sigma_1}(1) \circ f_{\sigma_2}(2) \circ \dots \circ f_{\sigma_n}(n)$$

with contraction ratios $r_\sigma = r_\sigma(\omega) := r_{\sigma_1}(1)r_{\sigma_2}(2) \dots r_{\sigma_n}(n)$.

Then the random compact subset of \overline{O}

$$F = F(\omega) := \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in \Sigma_n} f_\sigma(\overline{O}) \quad (3)$$

is \mathbb{P} -a.s. determined and measurable with respect to the Borel σ -algebra $\mathcal{B}(\mathcal{K})$ determined by the Hausdorff distance d_H on the space \mathcal{K} of nonempty compact subsets of \mathbb{R}^d . It is called the *associate homogeneous random fractal*. F is *stochastically self-similar* in the following sense (recall that $F^n(\omega) = F(\theta^n(\omega))$).

$$F = \bigcup_{i=1}^N f_i(1)(F^1) \quad , \quad \mathbb{P} - \text{a.s.} .$$

More generally, for all $n \in \mathbb{N}$,

$$F = \bigcup_{\sigma \in \Sigma_n} f_\sigma(F^n) \quad , \quad \mathbb{P} - \text{a.s.} , \quad (4)$$

where the random compact set F^n is independent of the random mappings $\{f_\sigma, \sigma \in \Sigma_n\}$ and has the same distribution as F .

In the sequel, many relationships between random elements are fulfilled only with probability 1. We will not mention this if it can be seen from the context.

In order to treat the mean Minkowski content of F we need the following notions. For $r > 0$ the r -parallel set of $K \in \mathcal{K}$ is given by

$$K_r := \{x \in \mathbb{R}^d : \min_{y \in K} |x - y| \leq r\} ,$$

and the *inner r -parallel set* of a bounded open set G by

$$G_{-r} := \{x \in G : \min_{y \in \partial G} |x - y| \geq r\} ,$$

where ∂G means the topological boundary of G .

The measurability properties of the random elements used in the sequel follow easily from their definitions or together with the next result, which will be proved at the end of the next section. (Note that the Hausdorff metric generates the so-called hit and miss topology on \mathcal{K} . For more details see, e.g., Matheron [19].)

Lemma 1. *The following mappings are continuous with respect to the corresponding (product) metrics:*

$$(i) \quad (r, (K_{11}, \dots, K_{k1}), (K_{12}, \dots, K_{k2}), \dots, (K_{1l}, \dots, K_{kl})) \mapsto \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (K_{ij})_r \right)$$

from $(0, \infty) \times \mathcal{K}^{k \cdot l}$ into $(0, \infty)$.

$$(ii) \quad r \mapsto \mathbb{E} \mathcal{L}^d(K_r) \text{ from } (0, \infty) \text{ into } (0, \infty) \text{ for any random nonempty compact set } K = K(\omega) \text{ with } \text{diam}(K) \leq c \text{ w.p.1 for some constant } c.$$

In order to formulate the main results concerning the Minkowski content suppose now that $1 < \mathbb{E}N < \infty$ and let D be the number determined by

$$\mathbb{E} \sum_{i=1}^N r_i^D = 1. \quad (5)$$

(The (UOSC) implies that $D \leq d$.)

$$\mu := \mathbb{E} \left(\sum_{i=1}^N \mathbf{1}_{(\cdot)}(|\ln r_i|) r_i^D \right) \quad (6)$$

is an associated probability distribution for the logarithmic contraction ratios r_i of the primary random IFS. The corresponding mean value is denoted by

$$\eta := \mathbb{E} \left(\sum_{i=1}^N |\ln r_i| r_i^D \right). \quad (7)$$

By definition, the random set F satisfies the *Uniform Strong Open Set Condition* (USOSC) if (UOSC) (see Equation (2)) is fulfilled and $\mathbb{P}(F \cap O \neq \emptyset) > 0$.

Then we get the following. In the sequel we will use the notation $\varphi(\varepsilon) := \mathbb{E} \mathcal{L}^d(F_\varepsilon)$.

Theorem 1. *Suppose (USOSC) for the homogeneous random fractal set F and $1 < \mathbb{E}N < \infty$.*

(i) *If the measure μ is non-arithmetic, then the finite limit*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{D-d} \mathbb{E} \mathcal{L}^d(F_\varepsilon)$$

exists and equals

$$M^D(F) = \frac{1}{\eta} \int_0^1 \varepsilon^{D-d-1} R(\varepsilon) d\varepsilon,$$

where the function $R(\varepsilon)$ is given by

$$R(\varepsilon) = \varphi(\varepsilon) - \mathbb{E} \sum_{i=1}^N (\mathbf{1}_{(0, r_i]}(\varepsilon) r_i^d \varphi(\frac{\varepsilon}{r_i})).$$

(ii) *For general μ we get for the average limit*

$$\lim_{\delta \rightarrow 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{D-d} \mathbb{E} \mathcal{L}^d(F_\varepsilon) \frac{1}{\varepsilon} d\varepsilon = M^D(F).$$

Theorem 2. *Under the conditions of Theorem 1 the constant $M^D(F)$ is positive.*

These two theorems show that $M^D(F)$ can be interpreted as the *mean D -dimensional Minkowski content* of the homogeneous random fractal F .

Note that according to Hambly [13] (for a special case) the almost sure *Hausdorff dimension* D_H of F is given by the equation

$$\mathbb{E} \ln \left(\sum_{i=1}^N r_i^{D_H} \right) = 0.$$

The general case can be included in an approach from the theory of dynamical systems by Roy and Urbanski [14], see also Barnsley, Hutchinson and Stenflo [16] in the context of V -variable fractals for $V = 1$. In [13] and in Troscheit [15] for the general case it is shown that D_H coincides with the a.s. box counting dimension. It is well known that the box counting dimension always agrees with the Minkowski dimension. By the above formulas $D_H \leq D$, where the equality is not valid in general and hence, the a.s. Minkowski dimension can be less than the mean version in the sense of the above theorems. (Note that in the deterministic case we have $D_H = D$.)

3. Proofs

Proof of Theorem 1. Recall that $\varphi(\varepsilon) = \mathbb{E} \mathcal{L}^d(F_\varepsilon)$ and

$$R(\varepsilon) = \mathbf{1}_{[0,1]}(\varepsilon) \varphi(\varepsilon) - \mathbb{E} \sum_{i=1}^N (\mathbf{1}_{(0,r_i]}(\varepsilon) r_i^d \varphi(\frac{\varepsilon}{r_i})).$$

By the scaling property of the Lebesgue measure and using that the random set F^1 is independent of the random contraction ratios (r_1, \dots, r_N) and has the same distribution as F we get the representation

$$R(\varepsilon) = \mathbf{1}_{(0,1]}(\varepsilon) \mathbb{E} \mathcal{L}^d(F_\varepsilon) - \mathbb{E} \sum_{i=1}^N (\mathbf{1}_{(0,r_i]}(\varepsilon) \mathcal{L}^d(f_i(F^1)_\varepsilon)). \quad (8)$$

Substituting $\varepsilon = e^{-t}$ we infer from the definition of $R(\varepsilon)$ that

$$\begin{aligned} & \mathbf{1}_{[0,\infty)}(t) e^{(d-D)t} \varphi(e^{-t}) \\ &= \mathbb{E} \sum_{i=1}^N \left(\mathbf{1}_{[0,\infty)}(t - |\ln r_i|) e^{(d-D)(t - |\ln r_i|)} \varphi(e^{-(t - |\ln r_i|)}) r_i^D \right) \\ &+ e^{(d-D)t} R(e^{-t}). \end{aligned}$$

Denoting $Z(t) := \mathbf{1}_{[0,\infty)}(t) e^{(d-D)t} \varphi(e^{-t})$ and $z(t) := e^{(d-D)t} R(e^{-t})$, i.e., $z(t) = 0$ for $t < 0$, the above equation can be rewritten as

$$Z(t) = \int_0^t Z(t-s) \mu(ds) + z(t).$$

Thus, the function $Z(t)$ satisfies the renewal equation with respect to the distribution μ . In view of Lemma 1 and dominated convergence, the function $z(t)$ is right continuous with left limits. Hence, it is Lebesgue-a.e. continuous. Below we will show that

$$|z(t)| \leq c e^{-t\delta} \quad (9)$$

for some constants c and $\delta > 0$, i.e., z is bounded by a directly Riemann integrable function. According to Asmussen ([20] Prop. 4.1, p. 118) z is then directly Riemann integrable, too. Therefore the classical Renewal theorem in Feller ([18] p. 363) can be applied. In the non-arithmetic case we get

$$\lim_{t \rightarrow \infty} Z(t) = \frac{1}{\eta} \int_0^\infty z(t) dt = \frac{1}{\eta} \int_0^\infty e^{(d-D)t} R(e^{-t}) dt,$$

i.e., assertion (i) after substituting $\varepsilon = e^{-t}$ under the integral.

Since $Z(t)$ is bounded on finite intervals, in the non-arithmetic case the corresponding average limit in (ii) is a consequence. In the lattice case, the Renewal theorem provides the limit in discrete steps with respect to the lattice constant. This implies the average convergence. (For more details see the end of the proof of Theorem 2.3 in Gatzouras [5].)

Now it remains to prove Equation (9), i.e., in view of Equation (8) that

$$|\mathbb{E} \mathcal{L}^d(F_\varepsilon) - \mathbb{E}(\sum_{i=1}^N \mathbf{1}_{(0,r_i]}(\varepsilon) \mathcal{L}^d(f_i(F^1)_\varepsilon))| \leq c \varepsilon^{d-D+\delta},$$

for some constants c and $\delta > 0$. For this it suffices to show that

$$|\mathbb{E}(\mathcal{L}^d(F_\varepsilon) - \sum_{i=1}^N \mathcal{L}^d(f_i(F^1)_\varepsilon))| \leq c \varepsilon^{d-D+\delta}, \quad (10)$$

since $|\mathbb{E}(\sum_{i=1}^N \mathbf{1}_{(r_i,1]}(\varepsilon) \mathcal{L}^d(f_i(F^1)_\varepsilon))|$ for $\varepsilon > r_{\min}$ is uniformly bounded.

To this aim we consider the auxiliary random sets

$$\begin{aligned} A(\varepsilon) &:= \bigcup_{i,j \in \{1,\dots,N\}, i \neq j} f_i(\overline{O})_\varepsilon \cap f_j(\overline{O})_\varepsilon, \\ B_i(\varepsilon) &:= f_i(F^1)_\varepsilon \setminus A(\varepsilon). \end{aligned}$$

Then $F_\varepsilon = \bigcup_{i=1}^N B_i(\varepsilon) \cup F_\varepsilon \cap A(\varepsilon)$ is a disjoint union and thus,

$$\mathcal{L}^d(F_\varepsilon) = \sum_{i=1}^N \mathcal{L}^d(F_\varepsilon \cap B_i(\varepsilon)) + \mathcal{L}^d(F_\varepsilon \cap A(\varepsilon)).$$

Similarly,

$$\mathcal{L}^d(f_i(F^1)_\varepsilon) = \mathcal{L}^d(f_i(F^1)_\varepsilon \cap B_i(\varepsilon)) + \mathcal{L}^d(f_i(F^1)_\varepsilon \cap A(\varepsilon)),$$

since $B_i(\varepsilon) \cap f_j(F^1)_\varepsilon = \emptyset$ for $i \neq j$. Furthermore, $F_\varepsilon \cap B_i(\varepsilon) = f_i(F^1)_\varepsilon \cap B_i(\varepsilon)$, so that

$$\mathcal{L}^d(F_\varepsilon) = \sum_{i=1}^N \mathcal{L}^d(f_i(F^1)_\varepsilon \cap B_i(\varepsilon)) + \mathcal{L}^d(F_\varepsilon \cap A(\varepsilon)).$$

Substituting the last two relationships in the left hand side of Equation (10) we obtain

$$\begin{aligned} & |\mathbb{E}(\mathcal{L}^d(F_\varepsilon) - \sum_{i=1}^N \mathcal{L}^d(f_i(F^1)_\varepsilon))| \\ &= |\mathbb{E} \mathcal{L}^d(F_\varepsilon \cap A(\varepsilon)) - \sum_{i=1}^N \mathbb{E} \mathcal{L}^d((f_i(F^1)_\varepsilon) \cap A(\varepsilon))| \\ &\leq \mathbb{E} \mathcal{L}^d(F_\varepsilon \cap A(\varepsilon)) + \mathbb{E} \sum_{i=1}^N \mathcal{L}^d((f_i(F^1)_\varepsilon) \cap A(\varepsilon)) =: S_1(\varepsilon) + S_2(\varepsilon). \end{aligned}$$

Denote

$$f(O) := \bigcup_{i=1}^N f_i(O). \quad (11)$$

Taking into regard that $A(\varepsilon) \subset (f(O)^c)_\varepsilon$, which follows from (UOSC), Lemma 2 below provides the estimate

$$\sup_{\varepsilon} \frac{S_1(\varepsilon)}{\varepsilon^{d-D+\delta}} \leq \mathbb{E} \sup_{\varepsilon} \frac{\mathcal{L}^d(F_\varepsilon \cap (f(O)^c)_\varepsilon)}{\varepsilon^{d-D+\delta}} < \infty$$

with some $\delta > 0$. For the above summand $S_2(\varepsilon)$ the problem can also be reduced to Lemma 2 by the following arguments. Using the scaling property of \mathcal{L}^d and $f_i^{-1}(A(\varepsilon)) \subset (O^c)_{\varepsilon/r_i}$ we get for $\delta < D$,

$$\begin{aligned} \sup_{\varepsilon} \frac{S_2(\varepsilon)}{\varepsilon^{d-D+\delta}} &\leq \mathbb{E} \sup_{\varepsilon} \sum_{i=1}^N \frac{\mathcal{L}^d((f_i(F^1)_\varepsilon) \cap A(\varepsilon))}{\varepsilon^{d-D+\delta}} \\ &= \mathbb{E} \sup_{\varepsilon} \sum_{i=1}^N \frac{r_i^d}{\varepsilon^{d-D+\delta}} \mathcal{L}^d(F_{\varepsilon/r_i}^1 \cap f_i^{-1}(A(\varepsilon))) \\ &\leq \mathbb{E} \sup_{\varepsilon} \sum_{i=1}^N r_i^{D-\delta} \frac{r_i^{d-D+\delta}}{\varepsilon^{d-D+\delta}} \mathcal{L}^d(F_{\varepsilon/r_i}^1 \cap ((O^c)_{\varepsilon/r_i})) \\ &\leq \mathbb{E} \sum_{i=1}^N r_i^{D-\delta} \sup_{\varepsilon} \frac{\mathcal{L}^d(F_\varepsilon^1 \cap (O^c)_\varepsilon)}{\varepsilon^{d-D+\delta}} \\ &\leq \mathbb{E} N \mathbb{E} \sup_{\varepsilon} \frac{\mathcal{L}^d(F_\varepsilon \cap (O^c)_\varepsilon)}{\varepsilon^{d-D+\delta}} \leq \mathbb{E} N \mathbb{E} \sup_{\varepsilon} \frac{\mathcal{L}^d(F_\varepsilon \cap (f(O)^c)_\varepsilon)}{\varepsilon^{d-D+\delta}}. \end{aligned}$$

In the last two inequalities we have used that the random set F^1 is independent of the random number N and has the same distribution as F , and then the set inclusion $O^c \subset f(O)^c$. In view of Lemma 2 below the last expression is finite for some $\delta > 0$, which completes the proof of (10). \square

Lemma 2. *There exists some $0 < \delta < D$ such that*

$$\mathbb{E} \sup_{\varepsilon} \varepsilon^{-(d-D+\delta)} \mathcal{L}^d(F_\varepsilon \cap (f(O)^c)_\varepsilon) < \infty.$$

In the proof of this lemma we will use special random *Markov stoppings* for the coding tree Σ_* , i.e.,

$$\Sigma(r) := \{\sigma \in \Sigma_* : r_\sigma \leq r < r_{\sigma||\sigma|-1}\}, \quad r > 0. \quad (12)$$

From the construction of the random fractal F it follows easily that for all $r > 0$,

$$F = \bigcup_{\sigma \in \Sigma(r)} f_\sigma(F^{|\sigma|}), \quad \text{a.s.} \quad (13)$$

Furthermore, $\Sigma(r)$ satisfies the following.

Proposition 1.

$$\mathbb{E} \sum_{\sigma \in \Sigma(r)} r_\sigma^D = 1.$$

Proof. From Equation (5) and the product structure of the basis probability space we get for all $k \in \mathbb{N}$ that

$$\mathbb{E} \sum_{\sigma \in \Sigma_k} (r_\sigma)^D = 1,$$

since $\sum_{\sigma \in \Sigma_k} (r_\sigma)^D = \prod_{n=1}^k \sum_{\sigma \in \Sigma_1^n} (r_\sigma)^D$ is the product of n independent random variables, each with expectation 1. (Recall the notation $\Sigma_l^n(\omega) = \Sigma_l(\theta^n \omega)$ and $r_{\sigma'}^n(\omega) = r_{\sigma'}(\theta^n \omega)$, $l, n \in \mathbb{N}$.) Then we infer for $M \geq |\ln r|(\ln r_{\max})^{-1} + 1$,

$$\begin{aligned} 1 &= \mathbb{E} \sum_{\sigma \in \Sigma_M} (r_\sigma)^D = \mathbb{E} \left(\sum_{n=1}^{M-1} \sum_{\sigma \in \Sigma(r), |\sigma|=n} (r_\sigma)^D \sum_{\sigma' \in \Sigma_{M-n}^n} (r_{\sigma'}^n)^D \right) \\ &= \sum_{n=1}^{M-1} \mathbb{E} \left(\sum_{\sigma \in \Sigma(r), |\sigma|=n} (r_\sigma)^D \sum_{\sigma' \in \Sigma_{M-n}^n} (r_{\sigma'}^n)^D \right) \\ &= \sum_{n=1}^{M-1} \mathbb{E} \sum_{\sigma \in \Sigma(r), |\sigma|=n} (r_\sigma)^D \mathbb{E} \sum_{\sigma' \in \Sigma_{M-n}^n} (r_{\sigma'}^n)^D \\ &= \sum_{n=1}^{M-1} \mathbb{E} \sum_{\sigma \in \Sigma(r), |\sigma|=n} (r_\sigma)^D = \mathbb{E} \sum_{n=1}^{M-1} \sum_{\sigma \in \Sigma(r), |\sigma|=n} (r_\sigma)^D = \mathbb{E} \sum_{\sigma \in \Sigma(r)} (r_\sigma)^D, \end{aligned}$$

where we have used in the fourth equation that in the product under the expectation the second sum $\sum_{\sigma' \in \Sigma_{M-n}^n} (r_{\sigma'}^n)^D$ is independent of the first one and after this that

$$\mathbb{E} \sum_{\sigma' \in \Sigma_{M-n}^n} (r_{\sigma'}^n)^D = \mathbb{E} \sum_{\sigma' \in \Sigma_{M-n}} (r_{\sigma'})^D = 1.$$

Then the above equalities lead to the assertion. \square

Proof of Lemma 2. Equation (13) implies

$$\mathcal{L}^d(F_\varepsilon \cap (f(O)^c)_\varepsilon) \leq \sum_{\sigma \in \Sigma(\varepsilon^*)} \mathcal{L}^d(f_\sigma(F^{|\sigma|})_\varepsilon \cap (f(O)^c)_\varepsilon) \leq \sum_{\sigma \in \Sigma_b(\varepsilon)} \mathcal{L}^d(f_\sigma(F^{|\sigma|})_\varepsilon),$$

where the random *boundary code tree* $\Sigma_b(\varepsilon)$ is defined as

$$\Sigma_b(\varepsilon) := \{\sigma \in \Sigma(\varepsilon^*) : f_\sigma(F^{|\sigma|})_\varepsilon \cap (f(O)^c)_\varepsilon \neq \emptyset\}, \quad (14)$$

and $\varepsilon^* := c\varepsilon$ for some constant $c > 0$, which will be determined below (see Equation (18)).

For $\sigma \in \Sigma(\varepsilon^*)$ we get

$$\mathcal{L}^d(f_\sigma(F^{|\sigma|})_\varepsilon) = r_\sigma^d \mathcal{L}^d((F^{|\sigma|})_{\varepsilon/r_\sigma}) \leq \text{const } \varepsilon^d,$$

since $(F^{|\sigma|})_{\varepsilon/r_\sigma} \subset O_{1/cr_{\min}}$. Therefore, the right hand side of the above estimates does not exceed $\text{const} \#(\Sigma_b(\varepsilon)) \varepsilon^d$, where $\#(\cdot)$ denotes the number of elements of a finite set, and $\#(\emptyset) := 0$. Hence, it suffices to show that

$$\sup_{\varepsilon} \varepsilon^{D-\delta} \mathbb{E} \#(\Sigma_b(\varepsilon)) < \infty \quad \text{for some } \delta > 0. \quad (15)$$

To this end we now will use (USOSC), i.e., (UOSC) and $\mathbb{P}(F \cap O \neq \emptyset) > 0$, which implies that there exist some constants $\alpha > 0$ and $0 < \rho < 1$ such that

$$\mathbb{P}(\Sigma(\rho, \alpha) \neq \emptyset) > 0 \quad \text{for } \Sigma(\rho, \alpha) := \{\tau \in \Sigma(\rho) : d(x, \partial O) > \alpha, x \in f_\tau(F^{|\tau|})\}. \quad (16)$$

(Otherwise, by construction, the random fractal set F would concentrate on the boundary of O , which is a contradiction.)

Recall from Proposition 1 that $\mathbb{E} \sum_{\tau \in \Sigma(\rho)} r_\tau^D = 1$. Then let δ be determined by

$$\mathbb{E} \left(\sum_{\tau \in \Sigma(\rho) \setminus \Sigma(\rho, \alpha)} r_\tau^{D-\delta} \right) = 1. \quad (17)$$

We now choose

$$\varepsilon^* := 2(\alpha r_{\min})^{-1} \varepsilon. \quad (18)$$

Then we get for $i = 1, \dots, N$ and $i\sigma \in \Sigma(\varepsilon^*)$ with $\sigma = \tau\sigma'$ for some $\tau \in \Sigma^1(\rho, \alpha)$ that

$$f_{i\sigma}(F^{[i\sigma]})_\varepsilon \cap (f(O)^c)_\varepsilon = \emptyset.$$

(To see this note that for any $x \in f_{i\sigma}(F^{[i\sigma]})_\varepsilon$ there exists a $y \in f_{i\sigma}(F^{[i\sigma]})$ such that $|x - y| \leq \varepsilon$. Furthermore, $y \in f_{i\sigma}(F^{[i\sigma]}) \subset f_i(F^1) \subset f_i(O) \subset f(O)$, and $d(y, \partial f(O)) \geq d(y, f_i(O)) > r_i \alpha > r_{i\sigma} \alpha > \varepsilon^* r_{\min} \alpha = 2\varepsilon$. Consequently, $d(x, \partial f(O)^c) \geq d(y, \partial f(O)^c) - |x - y| > 2\varepsilon - \varepsilon = \varepsilon$, i.e., $x \notin (f(O)^c)_\varepsilon$.)

From this we obtain

$$\#(\Sigma_b(\varepsilon)) = \sum_{i=1}^N \#(\{w \in \Sigma(\varepsilon^*) : w = i\sigma, \sigma \in \Xi^1(\varepsilon^*/r_i)\}),$$

where the random sets $\Xi(r)$, $r > 0$, are defined as

$$\Xi(r) := \Sigma(r) \setminus \{\sigma \in \Sigma(r) : \sigma = \tau\sigma' \text{ for some } \tau \in \Sigma(\rho, \alpha)\}.$$

With these notations we get

$$\begin{aligned} \varepsilon^{D-\delta} \mathbb{E} \#(\Sigma_b(\varepsilon)) &\leq \varepsilon^{D-\delta} \mathbb{E} \sum_{i=1}^N \#(\Xi^1(\varepsilon^*/r_i)) \\ &= 1/2(\alpha r_{\min})^{D-\delta} \mathbb{E} \sum_{i=1}^N r_i^{D-\delta} (\varepsilon^*/r_i)^{D-\delta} \#(\Xi^1(\varepsilon^*/r_i)) \\ &= 1/2(\alpha r_{\min})^{D-\delta} \mathbb{E} \sum_{i=1}^N r_i^{D-\delta} (\varepsilon^*/r_i)^{D-\delta} \mathbb{E} \#(\Xi^1(\varepsilon^*/r_i)) \\ &= \text{const} \mathbb{E} \sum_{i=1}^N r_i^{D-\delta} \psi(\varepsilon^*/r_i), \end{aligned}$$

where we have used that Ξ^1 is independent of the events in the first step and has the same distribution as Ξ and then the notation $\psi(r) := r^{D-\delta} \mathbb{E} \#(\Xi(r))$. Now it suffices to show that the function ψ is bounded.

Similarly as above, using Equation (16) and the definition of $\Xi(\varepsilon)$ we infer for sufficiently large M and $\varepsilon < \rho$,

$$\begin{aligned} \psi(\varepsilon) &= \mathbb{E} \sum_{\tau \in \Sigma(\rho) \setminus \Sigma(\rho, \alpha)} r_\tau^{D-\delta} (\varepsilon/r_\tau)^{D-\delta} \#(\Xi^{|\tau|}(\varepsilon/r_\tau)) \\ &= \sum_{n=1}^M \mathbb{E} \sum_{\tau \in \Sigma(\rho) \setminus \Sigma(\rho, \alpha), |\tau|=n} r_\tau^{D-\delta} (\varepsilon/r_\tau)^{D-\delta} \#(\Xi^n(\varepsilon/r_\tau)) \\ &= \sum_{n=1}^M \mathbb{E} \sum_{\tau \in \Sigma(\rho) \setminus \Sigma(\rho, \alpha), |\tau|=n} r_\tau^{D-\delta} \varphi(\varepsilon/r_\tau) = \mathbb{E} \sum_{\tau \in \Sigma(\rho) \setminus \Sigma(\rho, \alpha)} r_\tau^{D-\delta} \varphi(\varepsilon/r_\tau) \\ &\leq \mathbb{E} \sum_{\tau \in \Sigma(\rho) \setminus \Sigma(\rho, \alpha)} r_\tau^{D-\delta} \sup_{\varepsilon' \geq \varepsilon/\rho} \psi(\varepsilon') = \sup_{\varepsilon' \geq \varepsilon/\rho} \psi(\varepsilon'), \end{aligned}$$

where we have used that the random sets $\Xi^n(r)$ are independent of the behaviour of the system up to the step n via conditional expectation, that they have the same distribution as $\Xi(r)$, and then Equation (17). Hence, $\psi(\varepsilon) \leq \sup_{\varepsilon' \geq \varepsilon/\rho} \psi(\varepsilon')$ for any $\varepsilon < \rho$, which implies

$$\sup_{\varepsilon \geq \rho^{k+1}} \psi(\varepsilon) \leq \sup_{\varepsilon \geq \rho^k} \psi(\varepsilon) \text{ for all } k.$$

Since the function ψ is bounded on any interval away from zero it is bounded on $(0, \infty)$. This completes the proof of Equation (15). \square

Proof of Theorem 2. Recall that O_{-r} is the inner parallel set of O of distance r . Suppose that $\varepsilon_0 < \text{diam } O$ and $C \subset O_{-\varepsilon_0}$ for some non-empty random compact set C defined on our basic probability space. By (UOSC) for different $\sigma \in \Sigma(r)$ the random sets $f_\sigma(C^{|\sigma|}) \subset f_\sigma(O_{-\varepsilon_0})$ are disjoint and hence,

$$\mathcal{L}^d(F_\varepsilon) \geq \sum_{\sigma \in \Sigma(r)} \mathcal{L}^d(F_\varepsilon \cap f_\sigma(C^{|\sigma|})) =: S.$$

Choosing now

$$r = \varepsilon^* := (r_{\min} \varepsilon_0)^{-1} \varepsilon$$

we get for all $\sigma \in \Sigma(\varepsilon^*)$ that $r_\sigma \leq (r_{\min} \varepsilon_0)^{-1} \varepsilon < r_\sigma (r_{\min})^{-1}$, i.e., $\varepsilon < r_\sigma \varepsilon_0$, and thus,

$$f_\sigma(C^{|\sigma|}) \subset f_\sigma(O_{-\varepsilon_0}) = f_\sigma(O)_{-r_\sigma \varepsilon_0} \subset f_\sigma(O)_{-\varepsilon}.$$

Furthermore, $F_\varepsilon \cap f_\sigma(O)_{-\varepsilon} = f_\sigma(F^{|\sigma|})_\varepsilon \cap f_\sigma(O)_{-\varepsilon}$. Therefore,

$$F_\varepsilon \cap f_\sigma(C^{|\sigma|}) = f_\sigma(F^{|\sigma|})_\varepsilon \cap f_\sigma(C^{|\sigma|}).$$

Hence,

$$\mathcal{L}^d(F_\varepsilon \cap f_\sigma(C^{|\sigma|})) = \mathcal{L}^d(f_\sigma(F^{|\sigma|})_\varepsilon \cap f_\sigma(C^{|\sigma|})),$$

and for $M \geq \lceil \ln \varepsilon^* / (\ln r_{\max}) \rceil + 1$ the above sum is equal to

$$\begin{aligned} S &= \sum_{\sigma \in \Sigma(\varepsilon^*)} \mathcal{L}^d(f_\sigma(F^{|\sigma|})_\varepsilon \cap f_\sigma(C^{|\sigma|})) = \sum_{\sigma \in \Sigma(\varepsilon^*)} r_\sigma^d \mathcal{L}^d((F^{|\sigma|})_\varepsilon / r_\sigma \cap C^{|\sigma|}) \\ &= \sum_{n=1}^M \sum_{\sigma \in \Sigma(\varepsilon^*), |\sigma|=n} r_\sigma^d \mathcal{L}^d((F^n)_{\varepsilon/r_\sigma} \cap C^n). \end{aligned}$$

Using that $\varepsilon/r_\sigma \geq r_{\min} \varepsilon_0$ and $r_\sigma > \varepsilon/\varepsilon_0$ for $\sigma \in \Sigma(\varepsilon^*)$ we infer the following.

$$\begin{aligned} \mathbb{E} \mathcal{L}^d(F_\varepsilon) &\geq \sum_{n=1}^M \mathbb{E} \sum_{\sigma \in \Sigma(\varepsilon^*), |\sigma|=n} r_\sigma^d \mathcal{L}^d((F^n)_{r_{\min} \varepsilon_0} \cap C^n) \\ &= \mathbb{E} \left(\sum_{\sigma \in \Sigma(\varepsilon^*)} r_\sigma^d \right) \mathbb{E} \mathcal{L}^d(F_{r_{\min} \varepsilon_0} \cap C) \\ &\geq (\varepsilon/\varepsilon_0)^d \mathbb{E}(\#(\Sigma(\varepsilon^*))) \mathbb{E} \mathcal{L}^d(F_{r_{\min} \varepsilon_0} \cap C) \\ &\geq \varepsilon^{d-D} \text{const} \mathbb{E} \mathcal{L}^d(F_{r_{\min} \varepsilon_0} \cap C), \end{aligned}$$

where we have used that the random sets F^n and C^n are independent of the behaviour of the system up to the step n and in the last inequality that

$$\mathbb{E} \#(\Sigma(\varepsilon^*)) \geq \text{const} \varepsilon^{-D}$$

which follows from

$$1 = \mathbb{E} \sum_{\sigma \in \Sigma(\varepsilon^*)} (r_\sigma)^D \leq \varepsilon^D \text{const} \mathbb{E} \#(\Sigma(\varepsilon^*)).$$

(Here and below, const stands for different positive constants.) Therefore, it remains to show that

$$\mathbb{E} \mathcal{L}^d(F_{r_{\min} \varepsilon_0} \cap C) > 0 \quad (19)$$

for some $\varepsilon_0 < \text{diam } O$ and some random compact set $C \subset O_{-\varepsilon_0}$.

Recall Equation (16) for the definition of the code set $\Sigma(\rho, \alpha)$. Choose $\varepsilon_0 := \alpha(1 + r_{\min})^{-1}$, where α in Equation (16) can be taken such that $\varepsilon_0 < \text{diam}(O)$, and

$$C := \bigcup_{\tau \in \Sigma(\rho, \alpha)} f_\tau(F^{|\tau|})_{r_{\min} \varepsilon_0}.$$

Since $F^{|\tau|} \subset O_{-\alpha}$ for $\tau \in \Sigma(\rho, \alpha)$ and $\alpha - r_{\min} \varepsilon_0 = \varepsilon_0$, we obtain that $C \subset O_{-\varepsilon_0}$. Furthermore,

$$\mathbb{E} \mathcal{L}^d(F_{r_{\min} \varepsilon_0} \cap C) \geq \text{const}(r_{\min} \varepsilon_0)^d \mathbb{P}(\Sigma(\rho, \alpha) \neq \emptyset) > 0.$$

Consequently, our C and ε_0 satisfy Equation (19). \square

Proof of Lemma 1. Recall that the Hausdorff distance between two nonempty compact sets K and L in \mathbb{R}^d is given by

$$d_H(K, L) := \max \left(\max_{x \in K} d(x, L), \max_{y \in L} d(y, K) \right),$$

where $d(x, L) := \min_{y \in L} |x - y|$, or equivalently by

$$d_H(K, L) = \min \{ r \geq 0 : K \subset L_r, L \subset K_r \}.$$

In order to prove (i) let for $\varepsilon < r$ the sets $K_{i,j}(\varepsilon), L_{i,j} \in \mathcal{K}, 1 \leq i \leq k, 1 \leq j \leq l$, be such that

$$\max_{i,j} d_H(K_{i,j}(\varepsilon), L_{i,j}) < \varepsilon.$$

Then we get

$$\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_{r-\varepsilon} \subset \bigcup_{i=1}^k \bigcap_{j=1}^l (K_{i,j}(\varepsilon))_r \subset \bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_{r+\varepsilon}$$

and consequently,

$$\begin{aligned} & \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_{r-\varepsilon} \right) - \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_r \right) \\ & \leq \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (K_{i,j}(\varepsilon))_r \right) - \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_r \right) \\ & \leq \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_{r+\varepsilon} \right) - \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_r \right). \end{aligned}$$

Using Rataj and Winter ([21] Prop. 2.3) one obtains that the boundary of the set $\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_r$ is $(d-1)$ -rectifiable, and thus it has vanishing Lebesgue measure. Therefore, the left and right hand sides of the above inequalities tend to zero as $\varepsilon \rightarrow 0$. Hence,

$$\lim_{\varepsilon \rightarrow 0} \left| \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (K_{i,j}(\varepsilon))_r \right) - \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_r \right) \right| = 0.$$

Similarly one infers

$$\lim_{s \rightarrow r} \left| \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_r \right) - \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_s \right) \right| = 0.$$

Finally,

$$\begin{aligned} & \left| \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (K_{i,j}(\varepsilon))_r \right) - \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_s \right) \right| \\ & \leq \left| \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (K_{i,j}(\varepsilon))_r \right) - \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_r \right) \right| \\ & \quad + \left| \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_r \right) - \mathcal{L}^d \left(\bigcup_{i=1}^k \bigcap_{j=1}^l (L_{i,j})_s \right) \right|, \end{aligned}$$

which tends to zero as $\varepsilon \rightarrow 0$ and $s \rightarrow r$ by the above arguments. This proves (i).

In order to show (ii) we apply (i) for $k = l = 1$ to the random set K in order to see the continuity of the random function $r \mapsto \mathcal{L}^d(K_r)$ and thus, that of $r \mapsto \mathbb{E}\mathcal{L}^d(K_r)$, since $\text{diam}(K) \leq c$ and thus $\mathcal{L}^d(K_r) \leq \text{const}(c+r)^d$ with probability 1. \square

Funding: This research received no external funding.

Acknowledgments: This work was supported by DFG Grant ZA 242/8-1.

Conflicts of Interest: The author declares no conflict of interest.

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