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Noncommutative Functional Calculus and Its Applications on Invariant Subspace and Chaos

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Abstract: Let $T : \mathbb{H} \to \mathbb{H}$ be a bounded linear operator on a separable Hilbert space $\mathbb{H}$. In this paper, we construct an isomorphism $F_{xx^*} : \mathcal{L}^2(\sigma(|T-a|), \mu_{|T-a|,\xi}) \to \mathcal{L}^2(\sigma(|(T-a)^*|), \mu_{|(T-a)^*|, F^{\mathbb{H}}_{xx*}\xi})$ such that $(F_{xx^*})^2 = identity$ and $F^{\mathbb{H}}_{xx*}$ is a unitary operator on $\mathbb{H}$ associated with F_{xx^*}. With this construction, we obtain a noncommutative functional calculus for the operator T and $F_{xx^*} = identity$ is the special case for normal operators, such that $S = R_{|(S-a)|,\xi}(M_{z\phi(z)} + a)R^{-1}_{|S-a|,\xi}$ is the noncommutative functional calculus of a normal operator S, where $a \in \rho(T)$, $R_{|T-a|,\xi} : \mathcal{L}^2(\sigma(|T-a|), \mu_{|T-a|,\xi}) \to \mathbb{H}$ is an isomorphism and $M_{z\phi(z)} + a$ is a multiplication operator on $\mathcal{L}^2(\sigma(|S-a|), \mu_{|S-a|,\xi})$. Moreover, by F_{xx*} we give a sufficient condition to the invariant subspace problem and we present the Lebesgue class $\mathcal{B}_{Leb}(\mathbb{H}) \subset \mathcal{B}(\mathbb{H})$ such that T is Li-Yorke chaotic if and only if T^{*-1} is for a Lebesgue operator T.

Keywords: chaos; invariant subspace; Lebesgue operator; noncommutative functional calculus

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1. Introduction

1.1. Invariant Subspace

The invariant subspace problem has been stated by Beurling and von Neumann [1]. It can be formulated as follows.

Problem 1. *Does every bounded linear operator on a given linear space have a non-trivial invariant subspace?*

In 1966, Bernstein et al. [2] showed that if T is a bounded linear operator on a complex Hilbert space $\mathbb{H}$ and p is a nonzero polynomial such that $p(T)$ is compact, then T has non-trivial invariant subspace. Especially, when $p(t) = t$, which is, T itself is compact, the result was proved independently by von Neumann and N. Aronszajn, and in [3], this result was extended to compact operators on a Banach space.

Let T be a bounded linear operator on a Banach space. In 1973, Lomonosov [4] proved that if T is not a scalar multiple of the identity and commutes with a nonzero compact operator, then T has a non-trivial hyperinvariant subspace, which is, any bounded linear operator commuting with T has a non-trivial invariant subspace (other results see [5–7]).

In 1976, Enflo [8] was the first to construct an operator on a Banach space having no non-trivial invariant subspace and Nordgren et al. [9] proved that every operator has an invariant subspace if and only if every pair of idempotents has a common invariant subspace.

In 1983, Atzmon [10] constructed a nuclear Fréchet space $\mathbb{F}$ and a bounded linear operator, which has no non-trivial invariant subspace. Especially, in 1984, C. J. Read made an example, such that there is a bounded linear operator without non-trivial invariant subspace on ℓ_1 [11].

In 2011, Argyros et al. [12] constructed the first example of a Banach space for which every bounded linear operator on the space has the form $\lambda + K$ where λ is a real scalar and K is a compact operator, such that every bounded linear operator on the space has a non-trivial invariant subspace.

In 2013, Marcoux et al. [13] showed that, if a closed algebra of operators on a Hilbert space has a non-trivial almost-invariant subspace, then it has a non-trivial invariant subspace (more results see [14–18]).

In 2019, Tcaciuc [19] proved that, for any bounded operator T acting on an infinite-dimensional Banach space, there exists an operator F of rank at most one such that $T + F$ has an invariant subspace of infinite dimension and codimension.

For finite-dimensional vector spaces or nonseparable Hilbert spaces, the result is trivial. However, for infinite-dimensional separable Hilbert spaces, the problem is, after a long period of time, not yet completely solved.

1.2. Linear Dynamics

With the development of operator theory and dynamics progress, there are many papers about C^*-algebras and dynamics. Additionally, "the fundamental theorem of C^*-algebras [20]" is Gelfand-Naimark theorem [21]. Subsequently, in [22], Fujimoto said that this theorem eventually opened the gate to the subject of C^*-algebras. Hence, there are various attempts to generalize this theorem [23–27].

For the research on Problem 1 and with the development of chaos, Operator Dynamics or Linear Dynamics has aroused extensive attention as an important branch of functional analysis, which was probably born in 1982 with the Toronto Ph. D. thesis of C. Kitai [28]. More details of this subject can be found in [29–33].

If X is a metric space and T is a continuous self-map on X, then the pair (X, T) is called a topological dynamic systems, which is induced by the iteration

$$T^n = \underbrace{T \circ \cdots \circ T}_{n}, \qquad n \in \mathbb{N}, \text{ where } 0 \in \mathbb{N}.$$

Moreover, if T is a continuous invertible self-map on X, then (X, T) is called an invertible dynamic and if the metric space X and the continuous self-map T are both linear, then the topological dynamic systems (X, T) is called a linear dynamic.

For invertible dynamics, the relationship of Li-Yorke chaos between (X, f) and (X, f^{-1}) was raised by Stockman as an open question [34]. Additionally, in [35,36] and [37], the authors give counterexamples for this question in noncompact spaces and compact spaces, respectively. For an invertible bounded linear operator $T \in \mathcal{B}(\mathbb{H})$, the chaotic relationship between $(\mathbb{H}, T)$ and $(\mathbb{H}, T^{*-1})$ is also interesting.

Next, we give the following definition

Definition 1 (Li-Yorke chaos)**.** *Let $T \in \mathcal{B}(\mathbb{H})$. If there exists $x \in \mathbb{H}$, such that satisfies:*

$$(a)\ \overline{\lim_{n\to\infty}} |T^n(x)\| > 0 \quad \textit{and}$$
$$(b)\ \underline{\lim_{n\to\infty}} \|T^n(x)\| = 0,$$

then the operator T is said to be Li-Yorke chaotic, and x is called a Li-Yorke chaotic point of T.

An example of an operator T that is Li-Yorke chaotic but T^{*-1} is not can be found in [38]. However, presently there is no general method to do this research. In fact, the C*-algebra $\mathcal{A}(T)$ generated by T cannot be used for that.

1.3. Motivation and Main Results

For an n-tuple $\mathcal{T}$ of not necessarily commuting operators, Colombo et al. [39] put to use the notion of slice monogenic functions [40] to define a new functional calculus, which is consistent with the Riesz–Dunford calculus in the case of a single operator and that allows the explicit construction of the eigenvalue equation for the n-tuple $\mathcal{T}$ based on a new notion of spectrum for $\mathcal{T}$ (more results, see [41–44]).

In 2010, for bounded operators defined on quaternionic Banach spaces, Colombo et al. [45] developed a noncommutative functional calculus that is based on the new notion of slice-regularity and that is based on the key tools of a new resolvent operator and a new eigenvalue problem, also, they extended this calculus to the unbounded case [46] (more results, see [47]).

In 2018, Monguzzi et al. [48] characterized the closed invariant subspaces for the (*−) multiplier operator of the quaternionic space of slice $\mathcal{L}^2$ functions, obtained the inner-outer factorization theorem for the quaternionic Hardy space on the unit ball and provided a characterization of quaternionic outer functions in terms of cyclicity.

In this paper, we give a noncommutative functional calculus for $T \in \mathcal{B}(\mathbb{H})$. Additionally, by this construction, we give some applications, such as its applications on the invariant subspace problem and chaos. The precise meaning of the multiplication operator $M_{z\psi(z)} = M_z M_{\psi(z)} = M_{\psi(z)} M_z = M_{\psi(z)z}$ will become clear in Theorem 3.

Let $\mathbb{H}$ be a separable Hilbert space over $\mathbb{C}$, $\mathcal{B}(\mathbb{H})$ be the set of all bounded linear operator on $\mathbb{H}$. For any given $T \in \mathcal{B}(\mathbb{H})$, we obtain a C*-algebra $\mathcal{A}(|T-a|)$ associated with the polar decomposition $T - a = U|T-a|$, where $a \in \rho(T)$ and ξ is a $\mathcal{A}(|T-a|)$-cyclic vector, such that $\mathbb{H} = \overline{\mathcal{A}(|T-a|)\xi}$. In this paper, we construct an isomorphism F_{xx^*}, such that the following diagram is valid.

$$\begin{array}{ccc}
\mathcal{L}^2(\sigma(|T-a|), \mu_{|T-a|,\xi}) & \xrightarrow{\quad R_{|T-a|,\xi} \quad} & \mathbb{H} \\
F_{xx^*} \downarrow & & \downarrow F^{\mathbb{H}}_{xx^*} \\
\mathcal{L}^2(\sigma(|(T-a)^*|), \mu_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi}) & \xrightarrow[\quad R_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi} \quad]{} & \mathbb{H}
\end{array}$$

where $F^{\mathbb{H}}_{xx*}$ is the corresponding unitary operator associated with the isomorphism F_{xx^*} and $Fix(F^{\mathbb{H}}_{xx^*}) \neq \emptyset$, $(F^{\mathbb{H}}_{xx^*})^2 = identity$ and $(F_{xx^*})^2 = identity$. With this construction, we get a noncommutative functional calculus for the operator T such that

$$T - a = R_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T-a|,\xi}\,.$$

Especially, $F_{xx^*} = identity$, which is the special case for normal operators, will become clear in Corollary 3, and, in this special case, we get that the noncommutative functional calculus of a normal operator S is just only $S = R_{|(S-a)|,\xi}(M_{z\phi(z)} + a)R^{-1}_{|S-a|,\xi}$, which is compatible with the classical normal operator functional calculus of [49]. Where $\psi(z) \in \mathcal{L}^\infty(\sigma(|T-a|), \mu_{|T-a|,\xi})$ and $\phi(z) \in \mathcal{L}^\infty(\sigma(|S-a|), \mu_{|S-a|,\xi})$.

Moreover, from F_{xx*}, we deduce a sufficient condition to Problem 1 on infinite-dimensional separable Hilbert spaces and present the Lebesgue class $\mathcal{B}_{Leb}(\mathbb{H}) \subset \mathcal{B}(\mathbb{H})$, such that, if T is a Lebesgue operator, then T is Li-Yorke chaotic if and only if T^{*-1} is.

In fact, we get that

$$\mathcal{B}_{Leb}(\mathbb{H}) \cap \mathcal{B}_{Nor}(\mathbb{H}) \neq \emptyset$$

and

$$\mathcal{B}_{Leb}(\mathbb{H}) \cap (\mathcal{B}(\mathbb{H}) \setminus \mathcal{B}_{Nor}(\mathbb{H})) \neq \emptyset,$$

where $\mathcal{B}_{Nor}(\mathbb{H})$ is the set of all normal operator on $\mathbb{H}$.

2. Decomposition and Isomorphic Representation

In this paper, $\bar{f}(\cdot)$ means the conjugate of the complex function $f(\cdot)$. Let X be a compact subset of $\mathbb{C}$, $\mathcal{C}(X)$ be the set of all continuous function on X, and $\mathcal{P}(x)$ be the set of all polynomial on X. For any given $T \in \mathcal{B}(\mathbb{H})$, let $\sigma(T)$ be its spectrum.

Following the polar decomposition theorem [50] (p. 15), we get that

$$T = U|T| \quad \text{and} \quad |T|^2 = T^*T.$$

Let $\mathcal{A}(|T|)$ be the complex C^*-algebra generated by $|T|$ and 1. Obviously, if T is invertible, then U is a unitary operator.

Lemma 1. *Let $X \subseteq \mathbb{C}$ be a compact subset not containing zero. If $\mathcal{P}(x)$ is dense in $\mathcal{C}(X)$, then $\mathcal{P}(\frac{1}{x})$ is also dense in $\mathcal{C}(X)$.*

Proof. By the properties of complex polynomials, we get that $\mathcal{P}(\frac{1}{x})$ is a subalgebra of $\mathcal{C}(X)$, which is closed under the standard algebraic operations. In addition, we have:

(1) $1 \in \mathcal{P}(\frac{1}{x})$;
(2) $\mathcal{P}(\frac{1}{x})$ separate the points of X;
(3) If $p(\frac{1}{x}) \in \mathcal{P}(\frac{1}{x})$, then $\bar{p}(\frac{1}{x}) \in \mathcal{P}(\frac{1}{x})$.
We get the conclusion from the Stone–Weierstrass theorem [49] (p. 145). □

For $X \subseteq \mathbb{R}_+$, there is $x \neq y \iff x^2 \neq y^2$. With Lemma 1, we get the following result.

Lemma 2. *Let $X \subseteq \mathbb{R}_+$. If $\mathcal{P}(|x|)$ is dense in $\mathcal{C}(X)$, then $\mathcal{P}(|x|^2)$ is also dense in $\mathcal{C}(X)$.*

Using the GNS construction [49] (p. 250), for the C^*-algebra $\mathcal{A}(|T|)$, we have the following decomposition.

Lemma 3. *Let T be an invertible bounded linear operator on $\mathbb{H}$. Then there exists a sequence of nonzero $\mathcal{A}(|T|)$-invariant subspaces $\mathbb{H}_1, \mathbb{H}_2, \cdots, \mathbb{H}_i, \cdots$, such that:*

(1) $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \cdots \oplus \mathbb{H}_i \oplus \cdots$;
(2) For every $\mathbb{H}_i$, there is a $\mathcal{A}(|T|)$-cyclic vector ξ^i such that

$$\mathbb{H}_i = \overline{\mathcal{A}(|T|)\xi^i} = \overline{\mathcal{A}(|T|^{-1})\xi^i}$$

and

$$|T|\mathbb{H}_i = \mathbb{H}_i = |T|^{-1}\mathbb{H}_i \,.$$

Proof. The decomposition of (1) is obvious [51] (p. 54), Therefore,

$$|T|\mathbb{H}_i \subseteq \mathbb{H}_i \,,$$

that is,

$$\mathbb{H}_i \subseteq |T|^{-1}\mathbb{H}_i \,.$$

From Lemma 1, we get that

$$\mathbb{H}_i = \overline{\mathcal{A}(|T|)\xi^i} = \overline{\mathcal{A}(|T|^{-1})\xi^i}$$

and

$$|T|^{-1}\mathbb{H}_i \subseteq \mathbb{H}_i \,.$$

Hence,

$$|T|\mathbb{H}_i = \mathbb{H}_i = |T|^{-1}\mathbb{H}_i \,.$$

□

Let $\xi \in \mathbb{H}$ be a $\mathcal{A}(|T|)$-cyclic vector, such that $\mathcal{A}(|T|)\xi$ is dense in $\mathbb{H}$. Because the spectrum is closed and $\sigma(|T|) \neq \emptyset$, on $\mathcal{C}(\sigma(|T|))$, we can define the nonzero linear functional

$$\rho_{|T|,\xi} : \rho_{|T|,\xi}(f) = \langle f(|T|)\xi, \xi\rangle \,, \quad \forall f \in \mathcal{C}(\sigma(|T|)) \,.$$

It is easy to get that $\rho_{|T|,\xi}$ is a positive linear functional. By [51] (p. 54), and the Riesz–Markov theorem, on $\mathcal{C}(\sigma(|T|))$, we get that there is a uniquely finite positive Borel measure $\mu_{|T|,\xi}$, such that

$$\int\limits_{\sigma(|T|)} f(z)\mathrm{d}\mu_{|T|,\xi}(z) = \langle f(|T|)\xi, \xi\rangle \,, \quad \forall f \in \mathcal{C}(\sigma(|T|)) \,.$$

Theorem 1. *Let T be an invertible bounded linear operator on $\mathbb{H}$, $\mathcal{A}(|T^n|)$ be the complex C^*-algebra generated by $|T^n|$ and 1 and let ξ_n be a $\mathcal{A}(|T^n|)$-cyclic vector, such that $\overline{\mathcal{A}(|T^n|)\xi_n} = \mathbb{H}$, where $n \in \mathbb{N}$. Subsequently:*

(1) there is a uniquely positive linear functional

$$\int\limits_{\sigma(|T^n|)} f(z)\mathrm{d}\mu_{|T^n|,\xi_n}(z) = \langle f(|T^n|)\xi_n, \xi_n\rangle \,, \quad \forall f \in \mathcal{L}^1(\sigma(|T^n|), \mu_{|T^n|,\xi_n}) \,.$$

(2) there is a uniquely isomorphic representation $R_{|T^n|,\xi_n} : \mathcal{L}^2(\sigma(|T^n|), \mu_{|T^n|,\xi_n}) \to \mathbb{H}$ associated with the uniquely finite positive Borel measure $\mu_{|T^n|,\xi_n}$, which is complete.

Proof. (1) For $\mathcal{A}(|T^n|)$-cyclic vector ξ_n, we define the linear functional

$$\rho_{|T^n|,\xi_n}(f) = \langle f(|T^n|)\xi_n, \xi_n\rangle \,, \quad \forall f \in \mathcal{C}(\sigma(|T|)).$$

We get that, on $\mathcal{C}(\sigma(|T^n|))$, there is a uniquely finite positive Borel measure $\mu_{|T^n|,\xi_n}$, such that

$$\int\limits_{\sigma(|T^n|)} f(z)\mathrm{d}\mu_{|T^n|,\xi_n}(z) = \langle f(|T^n|)\xi_n, \xi_n\rangle \,, \quad \forall f \in \mathcal{C}(\sigma(|T^n|)).$$

Moreover, we can complete this Borel measure $\mu_{|T^n|,\xi_n}$ on $\sigma(|T^n|)$. For this completion, we keep the notation $\mu_{|T^n|,\xi_n}$. We know that this Borel measure is unique [52] .

For any $f \in \mathcal{L}^2(\sigma(|T^n|), \mu_{|T^n|,\xi_n})$, because of

$$\rho_{|T^n|,\xi_n}(|f|^2) = \rho_{|T^n|,\xi_n}(\bar{f}f) = \langle f(|T^n|)^* f(|T^n|)\xi_n, \xi_n\rangle = \|f(|T^n|)\xi_n\|^2_{\mathbb{H}} \geq 0,$$

we get that $\rho_{|T^n|,\xi_n}$ is a positive linear functional.

(2) We know that $\mathcal{C}(\sigma(|T^n|))$ is dense in $\mathcal{L}^2(\sigma(|T^n|), \mu_{|T^n|,\xi_n})$. For any $f, g \in \mathcal{C}(\sigma(|T^n|))$, we get

$$\begin{aligned}
&\langle f(|T^n|)\xi_n, g(|T^n|)\xi_n\rangle_{\mathbb{H}} \\
&= \langle g(|T^n|)^* f(|T^n|)\xi_n, \xi_n\rangle \\
&= \rho_{|T^n|,\xi_n}(\bar{g}f) = \int\limits_{\sigma(|T^n|)} f(z)\bar{g}(z)\mathrm{d}\mu_{|T^n|,\xi_n}(z) \\
&= \langle f, g\rangle_{\mathcal{L}^2(\sigma(|T^n|),\mu_{|T^n|,\xi_n})} \,.
\end{aligned}$$

Therefore,

$$R_{0,\xi_n} : \mathcal{C}(\sigma(|T^n|)) \to \mathbb{H}, \quad f(z) \to f(|T^n|)\xi_n$$

is a surjective isometry from $\mathcal{C}(\sigma(|T^n|))$ to $\mathcal{A}(|T^n|)\xi_n$.

Obviously, $\mathcal{C}(\sigma(|T^n|))$ and $\mathcal{A}(|T^n|)\xi_n$ are dense subspaces of $\mathcal{L}^2(\sigma(|T^n|), \mu_{|T^n|,\xi_n})$ and $\mathbb{H}$, respectively. Additionally, its closed extension

$$R_{|T^n|,\xi_n} : \mathcal{L}^2(\sigma(|T^n|), \mu_{|T^n|,\xi_n}) \to \mathbb{H}\,, \quad f(z) \to f(|T^n|)\xi_n$$

is an isomorphic operator.

Therefore, we get that $R_{|T^n|,\xi_n}$ is the uniquely isomorphic representation of $\mathbb{H}$ associated with the uniquely finite positive Borel measure $\mu_{|T^n|,\xi_n}$, which is complete. $\square$

Let T be an invertible bounded linear operator on $\mathbb{H} = \mathbb{H}_{\xi^1} \oplus \mathbb{H}_{\xi^2} \oplus \cdots \oplus \mathbb{H}_{\xi^i} \oplus \cdots$ and ξ^i be a $\mathcal{A}(|T|^{-1})$-cyclic vector such that $\mathbb{H}_{\xi^i} = \overline{\mathcal{A}(|T|^{-1})\xi^i} = \overline{\mathcal{A}(|T|)\xi^i}$. If there exists a unitary operator $U_0 \in \mathcal{B}(\mathbb{H})$, such that $U_0\mathcal{P}(|T|^{-1}) = \mathcal{P}(|T^{-1}|)U_0$, then $\mathbb{H}_{U_0\xi^i} = \overline{\mathcal{A}(|T^{-1}|)U_0\xi^i} = U_0\mathbb{H}_{\xi^i}$ and we get two series of isomorphic representations

$$R_{|T|^{-1},\xi^i} : \mathcal{L}^2(\sigma(|T|^{-1}|_{\mathbb{H}_{\xi^i}}), \mu_{|T|^{-1},\xi^i}) \to \mathbb{H}_{\xi^i}\,, \qquad f(z) \to f(|T|^{-1})\xi^i$$

and

$$R_{|T^{-1}|,U_0\xi^i} : \mathcal{L}^2(\sigma(|T^{-1}||_{\mathbb{H}_{U_0\xi^i}}), \mu_{|T^{-1}|,U_0\xi^i}) \to \mathbb{H}_{U_0\xi^i}\,, \qquad g(y) \to g(|T^{-1}|)U_0\xi^i\,.$$

Let $\xi = \xi^1 \oplus \xi^2 \oplus \cdots \oplus \xi^i \oplus \cdots$. Subsequently, ξ is a $\mathcal{A}(|T|^{-1})$-cyclic vector, such that $\mathbb{H} = \overline{\mathcal{A}(|T|^{-1})\xi}$ and we get the following equation

$$R_{|T|^{-1},\xi} = R_{|T|^{-1},\xi^1} \oplus R_{|T|^{-1},\xi^2} \oplus \cdots \oplus R_{|T|^{-1},\xi^i} \oplus \cdots\,,$$

$$R_{|T^{-1}|,U_0\xi} = R_{|T^{-1}|,U_0\xi^1} \oplus R_{|T^{-1}|,U_0\xi^2} \oplus \cdots \oplus R_{|T^{-1}|,U_0\xi^i} \oplus \cdots\,,$$

$$\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi}) = \mathcal{L}^2(\sigma(|T|^{-1}|_{\mathbb{H}_{\xi^1}}), \mu_{|T|^{-1},\xi^1}) \oplus \cdots \oplus \mathcal{L}^2(\sigma(|T|^{-1}|_{\mathbb{H}_{\xi^i}}), \mu_{|T|^{-1},\xi^i}) \oplus \cdots\,,$$

and

$$\mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi}) = \mathcal{L}^2(\sigma(|T^{-1}||_{\mathbb{H}_{U_0\xi^1}}), \mu_{|T^{-1}|,U_0\xi^1}) \oplus \cdots \oplus \mathcal{L}^2(\sigma(|T^{-1}||_{\mathbb{H}_{U_0\xi^i}}), \mu_{|T^{-1}|,U_0\xi^i}) \oplus \cdots\,.$$

3. Noncommutative Functional Calculus

We know that the spectral theory and functional calculus of normal operators [49] is very important in the study of operator theory and C^*-algebras [50]. Inspired by the Hua Loo-kang theorem on the automorphisms of a sfield [53], in this section, we give a useful construction from $\mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,\eta})$ to $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi})$ and with this construction, we give a noncommutative functional calculus for any given $T \in \mathcal{B}(\mathbb{H})$. However, there is valueless information just only from $R^{-1}_{|T|,\xi} \circ R_{|T^{-1}|,\eta}$ or $R^{-1}_{|T|^{-1},\xi} \circ R_{|T^{-1}|,\eta}$.

Lemma 4. *Let T be an invertible bounded linear operator on $\mathbb{H}$. Subsequently, we get*

$$\sigma(|T^{-1}|) = \sigma(|T|^{-1})\,.$$

Proof. Because of

$$\lambda \in \sigma(T^*T) \Longleftrightarrow \frac{1}{\lambda} \in \sigma(T^{*-1}T^{-1})\,,$$

we get

$$\lambda \in \sigma(|T|) \Longleftrightarrow \frac{1}{\lambda} \in \sigma(|T^{-1}|)\,.$$

That is, $\sigma(|T^{-1}|) = \sigma(|T|^{-1})$. $\square$

Let T be an invertible bounded linear operator on $\mathbb{H}$, ξ be a $\mathcal{A}(|T|)$-cyclic vector, such that $\mathbb{H} = \overline{\mathcal{A}(|T|)\xi}$. On $\mathcal{P}(z)$, with $z \in \sigma(|T|)$, we define the mapping

$$F_{z^{-1}} : \mathcal{P}(z) \to \mathcal{P}(z^{-1}),\ F_{z^{-1}}(f(z)) = f(z^{-1}) \ .$$

Following Lemma 3 and Theorem 1, $\mathcal{P}(z)$ and $\mathcal{P}(\frac{1}{z})$ are dense subspaces of $\mathcal{L}^2(\sigma(|T|), \mu_{|T|,\xi})$ and $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi})$, respectively. Its closed extension

$$F_{z^{-1}} : \mathcal{L}^2(\sigma(|T|), \mu_{|T|,\xi}) \to \mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi}) \ ,\ \ F_{z^{-1}}(f(z)) = f(z^{-1})$$

is linear and for this closed extension we keep the notation $F_{z^{-1}}$.

Subsequently, we obtain that

$$\int\limits_{\sigma(|T|)} f(z^{-1}) \mathrm{d}\mu_{|T|,\xi}(z) = \left\langle f(|T|^{-1})\xi, \xi \right\rangle = \int\limits_{\sigma(|T|^{-1})} f(z) \mathrm{d}\mu_{|T|^{-1},\xi}(z)$$

and

$$\mathrm{d}\mu_{|T|^{-1},\xi}(z) = |z|^2 \mathrm{d}\mu_{|T|,\xi}(z) \ .$$

By a simple computation, we get that

$$\begin{aligned}
&\|F_{z^{-1}}(f(z))\|^2_{\mathcal{L}^2(\sigma(|T|^{-1}),\mu_{|T|^{-1},\xi})} \\
&= \int\limits_{\sigma(|T|^{-1})} F_{z^{-1}}(f(z))\bar{F}_{z^{-1}}(f(z)) \mathrm{d}\mu_{|T|^{-1},\xi}(z) \\
&= \int\limits_{\sigma(|T|^{-1})} f(z^{-1})\bar{f}(z^{-1}) \mathrm{d}\mu_{|T|^{-1},\xi}(z) \\
&= \int\limits_{\sigma(|T|)} |z|^2 f(z)\bar{f}(z) \mathrm{d}\mu_{|T|,\xi}(z) \\
&\le \sup_{m\in\sigma(|T|)} m^2 \int\limits_{\sigma(|T|)} f(z)\bar{f}(z) \mathrm{d}\mu_{|T|,\xi}(z) \\
&\le \sup_{m\in\sigma(|T|)} m^2 \|f(z)\|^2_{\mathcal{L}^2(\sigma(|T|),\mu_{|T|,\xi})} \ .
\end{aligned}$$

Hence, it follows that

$$\|F_{z^{-1}}\| \le \sup_{m\in\sigma(|T|)} |m| \ .$$

By an application of the Banach inversion theorem [49] (p. 91), we get that $F_{z^{-1}}$ is an invertible bounded linear operator from $\mathcal{L}^2(\sigma(|T|), \mu_{|T|,\xi})$ to $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi})$.

Next, we define the operator

$$F^{\mathbb{H}}_{z^{-1}} : \mathcal{A}(|T|)\xi \to \mathcal{A}(|T|^{-1})\xi, \quad F^{\mathbb{H}}_{z^{-1}}(f(|T|)\xi) = f(|T|^{-1})\xi \ .$$

By Lemma 1 and [51] (p. 55), we get that $F^{\mathbb{H}}_{z^{-1}}$ is an invertible bounded linear operator on the Hilbert space $\overline{\mathcal{A}(|T|)\xi} = \mathbb{H}$ and

$$\|F^{\mathbb{H}}_{z^{-1}}\| \le \sup \sup_{m\in\sigma(|T|)} |m| \ .$$

Moreover, we obtain the following diagram.

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\quad |T| \quad} & \mathbb{H} \\ F^{\mathbb{H}}_{z^{-1}} \downarrow & & \downarrow F^{\mathbb{H}}_{z^{-1}} \\ \mathbb{H} & \xrightarrow[\quad |T|^{-1} \quad]{} & \mathbb{H} \end{array}$$

By [53] and the isomorphic representations $R_{|T|^{-1},\xi}$ and $R_{|T^{-1}|,U_0\xi}$ of $\mathbb{H}$, also, by Lemma 2 and 3, naturally, we give the following definition.

Definition 2. *For invertible $T \in \mathcal{B}(\mathbb{H})$, let the symbol xx^* stand for $T^{-1}T^{*-1}$. Subsequently, we get that there is a linear algebraic isomorphism from $\mathcal{P}(xx^*)$ to $\mathcal{P}(x^*x)$, such that*

$$\mathcal{F}_{xx^*} : \mathcal{P}(xx^*) \to \mathcal{P}(x^*x), \quad p_n(xx^*) \to p_n(x^*x) \,.$$

Let ξ be a $\mathcal{A}(|T|^{-1})$-cyclic vector, such that $\overline{\mathcal{A}(|T|^{-1})\xi} = \mathbb{H}$ and $U_0 \in \mathcal{B}(\mathbb{H})$ be a unitary operator such that $U_0\mathcal{P}(|T|^{-1}) = \mathcal{P}(|T^{-1}|)U_0$. Subsequently, on $\sigma(|T|^{-1})$ we define

$$F_{xx^*} : R^{-1}_{|T|^{-1}} p_n(xx^*)\xi \to R^{-1}_{|T^{-1}|} p_n(x^*x)U_0\xi \,\,.$$

Obviously, $\mathcal{P}(|y|^2)$ is dense in $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi})$ and $\mathcal{P}(|z|^2)$ is dense in $\mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi})$. Then its closed extension is

$$F_{xx^*} : \mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi})| \to \mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi}) \;, \quad R^{-1}_{|T|^{-1}} f(xx^*)\xi \to R^{-1}_{|T^{-1}|} f(x^*x)U_0\xi \,\,.$$

For this closed extension, we keep the notation F_{xx^*}.

With the polar decomposition theorem [50] (p. 15), there is $T = U|T|$. For invertible $T \in \mathcal{B}(\mathbb{H})$, we get that

$$U^*T^*TU = TT^* \quad \text{and} \quad U^*|T|^{-2}U = |T^{-2}| \,.$$

In fact, when T is invertible, we can choose a special unitary operator, which shows that the operators $|T|^{-1}$ and $|T^{-1}|$ are unitary equivalent. This is explained in the following theorem.

Theorem 2. *Let T be an invertible bounded linear operator on $\mathbb{H}$ and $U_0 \in \mathcal{B}(\mathbb{H})$ be a unitary operator, such that $U_0\mathcal{P}(|T|^{-1}) = \mathcal{P}(|T^{-1}|)U_0$. Afterwards, there is a unitary operator $F^{\mathbb{H}}_{xx^*}$, such that*

$$F^{\mathbb{H}}_{xx^*}|T|^{-1} = |T^{-1}|F^{\mathbb{H}}_{xx^*} \,.$$

Moreover, $F^{\mathbb{H}}_{xx^*}$ is the corresponding unitary operator associated with the almost everywhere nonzero function $|\phi_{|T|}(z)|$, such that

$$d\mu_{|T^{-1}|,U_0\xi} = |\phi_{|T|}(\frac{1}{z})| d\mu_{|T|^{-1},\xi} \,,$$

where $|\phi_{|T|}(z)| \in \mathcal{L}^1(\sigma(|T|), \mu_{|T|,\xi})$ and ξ is a $\mathcal{A}(|T|)$-cyclic vector, such that $\overline{\mathcal{A}(|T|)\xi} = \mathbb{H}$.

Proof. By Lemma 3, let ξ be a $\mathcal{A}(|T|)$-cyclic vector, such that $\mathbb{H} = \overline{\mathcal{A}(|T|)\xi}$. By Definition 2, we have the linear operator $F_{xx^*} : \mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi}) \to \mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi})$.

This construction yields that F_{xx^*} is an invertible linear operator from $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi})$ to $\mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi})$. Hence, $F_{xx^*} \circ F_{z^{-1}}$ is an invertible linear operator from $\mathcal{L}^2(\sigma(|T|), \mu_{|T|,\xi})$ to $\mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi})$.

By [53], we get that F_{xx^*} is a linear algebraic isomorphism from $\mathcal{P}(|y|^2)$ on $\sigma(|T|^{-1})$ to $\mathcal{P}(|z|^2)$ on $\sigma(|T^{-1}|)$. Additionally, by Lemma 2, $\mathcal{P}(|y|^2)$ is dense in $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi})$ and $\mathcal{P}(|z|^2)$ is dense in $\mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi})$.

Hence, we obtain

$$[\mathrm{d}\mu_{|T^{-1}|,U_0\xi}] = [\mathrm{d}\mu_{|T|^{-1},\xi}]\ ,$$

that is, $\mathrm{d}\mu_{|T^{-1}|,U_0\xi}$ and $\mathrm{d}\mu_{|T|^{-1},\xi}$ are mutually absolutely continuous. Following [49], (IX Theorem 3.6) and the construction $F_{z^{-1}}$, we get that there exists $\phi_{|T|}(z) \in \mathcal{L}^1(\sigma(|T|), \mu_{|T|,\xi})$, where $|\phi_{|T|}(z)| \neq 0$, a.e., such that

$$\mathrm{d}\mu_{|T^{-1}|,U_0\xi} = |\phi_{|T|}(\frac{1}{z})|\mathrm{d}\mu_{|T|^{-1},\xi} = |z|^2|\phi_{|T|}(z)|\mathrm{d}\mu_{|T|,\xi}\ .$$

From Lemma 4, for any $p_n \in \mathcal{P}(\sigma(|T|^{-1})) \subseteq \mathcal{A}(\sigma(|T|^{-1}))$, because of

$$T^{*-1}p_n(|T|^{-1}) = p_n(|T^{-1}|)T^{*-1}\ ,$$

with [50] (p. 60), we get that there is a unitary operator $U_0 \in \mathcal{B}(\mathbb{H})$, such that

$$U_0\mathcal{P}(|T|^{-1}) = \mathcal{P}(|T^{-1}|)U_0\ .$$

Hence, we conclude

$$U_0\mathcal{A}(|T|^{-1}) = \mathcal{A}(|T^{-1}|)U_0$$

and

$$\mathbb{H} = \overline{U_0\mathcal{A}(|T|^{-1})\xi} = \overline{\mathcal{A}(|T^{-1}|)U_0\xi}\ .$$

That is, $U_0\xi$ is a $\mathcal{A}(|T^{-1}|)$-cyclic vector. Additionally, with Theorem 1, we get

$$\int\limits_{\sigma(|T|^{-1})} f(z)\mathrm{d}\mu_{|T|^{-1},\xi}(z) = \left\langle f(|T|^{-1})\xi, \xi\right\rangle = \int\limits_{\sigma(|T|)} f(\frac{1}{z})\mathrm{d}\mu_{|T|,\xi}(z)$$

and

$$\int\limits_{\sigma(|T^{-1}|)} f(z)\mathrm{d}\mu_{|T^{-1}|,U_0\xi}(z) = \left\langle f(|T^{-1}|)U_0\xi, U_0\xi\right\rangle\ .$$

By a simple computation, we obtain that

$$\begin{aligned}
&\|F_{xx^*} \circ F_{z^{-1}}(f(z))\|^2_{\mathcal{L}^2(\sigma(|T^{-1}|),\mu_{|T^{-1}|,U_0\xi})}\\
=& \int\limits_{\sigma(|T^{-1}|)} F_{xx^*} \circ F_{z^{-1}}(f(z))\overline{F_{xx^*} \circ F_{z^{-1}}(f(z))}\mathrm{d}\mu_{|T^{-1}|,U_0\xi}(z)\\
=& \int\limits_{\sigma(|T^{-1}|)} F_{xx^*}(f(z^{-1}))\overline{F_{xx^*}(f(z^{-1}))}\mathrm{d}\mu_{|T^{-1}|,U_0\xi}(z)\\
\triangleq& \int\limits_{\sigma(|T|^{-1})} f(y^{-1})\bar{f}(y^{-1})\mathrm{d}\mu_{|T|^{-1},\xi}(y)\\
=& \int\limits_{\sigma(|T|^{-1})} F_{y^{-1}}(f(y))F_{y^{-1}}(\bar{f}(y))\mathrm{d}\mu_{|T|^{-1},\xi}(y)\\
=& \|F_{y^{-1}}(f(y))\|^2_{\mathcal{L}^2(\sigma(|T|^{-1}),\mu_{|T|^{-1},\xi})}
\end{aligned}$$

and $\triangleq$ is introduced by U_0.

Hence, F_{xx^*} is an isomorphism from $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi})$ to $\mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi})$.

With Theorem 1 and Definition 2, we have the operator

$$F_{xx^*}^{\mathbb{H}} : \mathbb{H} \to \mathbb{H}, \quad F_{xx^*}^{\mathbb{H}}(R_{|T|^{-1},\xi} f(|T|^{-2})) = R_{|T^{-1}|,U_0\xi}(F_{xx^*} f(|T|^{-2})).$$

That is, $F_{xx^*}^{\mathbb{H}} = R_{|T^{-1}|,U_0\xi} F_{xx^*} R_{|T|^{-1},\xi}^{-1}$, such that the following diagram is valid.

$$\begin{array}{ccc} \mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1},\xi}) & \xrightarrow{R_{|T|^{-1},\xi}} & \mathbb{H} \\ F_{xx^*} \downarrow & & \downarrow F_{xx^*}^{\mathbb{H}} \\ \mathcal{L}^2(\sigma(|T^{-1}|), \mu_{|T^{-1}|,U_0\xi}) & \xrightarrow[R_{|T^{-1}|,U_0\xi}]{} & \mathbb{H} \end{array}$$

Therefore, we said that the linear operator $F_{xx^*}^{\mathbb{H}}$ is associated with F_{xx^*} . Subsequently, we see that $F_{xx^*}^{\mathbb{H}}$ is a unitary operator and by Lemma 3, we obtain

$$\overline{\mathcal{A}(|T|^{-1})\xi} = \mathbb{H} = \overline{\mathcal{A}(|T^{-1}|)U_0\xi} \ .$$

Subsequently, we obtain

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{|T|^{-1}} & \mathbb{H} \\ F_{xx^*}^{\mathbb{H}} \downarrow & & \downarrow F_{xx^*}^{\mathbb{H}} \\ \mathbb{H} & \xrightarrow[|T^{-1}|]{} & \mathbb{H} \end{array}$$

Naturally, there is

$$F_{xx^*}^{\mathbb{H}} |T|^{-1} = |T^{-1}| F_{xx^*}^{\mathbb{H}} \ .$$

Afterwards, $F_{xx^*}^{\mathbb{H}}$ is the corresponding unitary operator associated with F_{xx^*} , which is, associated with the almost everywhere nonzero function $|\phi_{|T|}(\frac{1}{z})|$, such that

$$\mathrm{d}\mu_{|T^{-1}|,U_0\xi} = |\phi_{|T|}(\frac{1}{z})| \mathrm{d}\mu_{|T|^{-1},\xi} = |z|^2 |\phi_{|T|}(z)| \mathrm{d}\mu_{|T|,\xi} \ ,$$

where $|\phi_{|T|}(z)| \in \mathcal{L}^1(\sigma(|T|), \mu_{|T|,\xi})$. $\square$

We easily deduce $(F_{xx^*}^{\mathbb{H}})^* = F_{xx^*}^{\mathbb{H}}$ and the next results also readily follows. Let T be an invertible bounded linear operator on $\mathbb{H}$, ξ be a $\mathcal{A}(|T|)$-cyclic vector, such that $\mathbb{H} = \overline{\mathcal{A}(|T|)\xi}$ and let $U_0 \in \mathcal{B}(\mathbb{H})$ be a unitary operator, such that $U_0 \mathcal{P}(|T|^{-1}) = \mathcal{P}(|T^{-1}|) U_0$. In the proof of Theorem 2, and with the isomorphic representations $R_{|T|^{-1},\xi}$ and $R_{|T^{-1}|,U_0\xi}$ of $\mathbb{H}$, we provide that

$$F_{xx^*} = R_{|T^{-1}|,U_0\xi}^{-1} \circ F_{xx^*}^{\mathbb{H}} \circ R_{|T|^{-1},\xi} \ .$$

Especially, let $U_0 = F_{xx^*}^{\mathbb{H}}$. Subsequently,

$$F_{xx^*} = R_{|T^{-1}|,F_{xx^*}^{\mathbb{H}}\xi}^{-1} \circ F_{xx^*}^{\mathbb{H}} \circ R_{|T|^{-1},\xi} \ .$$

Corollary 1. *Let T be an invertible bounded linear operator on $\mathbb{H}$ and let ξ be a $\mathcal{A}(|T|)$-cyclic vector, such that $\mathbb{H} = \overline{\mathcal{A}(|T|)\xi}$. Then $\sigma(|T|) = \sigma(|T^*|)$ and the equality $F_{xx^*}^{\mathbb{H}}|T| = |T^*|F_{xx^*}^{\mathbb{H}}$ is valid. Moreover, $F_{xx^*}^{\mathbb{H}}$ is the corresponding unitary operator associated with F_{xx^*}, whih is, associated with the almost everywhere nonzero function $|\phi_{|T|}(z)|$, such that*

$$\mathrm{d}\mu_{|T^*|,F_{xx^*}^{\mathbb{H}}\xi} = |\phi_{|T|}(z)| \mathrm{d}\mu_{|T|,\xi} \ ,$$

where $|\phi_{|T|}(z)| \in \mathcal{L}^1(\sigma(|T|), \mu_{|T|,\xi})$.

Next, for any given $g(z) \in \mathcal{L}^\infty(\sigma(|T|), \mu_{|T|,\xi})$, we define

$$M_g : \mathcal{L}^2(\sigma(|T|), \mu_{|T|,\xi}) \to \mathcal{L}^2(\sigma(|T|), \mu_{|T|,\xi}), \quad M_g f(z) = g(z) f(z).$$

Theorem 3. *Let T be an invertible bounded linear operator on $\mathbb{H}$ and $T = U|T|$ be its Polar Decomposition. Let ξ be a $\mathcal{A}(|T|)$-cyclic vector, such that $\mathbb{H} = \overline{\mathcal{A}(|T|)\xi}$ and $|T| = R_{|T|,\xi} M_z R^{-1}_{|T|,\xi}$. Subsequently, there exists $\psi(z) \in \mathcal{L}^\infty(\sigma(|T|), \mu_{|T|,\xi})$, such that $U = R_{|T^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{\psi(z)} R^{-1}_{|T|,\xi}$ and $T = R_{|T^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T|,\xi}$. Here $M_{z\psi(z)} = M_z M_{\psi(z)} = M_{\psi(z)} M_z = M_{\psi(z)z}$.*

Proof. Let $U_0 = F^{\mathbb{H}}_{xx^*}$ in the proof of Theorem 2. Afterwards, we get

$$F_{xx^*} = R^{-1}_{|T^{-1}|, F^{\mathbb{H}}_{xx^*}\xi} \circ F^{\mathbb{H}}_{xx^*} \circ R_{|T|^{-1},\xi} \quad \text{and} \quad F^{\mathbb{H}}_{xx^*} = R_{|T^{-1}|, F^{\mathbb{H}}_{xx^*}\xi} \circ F_{xx^*} \circ R^{-1}_{|T|^{-1},\xi}.$$

By the polar decomposition theorem [50] (p. 15) we have $T = U|T|$. Hence, we get

$$T^*T = |T|^2 \quad \text{and} \quad TT^* = U|T|^2 U^*.$$

By Corollary 1, we get

$$TT^* = F^{\mathbb{H}}_{xx^*} |T|^2 (F^{\mathbb{H}}_{xx^*})^*,$$

that is,

$$F^{\mathbb{H}}_{xx^*} |T|^2 (F^{\mathbb{H}}_{xx^*})^* = TT^* = U|T|^2 U^*.$$

We see that

$$F^{\mathbb{H}}_{xx^*} U |T|^2 = |T|^2 F^{\mathbb{H}}_{xx^*} U.$$

With the fact that $\{M_{\psi(z)} : \psi(z) \in \mathcal{L}^\infty(\sigma(|T|), \mu_{|T|,\xi})\}$ is a maximal abelian von Neumann algebra in $\mathcal{B}(\mathcal{L}^2(\sigma(|T|), \mu_{|T|,\xi}))$ and the Fuglede–Putnam theorem [49] (p. 279), we obtain that there exists $\psi(z) \in \mathcal{L}^\infty(\sigma(|T|), \mu_{|T|,\xi})$ such that

$$F^{\mathbb{H}}_{xx^*} U = R_{|T|,\xi} M_{\psi(z)} R^{-1}_{|T|,\xi} = R_{|T|^{-1},\xi} M_{\psi(\frac{1}{z})} R^{-1}_{|T|^{-1},\xi}.$$

Therefore, we get that

$$U = F^{\mathbb{H}}_{xx^*} R_{|T|^{-1},\xi} M_{\psi(\frac{1}{z})} R^{-1}_{|T|^{-1},\xi} = R_{|T^{-1}|, F^{\mathbb{H}}_{xx^*}\xi} \circ F_{xx^*} \circ R^{-1}_{|T|^{-1},\xi} R_{|T|^{-1},\xi} M_{\psi(\frac{1}{z})} R^{-1}_{|T|^{-1},\xi}.$$

That is,

$$U = R_{|T^{-1}|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{\psi(\frac{1}{z})} R^{-1}_{|T|^{-1},\xi} = R_{|T^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{\psi(z)} R^{-1}_{|T|,\xi}$$

and

$$T = U|T| = R_{|T^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{\psi(z)} R^{-1}_{|T|,\xi} R_{|T|,\xi} M_z R^{-1}_{|T|,\xi} = R_{|T^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T|,\xi}.$$

□

Corollary 2. *Let $T \in \mathcal{B}(\mathbb{H})$. Suppose $a \in \rho(T) = \mathbb{C} \setminus \sigma(T)$ and let ξ be a $\mathcal{A}(|T-a|)$-cyclic vector, such that $\mathbb{H} = \overline{\mathcal{A}(|T-a|)\xi}$. Subsequently, there exists a function $\psi(z) \in \mathcal{L}^\infty(\sigma(|T-a|), \mu_{|T-a|,\xi})$, such that*

$$T = R_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T-a|,\xi} + a.$$

Proof. For $a \in \rho(T)$, $T - a$ is an invertible bounded linear operator on $\mathbb{H}$. By the proof of Theorem 3, we get that $T - a = R_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T-a|,\xi}$, that is,

$$T = R_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T-a|,\xi} + a.$$

□

The following definition is quite natural.

Definition 3. *For any given $T \in \mathcal{B}(\mathbb{H})$, we say that $R_{|(T-a)^*|,F^{\mathbb{H}}_{xx^*}\xi}F_{xx^*}M_{z\psi(z)}R^{-1}_{|T-a|,\xi}+a$ is the noncommutative functional calculus of T on $F_{xx^*}:\mathcal{L}^2(\sigma(|T|),\mu_{|T|,\xi})|\to\mathcal{L}^2(\sigma(|T^*|),\mu_{|T^*|,F^{\mathbb{H}}_{xx^*}\xi})$,, where ξ is a $\mathcal{A}(|T-a|)$-cyclic vector, such that $\mathbb{H}=\overline{\mathcal{A}(|T-a|)\xi}$, $\psi(z)\in\mathcal{L}^\infty(\sigma(|T-a|),\mu_{|T-a|,\xi})$ and $a\in\rho(T)$.*

In the final part of this section, we give some properties of normal operator through the noncommutative functional calculus.

Corollary 3. *For $T\in\mathcal{B}(\mathbb{H})$, if $TT^*=T^*T$, then there exists $\psi(z)\in\mathcal{L}^\infty(\sigma(|T-a|),\mu_{|T-a|,\xi})$ such that $R_{|(T-a)|,\xi}(M_{z\psi(z)}+a)R^{-1}_{|T-a|,\xi}$ is the noncommutative functional calculus of T on $\mathcal{L}^2(\sigma(|T-a|),\mu_{|T-a|,\xi})$. and we get $T\in\mathcal{A}'(|T-a|)$. Where $a\in\rho(T)$, ξ is a $\mathcal{A}(|T-a|)$-cyclic vector, such that $\mathbb{H}=\overline{\mathcal{A}(|T-a|)\xi}$ and*

$$\mathcal{A}'(|T-a|)=\{A\in\mathcal{B}(\mathbb{H}):AB=BA\ \text{for every}\ B\in\mathcal{A}(|T-a|)\}.$$

Proof. For $TT^*=T^*T$ and $a\in\rho(T)$, we get that

$$F^{\mathbb{H}}_{xx^*}=identity\quad\text{and}\quad|T-a|=[(T-a)^*(T-a)]^{\frac{1}{2}}=[(T-a)(T-a)^*]^{\frac{1}{2}}=|(T-a)^*|.$$

Therefore, we see that

$$R^{-1}_{|(T-a)^*|,F^{\mathbb{H}}_{xx^*}\xi}=R^{-1}_{|T-a|,\xi}$$

and there exists $\psi(z)\in\mathcal{L}^\infty(\sigma(|T-a|),\mu_{|T-a|,\xi})$ such that

$$T-a=R_{|(T-a)|,\xi}M_{z\psi(z)}R^{-1}_{|T-a|,\xi}\,.$$

That is,

$$T=R_{|(T-a)|,\xi}(M_{z\psi(z)}+a)R^{-1}_{|T-a|,\xi}\,.$$

With the proof of Theorem 3, we get $T-a\in\mathcal{A}'(|T-a|)$, which is, $T\in\mathcal{A}'(|T-a|)$. □

Corollary 4. *Let $T\in\mathcal{B}(\mathbb{H})$. Subsequently, the operator T is normal if and only if T is unitary equivalent to $M_{\psi(z)}+a$ on $\mathcal{L}^2(\sigma(|T-a|),\mu_{|T-a|,\xi})$, and if and only if $T\in\mathcal{A}'(|T-a|)$, where ξ is a $\mathcal{A}(|T-a|)$-cyclic vector, such that $\mathbb{H}=\overline{\mathcal{A}(|T-a|)\xi}$, $\psi(z)\in\mathcal{L}^\infty(\sigma(|T-a|),\mu_{|T-a|,\xi})$, and $a\in\rho(|T|)$.*

4. A Sufficient Condition

In this section, we study Problem 1 on infinite-dimensional separable Hilbert spaces. With the fact that the exist of non-trivial invariant subspace is unchanged by the similarity of bounded linear operators on Banach spaces [1], which is, for $R\in\mathcal{B}(\mathbb{B}_1)$ and $S\in\mathcal{B}(\mathbb{B}_2)$, if $T:\mathbb{B}_1\to\mathbb{B}_2$ is an invertible bounded linear operator and $S=TRT^{-1}$, then R has non-trivial invariant subspace if and only if S has, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces. Therefore, for any given $T\in\mathcal{B}(\mathbb{H})$, using the construction of F_{xx^*}, we give a sufficient condition to Problem 1 on infinite-dimensional separable Hilbert spaces.

For convenience, we define $Fix(F^{\mathbb{H}}_{xx^*})=\{F^{\mathbb{H}}_{xx^*}(f)=f;f\in\mathbb{H}\}$. Obviously, $Fix(F^{\mathbb{H}}_{xx^*})$ is a closed subspace of $\mathbb{H}$.

Theorem 4. *Let $\dim\mathbb{H}>1$, $T\in\mathcal{B}(\mathbb{H})$ and $R_{|(T-a)^*|,F^{\mathbb{H}}_{xx^*}\xi}F_{xx^*}M_{z\psi(z)}R^{-1}_{|T-a|,\xi}$ be the noncommutative functional calculus of $T-a$ on $F_{xx^*}:\mathcal{L}^2(\sigma(|T-a|),\mu_{|T-a|,\xi})|\to\mathcal{L}^2(\sigma(|(T-a)^*|),\mu_{|(T-a)^*|,F^{\mathbb{H}}_{xx^*}\xi})$, where $a\in\rho(|T|)$, ξ is a $\mathcal{A}(|T-a|)$-cyclic vector, such that $\mathbb{H}=\overline{\mathcal{A}(|T-a|)\xi}$ and $\psi(z)\in\mathcal{L}^\infty(\sigma(|T-a|),\mu_{|T-a|,\xi})$. If $R_{|T-a|,\xi}M_{z\psi(z)}R^{-1}_{|T-a|,\xi}Fix(F^{\mathbb{H}}_{xx^*})\subseteq Fix(F^{\mathbb{H}}_{xx^*})$, then T has a non-trivial invariant subspace.*

Proof. It is enough to prove the result for infinite-dimensional separable complex Hilbert space $\mathbb{H}$. Obviously, if $A \subset \mathbb{H}$ is a non-trivial invariant subspace of T if and only if A is a non-trivial invariant subspace of $T-a$, where $a \in \mathbb{C}$.

Let $a \in \rho(T)$ and let ξ be a $\mathcal{A}(|T-a|)$-cyclic vector such that $\mathbb{H} = \overline{\mathcal{A}(|T-a|)\xi}$. Subsequently, following Corollary 2, we get that there exists $\psi(z) \in \mathcal{L}^\infty(\sigma(|T-a|), \mu_{|T-a|,\xi})$, such that $R_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T-a|,\xi}$ is the noncommutative functional calculus of $T-a$ on

$$F_{xx^*} : \mathcal{L}^2(\sigma(|T-a|), \mu_{|T-a|,\xi})| \to \mathcal{L}^2(\sigma(|(T-a)^*|), \mu_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi}) .$$

By the construction of $F^{\mathbb{H}}_{xx^*}$ in Theorem 2, we get $(F^{\mathbb{H}}_{xx^*})^2 = identity$ and $Fix(F^{\mathbb{H}}_{xx^*}) \neq \emptyset$.

(1) If $Fix(F^{\mathbb{H}}_{xx^*}) = \mathbb{H}$, that is $F^{\mathbb{H}}_{xx^*} = identity$, by the proof of Corollary 3, then T is unitary equivalent to $M_{z\psi(z)} + a$. Because $M_{z\psi(z)} + a$ is a normal operator, it possesses a non-trivial invariant subspace and, hence, the same is true for T. For details, see, e.g., [49].

(2) If $Fix(F^{\mathbb{H}}_{xx^*}) \neq \mathbb{H}$ and $R_{|T-a|,\xi} M_{z\psi(z)} R^{-1}_{|T-a|,\xi} Fix(F^{\mathbb{H}}_{xx^*}) \subseteq Fix(F^{\mathbb{H}}_{xx^*})$, then $Fix(F^{\mathbb{H}}_{xx^*})$ is a non-trivial invariant subspace of $F^{\mathbb{H}}_{xx^*}$ and we get

$$F^{\mathbb{H}}_{xx^*} R_{|T|,\xi} M_{z\psi(z)} R^{-1}_{|T|,\xi} Fix(F^{\mathbb{H}}_{xx^*}) \subseteq Fix(F^{\mathbb{H}}_{xx^*}).$$

Hence, $Fix(F^{\mathbb{H}}_{xx^*})$ is a non-trivial invariant subspace of $F^{\mathbb{H}}_{xx^*} R_{|T|,\xi} M_{z\psi(z)} R^{-1}_{|T|,\xi}$. With the proof of Theorem 3, we get that

$$F^{\mathbb{H}}_{xx^*} R_{|T|,\xi} M_{z\psi(z)} R^{-1}_{|T|,\xi} = R_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T-a|,\xi} = T - a.$$

That is,

$$(T-a) Fix(F^{\mathbb{H}}_{xx^*}) \subseteq Fix(F^{\mathbb{H}}_{xx^*}).$$

□

5. Lebesgue Operator

In this section, we study chaos of an invertible bounded linear operator on an infinite-dimensional separable Hilbert space. For the example of integral calculus in mathematical analysis, we know that the convergence or the divergence of the weighted integral calculus of x and x^{-1} should be independent of each other; however, sometimes it happens that this indeed depends on a special choice of the weight function.

In the view of integral calculus, we define the Lebesgue class and prove that if T is a Lebesgue operator, then T is Li-Yorke chaotic if and only if T^{*-1} is. With the idea of the noncommutative functional calculus $R_{|(T-a)^*|, F^{\mathbb{H}}_{xx^*}\xi} F_{xx^*} M_{z\psi(z)} R^{-1}_{|T-a|,\xi}$, we give an example of a Lebesgue operator that is not a normal operator.

Let $\mathrm{d}x$ be the Lebesgue measure on $\mathcal{L}^2(\mathbb{R}_+)$. By Theorem 1, there exists a Borel measure $\mathrm{d}\mu_{|T^n|,\xi_n}$, which is complete, such that $\mathcal{L}^2(\sigma(|T^n|), \mathrm{d}\mu_{|T^n|,\xi_n})$ is a Hilbert space. If there exists $N > 0$, such that, for all $n \geq N$, the measure $\mathrm{d}\mu_{|T^n|,\xi_n}$ is absolutely continuity with respect to $\mathrm{d}x$, then using the Radon–Nikodym theorem [49] (p. 380), there exists $f_n \in \mathcal{L}^1(\mathbb{R}_+)$, such that $\mathrm{d}\mu_{|T^n|,\xi_n} = f_n(x)\mathrm{d}x$, where $n \in \mathbb{N}$, $n \geq N$ and $\mathbb{H} = \overline{\mathcal{A}(|T^n|)\xi_n}$.

Definition 4. *Let T be an invertible bounded linear operator on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$. Suppose that the operator T satisfies the following conditions:*

(1) *There exists $N \in \mathbb{N}$, such that, for all $n \geq N$*

$$\begin{cases} \mathrm{d}\mu_{|T^n|,\xi_n} = f_n(x)\mathrm{d}x, & f_n \in \mathcal{L}^1(\mathbb{R}_+) \\ x^2 f_n(x) = f_n(x^{-1}), & 0 < x < 1 \end{cases}$$

(2) *There exists $N \in \mathbb{N}$, such that for all $n \geq N$ and for any given nonzero $x \in \mathbb{H}$, there exists a nonzero function $g_n(t) \in \mathcal{L}^2(\sigma(|T^n|), \mathrm{d}\mu_{|T^n|,\xi_n})$ and a nonzero vector $y \in \mathbb{H}$, such that $y = g_n(|T^n|^{-1})\xi_n$ whenever $x = g_n(|T^n|)\xi_n$.*

Subsequently, the operator T is said to be a Lebesgue operator, and the family of all Lebesgue operators on $\mathbb{H}$ is denoted by $\mathcal{B}_{Leb}(\mathbb{H})$.

Theorem 5. *Let T be a Lebesgue operator on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$. Subsequently, T is Li-Yorke chaotic if and only if T^{*-1} is.*

Proof. Let ξ_n be a $\mathcal{A}(|T^n|)$-cyclic vector such that

$$\overline{\mathcal{A}(|T^n|)\xi_n} = \mathbb{H}.$$

If x_0 is a Li-Yorke chaotic point of T, then by Definition 4, we see that, for $n \in \mathbb{N}$ large enough, there exist $g_n(x) \in \mathcal{L}^2(\sigma(|T^n|), \mathrm{d}\mu_{|T^n|,\xi_n})$, $f_n(x) \in \mathcal{L}^1(\mathbb{R}_+)$ and $y_0 \in \mathbb{H}$, such that $x_0 = g_n(|T^n|)\xi_n$, $y_0 = g_n(|T^n|^{-1})\xi_n$, and

$$\mathrm{d}\mu_{|T^n|,\xi_n} = f_n(x)\mathrm{d}x.$$

Therefore, we get the following

$$\begin{aligned}
\|T^n x_0\|^2_{\mathbb{H}} &= \langle T^{n*}T^n x_0, x_0\rangle = \left\langle |T^n|^2 g_n(|T^n|)\xi_n, g_n(|T^n|)\xi_n\right\rangle = \left\langle g_n(|T^n|)^*|T^n|^2 g_n(|T^n|)\xi_n, \xi_n\right\rangle \\
&= \int\limits_{\sigma(|T^n|)} x^2 g_n(x)\bar{g}(x)\mathrm{d}\mu_{|T^n|,\xi_n}(x) = \int_0^{+\infty} x^2|g_n(x)|^2 f_n(x)\mathrm{d}x \\
&= \int_0^1 x^2|g_n(x)|^2 f_n(x)\mathrm{d}x + \int_1^{+\infty} x^2|g_n(x)|^2 f_n(x)\mathrm{d}x \\
&= \int_0^1 x^2|g_n(x)|^2 f_n(x)\mathrm{d}x + \int_0^1 x^{-4}|g_n(x^{-1})|^2 f_n(x^{-1})\mathrm{d}x \\
&\triangleq \int_0^1 |g_n(x)|^2 f_n(x^{-1})\mathrm{d}x + \int_0^1 x^{-2}|g_n(x^{-1})|^2 f_n(x)\mathrm{d}x \\
&= \int_1^{+\infty} x^{-2}|g_n(x^{-1})|^2 f_n(x)\mathrm{d}x + \int_0^1 x^{-2}|g_n(x^{-1})|^2 f_n(x)\mathrm{d}x \\
&= \int_0^{+\infty} x^{-2}|g_n(x^{-1})|^2 f_n(x)\mathrm{d}x = \int\limits_{\sigma(|T^n|)} x^{-2} g_n(x^{-1})\bar{g}_n(x^{-1})\mathrm{d}\mu_{|T^n|,\xi_n}(x) \\
&= \left\langle g_n(|T^n|^{-1})^*|T^n|^{-2} g_n(|T^n|^{-1})\xi_n, \xi_n\right\rangle = \left\langle |T^n|^{-2} g_n(|T^n|^{-1})\xi_n, g_n(|T^n|^{-1})\xi_n\right\rangle \\
&= \left\langle |T^n|^{-2} y_0, y_0\right\rangle = \langle T^{-n}T^{-n*} y_0, y_0\rangle \\
&= \|T^{*-n} y_0\|^2_{\mathbb{H}}
\end{aligned}$$

where $\triangleq$ is following Definition 4. By Definition 1, we get that T is Li-Yorke chaotic if and only if T^{*-1} is. □

Following [54], for $T \in \mathcal{B}(\mathbb{H})$, $x \in \mathbb{H}$ and $n \in \mathbb{N}$, we introduce the distributional function

$$F_x^n(\tau) = \frac{1}{n}\sharp\{0 \leq i \leq n : \|T^n(x)\| < \tau\}.$$

In addition, we denote

$$F_x(\tau) = \liminf_{n\to\infty} F_x^n(\tau), \qquad F_x^*(\tau) = \limsup_{n\to\infty} F_x^n(\tau),$$

and introduce the following definition.

Definition 5. *Let $T \in \mathcal{B}(\mathbb{H})$. If there exists $x \in \mathbb{H}$ and*

(1) If $F_x(\tau) = 0$, for some $\tau > 0$ and $F_x^*(\epsilon) = 1$ for $\forall \epsilon > 0$, then we say that T is distributionally chaotic or I-distributionally chaotic.

(2) If $F_x^*(\epsilon) > F_x(\tau)$ for $\forall \tau > 0$ and $F_x^*(\epsilon) = 1$ for $\forall \epsilon > 0$, then we say that T is II-distributionally chaotic.

(3) If $F_x^*(\epsilon) > F_x(\tau)$ for $\forall \tau > 0$, then we say that T is III-distributionally chaotic.

Corollary 5. *Let T be a Lebesgue operator on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$. Then T is I-distributionally chaotic (or II-distributionally chaotic or III-distributionally chaotic) if and only if T^{*-1} is I-distributionally chaotic (or II-distributionally chaotic or III-distributionally chaotic).*

Theorem 6. *There exists an invertible bounded linear operator T on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$, such that T is Lebesgue operator that is not a normal operator.*

Proof. Let $0 < a < b < +\infty$. Subsequently, $\mathcal{L}^2([a,b])$ is a separable Hilbert space over $\mathbb{C}$. Any separable Hilbert space over $\mathbb{R}$ can be expanded to a separable Hilbert space over $\mathbb{C}$. Without loss of generality, let $\mathcal{L}^2([a,b])$ be the separable Hilbert space over $\mathbb{R}$. We prove the conclusion by six parts:

(1) Let $0 < a < 1 < b = \frac{1}{a} < +\infty$. We construct a measure preserving transformation on $[a,b]$. Let $M = \{[a, \frac{b-a}{2}], [\frac{b-a}{2}, b]\}$. We get a Borel algebra $\xi(M)$ generated by M . We define $\Phi : [a,b] \to [a,b]$,

$$\Phi([a, \frac{b-a}{2}]) = [\frac{b-a}{2}, b], \qquad \Phi([\frac{b-a}{2}, b]) = [a, \frac{b-a}{2}].$$

Subsequently, Φ is an invertible measure preserving transformation on the Borel algebra $\xi(M)$. With [55] (p. 63), $U_\Phi \neq 1$ and U_Φ is a unitary operator associated with Φ, where U_Φ is the operation of composition

$$U_\Phi h = h \circ \Phi, \quad \forall h \in \mathcal{L}^2([a,b]).$$

(2) Define $M_x h = xh$ on $\mathcal{L}^2([a,b])$. Subsequently, M_x is an invertible positive operator.

(3) For $f(x) = \dfrac{|\ln x|}{x}$, $x > 0$, we define $\mathrm{d}\mu = f(x)\mathrm{d}x$. Afterwards, $f(x)$ is continuous and $f(x) > 0$, a.e., $x \in [a,b]$. Hence, $\mathrm{d}\mu$ that is absolutely continuous with respect to $\mathrm{d}x$ is a finite positive Borel measure that is complete. That is, $\mathcal{L}^2([a,b], \mathrm{d}\mu)$ is a separable Hilbert space over $\mathbb{R}$. Moreover, $\mathcal{L}^2([a,b])$ and $\mathcal{L}^2([a,b], \mathrm{d}\mu)$ are unitary equivalence.

(4) Let $T = U_\Phi M_x$. We get

$$T^*T = U_\Phi T T^* U_\Phi^* \qquad \text{and} \qquad U_\Phi \neq 1.$$

Because of

$$U_\Phi M_x \neq M_x U_\Phi \quad \text{and} \quad U_\Phi M_{x^2} \neq U_\Phi M_{x^2},$$

we get that T is not a normal operator and $\sigma(|T|) = [a,b]$.

(5) Let the operator $T = U_\Phi M_x$ on $\mathcal{L}^2([a,b])$ be corresponding to the operator T' on $\mathcal{L}^2([a,b], \mathrm{d}\mu)$. Subsequently, T' is an invertible bounded linear operator that is not a normal operator and $\sigma(|T'|) = [a,b]$.

(6) From

$$\int_a^b x^n f(x)\mathrm{d}x = \int_{a^n}^{b^n} t f(t^{\frac{1}{n}}) \frac{1}{n t^{\frac{n-1}{n}}} \mathrm{d}t \ .$$

Let

$$f_n(t) = \frac{1}{n} I_{[a^n, b^n]} f(t^{\frac{1}{n}}) \frac{1}{t^{\frac{n-1}{n}}} \ .$$

We get that $f_n(t)$ is continuous and almost everywhere positive. Hence, $f_n(t)\mathrm{d}t$ is a finite positive Borel measure that is complete.

For any $E \subseteq \mathbb{R}_+$, we define $I_E = 1$ when $x \in E$ else $I_E = 0$. Subsequently, I_E is the identity function on E. With a simple computing, we get that

$$f_n(t^{-1}) = \frac{1}{n} I_{[a^n,b^n]} f(t^{-\frac{1}{n}}) \frac{1}{t^{-\frac{n-1}{n}}} = \frac{1}{n} I_{[a^n,b^n]} \frac{|\ln t^{-\frac{1}{n}}|}{t^{-\frac{1}{n}}} \frac{1}{t^{-\frac{n-1}{n}}}$$
$$= \frac{1}{n} I_{[a^n,b^n]} t |\ln t^{\frac{1}{n}}|$$

and

$$t^2 f_n(t) = \frac{1}{n} I_{[a^n,b^n]} f(t^{\frac{1}{n}}) \frac{t^2}{t^{\frac{n-1}{n}}} = \frac{1}{n} I_{[a^n,b^n]} \frac{|\ln t^{\frac{1}{n}}|}{t^{\frac{1}{n}}} \frac{t^2}{t^{\frac{n-1}{n}}}$$
$$= \frac{1}{n} I_{[a^n,b^n]} t |\ln t^{\frac{1}{n}}| \,.$$

We see that $x^2 f_n(x) = f_n(x^{-1})$. From $\sigma(|T'^n|) = [a^n, b^n]$ and

$$\int_{a^n}^{b^n} t^2 f(t^{\frac{1}{n}}) \frac{1}{nt^{\frac{n-1}{n}}} \mathrm{d}t = \int_0^{+\infty} t^2 f_n(t) \mathrm{d}t \,,$$

let $\mathrm{d}\mu_{|T'^n|} = f_n(t)\mathrm{d}t$.

Afterwards, $\mathrm{d}\mu_{|T'^n|}$ is a finite positive Borel measure that is complete. For any given nonzero $h(x) \in \mathcal{L}^2([a,b])$, we get the nonzero function $h(x^{-1}) \in \mathcal{L}^2([a,b])$.

Easily, we get that $I_{[a,b]}$ is a $\mathcal{A}(|M_x^n|)$-cyclic vector of the multiplication $M_x^n = M_{x^n}$ and $I_{[a^n,b^n]}$ is a $\mathcal{A}(|T'^n|)$-cyclic vector of $|T'^n|$. By Definition 4, we get that T' is Lebesgus operator, but is not a normal operator. □

Corollary 6. *There exists an invertible bounded linear operator T on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$, such that T is a Lebesgue operator that is a positive operator.*

Corollary 7. *Let $\mathcal{B}_{Nor}(\mathbb{H})$ be the subspace of all normal bounded linear operator on an infinite-dimensional separable Hilbert space $\mathbb{H}$. Subsequently, the following families of linear operators are non-empty:*

$$\mathcal{B}_{Leb}(\mathbb{H}) \cap \mathcal{B}_{Nor}(\mathbb{H}) \ \text{ and } \ \mathcal{B}_{Leb}(\mathbb{H}) \cap (\mathcal{B}(\mathbb{H}) \setminus \mathcal{B}_{Nor}(\mathbb{H})).$$

In fact, both these families contain non-trivial members.

6. Conclusions

By the idea of the isomorphism construction F_{xx*} of this paper, we could study the operator using the integral calculus on $\mathbb{R}$. This way maybe neither change the properties of chaos nor the difficulty of computing, but with this we should find some operator class and study its properties, as we give the Lebesgue class in this section. Hence, if some properties of operators on $\mathbb{H}$ only depending the norm that is compatible with the inner product, then these properties only depend on the corresponding properties of elements in

$$\{M_{\psi_n(z)} + a_n : \psi_n(z) \in \mathcal{L}^\infty(\sigma(|T^n - a_n|), \mu_{|T^n - a_n|, \xi_n}), a_n \in \rho(T^n), n \in \mathbb{N}, \mathbb{H} = \overline{\mathcal{A}(|T^n - a_n|)\xi_n}\},$$

just keeping the noncommutative functional calculus in mind.

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References

1. Kaiser, J. Invariant Subspace Problem. Bachelor Thesis, Technische Universität Wien', Wien, NY, USA, 2016.
2. Bernstein, A.R.; Robinson, A. Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos. *Pacific J. Math.* **1966**, *16*, 421–431. [CrossRef]
3. Aronszajn, N.; Smith, K.T. Invariant subspaces of completely continuous operators. *Ann. Math.* **1954**, *60*, 345–350. [CrossRef]
4. Lomonosov, V.I. Invariant subspaces of the family of operators that commute with a completely continuous operator. *Funkcional. Anal. Priložen.*, **1973**, *7*, 55–56. (In Russian) [CrossRef]
5. Fang, Q.; Xia, J. Invariant subspaces for certain finite-rank perturbations of diagonal operators. *J. Funct. Anal.* **2012**, *263*, 1356–1377. [CrossRef]
6. Kim, H. Hyperinvariant subspaces for operators having a normal part. *Oper. Matrices* **2011**, *5*, 487–494. [CrossRef]
7. Shi, L.; Wu, Y. Invariant and hyperinvariant subspaces for amenable operators. *J. Oper. Theory* **2013**, *69*, 87–100. [CrossRef]
8. Enflo, P.H. On the invariant subspace problem in Banach spaces. *Acta Math.* **1987**, *158*, 213–313. [CrossRef]
9. Nordgren, E.A.; Radjavi, H.; Rosenthal, P. A geometric equivalent of the invariant subspace problem. *Proc. Am. Math. Soc.* **1976**, *61*, 66–68. [CrossRef]
10. Atzmon, A. An operator without invariant subspaces on a nuclear Fréchet space. *Ann. Math.* **1983**, *117*, 669–694. [CrossRef]
11. Read, C.J. A solution to the invariant subspace problem. *Bull. Lond. Math. Soc.* **1984**, *16*, 337–401. [CrossRef]
12. Argyros, S.A.; Haydon, R.G. A hereditarily indecomposable $\mathscr{L}_\infty$-space that solves the scalar-plus-compact problem. *Acta Math.* **2011**, *206*, 1–54. [CrossRef]
13. Marcoux, L.W.; Popov, A.I.; Radjavi, H. On almost-invariant subspaces and approximate commutation. *J. Funct. Anal.* **2013**, *264*, 1088–1111. [CrossRef]
14. Elliott, G.A.; Yahaghi, B.R. Operators compatible with a chain of subspaces of a Banach space. *J. Ramanujan Math. Soc.* **2005**, *20*, 1–17.
15. Yahaghi, B.R. On injective or dense-range operators leaving a given chain of subspaces invariant. *Proc. Am. Math. Soc.* **2004** *132*, 1059–1066. [CrossRef]
16. Allan, G.R.; Zemánek, J. Invariant subspaces for pairs of projections. *J. Lond. Math. Soc.* **1998**, *57*, 449–468. [CrossRef]
17. Bernik, J.; Radjavi, H. Invariant and almost-invariant subspaces for pairs of idempotents. *Integral Equ. Oper. Theory* **2016**, *84*, 283–288. [CrossRef]
18. Popov, A.I.; Tcaciuc, A. Every operator has almost-invariant subspaces. *J. Funct. Anal.* **2013**, *265*, 257–265. [CrossRef]
19. Tcaciuc, A. The invariant subspace problem for rank-one perturbations. *Duke Math. J.* **2019**, *168*, 1539–1550. [CrossRef]
20. Doran, R.S. C^*-algebras: 1943–1993. A fifty year celebration. In *Proceedings of the AMS Special Session Held in San Antonio, Texas, 13–14 January 1993*; Contemporary Mathematics, 167; Doran, R.S., Ed.; American Mathmatical Society: New York, NY, USA, 1994.
21. Gelfand, I.M.; Naimark, M.A. On the embedding of normed linear rings into the ring of operators in Hilbert space. *Mat. Sbornik* **1943**, *12*, 197–213.
22. Fujimoto, I. A Gelfand-Naimark Theorem FOR C^*-Algebras. *Pac. J. Math.* **1998**, *184*, 95–119. [CrossRef]
23. Fell, J.M.G. The structure of algebras of operator fields. *Acta Math.* **1961**, *106*, 233–280. [CrossRef]
24. Fujimoto, I. A Gelfand representation for C^*-algebras. *Pac. J. Math.* **1971**, *39*, 1–11.
25. Kruszynski, P.; Woronowicz, S.L. A non-commutative Gelfand-Naimark theorem. *J. Oper. Theory* **1982**, *8*, 361–389.
26. Alfsen, E.M. On the Dirichlet problem of the Choquet boundary. *Acta Math.* **1968**, *120*, 149–159. [CrossRef]
27. Takesaki, M. A duality in the representation theory of C^*-algebras. *Ann. Math.* **1967**, *85*, 370–382. [CrossRef]

28. Kitai, C. Invariant Closed Sets for Linear Operators. Ph.D. Thesis, University of Toronto, Toronto, ON, Canada, 1982.
29. Bayart, F.; Matheron, E. *Dynamics of Linear Operators*; Cambridge University Press: Cambridge, UK, 2009.
30. Godefroy, G. Renorming of Banach spaces. In *Handbook of the Geometry of Banach Spaces*; North Holland: Amsterdam, The Netherlands, 2003; Volume 1, pp. 781–835.
31. Grosse-Erdmann, K.-G. Universal families and hypercyclic vectors. *Bull. Amerg. Math. Soc.* **1999**, *36*, 345–381. [CrossRef]
32. Grosse-Erdmann, K.-G.; Peris, A. *Linear Chaos*; Springer: London, UK, 2011.
33. Shapiro, J.H. Notes on the Dynamics of Linear Operators. 2001. Available online: http://joelshapiro.org/Pubvit/Downloads/LinDynamics/lindynamics.pdf (accessed on 5 May 2020).
34. Stockman, D. Li-Yorke Chaos in Models with Backward Dynamics. 2012. Available online: http://sites.udel.edu/econseminar/files/2012/03/lyc-backward.pdf (accessed on 5 May 2020).
35. Hou, B.; Luo, L. Li-Yorke chaos for invertible mappings on noncompact spaces. *Turk. J. Math.* **2016**, *40*, 411–416. [CrossRef]
36. Luo, L.; Hou, B. Some remarks on distributional chaos for bounded linear operators. *Turk. J. Math.* **2015**, *39*, 251–258. [CrossRef]
37. Luo, L.; Hou, B. Li-Yorke chaos for invertible mappings on compact metric spaces. *Arch. Math. (Basel)* **2017**, *108*, 65–69. [CrossRef]
38. Luo, L. Cowen-Douglas function and its application on chaos. *Ann. Funct. Anal.* **2020**, *11*, 897–913. [CrossRef]
39. Colombo, F.; Sabadini, I.; Struppa, D.C. A new functional calculus for noncommuting operators. *J. Funct. Anal.* **2008**, *254*, 2255–2274. [CrossRef]
40. Colombo, F.; Sabadini, I.; Struppa, D.C. Slice monogenic functions. *Isr. J. Math.* **2009**, *171*, 385–403. [CrossRef]
41. Alpay, D.; Colombo, F.; Qian, T.; Sabadini, I. The H^∞ functional calculus based on the S-spectrum for quaternionic operators and for n-tuples of noncommuting operators. *J. Funct. Anal.* **2016**, *271*, 1544–1584. [CrossRef]
42. Colombo, F.; Sabadini, I.; Struppa, D.C. A functional calculus for n-tuples of noncommuting operators. *Adv. Appl. Clifford Algebr.* **2009**, *19*, 225–236. [CrossRef]
43. Colombo, F.; Sabadini, I. The Cauchy formula with s-monogenic kernel and a functional calculus for noncommuting operators. *J. Math. Anal. Appl.* **2011**, *373*, 655–679. [CrossRef]
44. Colombo, F.; Gantner, J. An application of the S-functional calculus to fractional diffusion processes. *Milan J. Math.* **2018**, *86*, 225–303. [CrossRef]
45. Colombo, F.; Gentili, G.; Sabadini, I.; Struppa, D. C. Non commutative functional calculus: bounded operators. *Complex Anal. Oper. Theory* **2010**, *4*, 821–843. [CrossRef]
46. Colombo, F.; Gentili, G.; Sabadini, I.; Struppa, D.C. Non-commutative functional calculus: unbounded operators. *J. Geom. Phys.* **2010**, *60*, 251–259. [CrossRef]
47. Colombo, F.; Sabadini, I.; Struppa, D.C. *Noncommutative Functional Calculus: Theory and Applications of Slice Hyperholomorphic Functions*; Progress in Mathematics, 289; Birkhäuser/Springer Basel AG: Basel, Switzerland, 2011.
48. Monguzzi, A.; Sarfatti, G. Shift invariant subspaces of slice $\mathcal{L}^2$ functions. *Ann. Acad. Sci. Fenn. Math.* **2018**, *43*, 1045–1061. [CrossRef]
49. Conway, J.B. *A Course in Functional Analysis*, 2nd ed.; Springer-Verlag: New York, NY, USA, 1990.
50. Conway, J.B. *A Course in Operator Theory*; American Mathmatical Society: New York, NY, USA, 2000.
51. Arveson, W. *A Short Course on Spectral Theory*; Springer Science+Businee Media, LLC: New York, NY, USA, 2002.
52. Halmos, P.R. *Measure Theory*; Springer-Verlag: New York, NY, USA, 1974.
53. Hua, L. On the automorphisms of a sfield. *Proc. Nat. Acad. Sci. USA* **1949**, *35*, 386–389. [CrossRef]
54. Schweizer, B.; Smítal, J. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Amer. Math. Soc.* **1994**, *344*, 737–754. [CrossRef]
55. Walters, P. *An Introduction to Ergodic Theory*; Springer-Verlag: New York, NY, USA, 1982.