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M-Hazy Vector Spaces over M-Hazy Field

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Abstract: The generalization of binary operation in the classical algebra to fuzzy binary operation is an important development in the field of fuzzy algebra. The paper proposes a new generalization of vector spaces over field, which is called M-hazy vector spaces over M-hazy field. Some fundamental properties of M-hazy field, M-hazy vector spaces, and M-hazy subspaces are studied, and some important results are also proved. Furthermore, the linear transformation of M-hazy vector spaces is studied and their important results are also proved. Finally, it is shown that M-fuzzifying convex spaces are induced by an M-hazy subspace of M-hazy vector space.

Keywords: M-hazy group; M-hazy ring; M-hazy field; M-hazy vector space; M-hazy subspace; M-fuzzifying convex space

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1. Introduction

In 1971, Rosenfeld [1] published an innovative paper on fuzzy subgroups. This article introduced the new field of abstract algebra and the new field of fuzzy mathematics. Many scientists and researchers worked in this field and obtained fruitful research. Liu [2,3] gave an important generalization in the field of fuzzy algebra by introducing fuzzy subrings of a ring and fuzzy ideals. Demirci [4] firstly introduced the fuzzification of binary operation to group structure through fuzzy equality [5] and introduced "vague groups." After this work, many researchers used this concept and extended it to several other useful directions such as [6–10]. In Demirci's approach, the characteristic of the degree between the fuzzy binary operation is not used, and the identity and inverse element of an element are also not unique. Liu and Shi [11] proposed a new approach to fuzzify the group structure by characterizing the degree of fuzzy binary operation, which is called M-hazy groups. It is important to mention that M-hazy associative law has been defined in order to obtain M-hazy groups. Mehmood et al. [12] extended this concept to the ring structure and gave a new method to the fuzzification of rings, which is defined by M-hazy rings. It is also worth mentioning that an M-hazy distributive law has been proposed so as to define M-hazy rings. Furthermore, Mehmood et al. [13] also provided the homomorphism theorems of M-hazy rings with its induced fuzzifying convexities. Liu and Shi [14] proposed M-hazy lattices. Fan et al. [15] introduced an M-hazy Γ-semigroup.

Vector space has been the most widely studied and used in linear algebra theory. A vector space is a set of elements with a binary addition operator and a multiplication operator that has closure under these two operations over a field, all while satisfying a set of axioms. Vector spaces are the realm of linear combinations, also known as superpositions, weighted sums, and sums with coefficients. Such sums occur throughout mathematics, both pure and applied, including statistics, science, engineering, and economics. The key word is "linear". Even when studying nonlinear phenomena, it is often useful to approximate with a simpler linear model. You can say that vector spaces are one of the great organizing tools of mathematics, helping reveal a structural similarity in a wide variety of topics found in such different contexts that they may seem completely different. Suppose you stand in front of a house. It is rather old but beautifully constructed of masonry that exhibits excellent craftsmanship. You point at a brick in one of the lower layers and ask "What is the need for this brick?" Short answer: it helps the structure. Long answer: it can be missed in the sense that the building won't fall apart if you take it away, but it will damage it in various ways that will become clear if you live there for a couple of years. This house is a metaphor for mathematics. A vector space is a lot more than just a brick. It is one of the fundamental notions, and so is part of the foundation. The most fundamental notion is "Set," and a vector space is one notch higher, a set with a specific structure. Can you do without it? Not if you want to do any serious mathematics. You can refine the structure to get topological vector space, metric vector space, complete vector space, normed vector space, and inner product vector space, each a refinement of the former. The beauty of this is that a refinement inherits all properties of its ancestor, so you are saved a lot of groundwork and can explore additional properties. The need for refinement is usually triggered by questions from physics or engineering. A nice example is Fourier analysis that fits smoothly in a Hilbert space structure.

Since Katsaras and Liu [16] presented the notion of fuzzy vector spaces, many scientists and researchers explored its properties and obtained fruitful research such as [17–21]. Mordeson [22] defined bases of fuzzy vector spaces. Shi and Huang [23] defined fuzzy bases and fuzzy dimension of fuzzy vector spaces. Nanda [24,25] introduced fuzzy fields and linear spaces. Malik and Mordeson [26] defined fuzzy subfield of a field. Fang and Yan [27] introduced the notion of L-fuzzy topological vector spaces. Zhang and Xu [28,29] presented the concept of topological L-fuzzifying neighborhood structures. Furthermore, Yan and Wu [30] extended this concept by introducing fuzzifying topological vector spaces on completely distributive lattices. Wen et al. [31] gave the degree to which an L-subset of a vector space is an L-convex set.

In the past few years, theory of convexity has emerged more and more important study for exceptional problems in many fields of applied mathematics. Since the 1950s, convexity theory has developed into several related theories. Van de Vel [32] conducted an inventive investigation. His work was praised as excellence. An interesting question about the application part of convex theory attempts to include determining the computational complexity of convexity, pattern recognition problems, optimization, etc. Fuzzy set theory is an emerging discipline in different fields of abstract algebra such as topology and convexity theory, etc. Rosa [33] firstly presented the notion of fuzzy convex spaces. Shi and Xiu [34] proposed a new technique for the fuzzification of convex structures, which is described as M-fuzzifying convex structures. In this technique, each subset of a set X has a certain degree of convexity. Furthermore, Shi and Xiu [35] gave the generalization of L-convex structure and M-fuzzifying convex structure by introducing (L, M)-fuzzy convex structure. In their approach, every L-fuzzy subset can be considered as an L-convex set to some extent. With repeated progress in the area of convexity theory, fuzzy convex structures have become a major research area such as [36–52]. Pang and Xiu [53] introduced the notion of lattice-valued interval operators and described their connection between L-fuzzifying convex structures. Liu and Shi [54] proposed M-fuzzifying median algebra, which is obtained through fuzzy binary operation. This study provided the characteristics of M-fuzzifying median algebra and M-fuzzifying convex spaces. This work provided motivation to extend it on more algebraic structures like groups, rings, lattices, and vector spaces. Liu and Shi [11] introduced M-hazy groups by using the M-hazy binary operation. Mehmood et al. [12,13] extended this idea by defining M-hazy rings and obtained its induced fuzzifying convexities. By getting the motivations of these new proposed concepts through M-hazy operations, we proposed a new generalization of vector spaces over field based on M-hazy binary operation, which is denoted as M-hazy vector spaces over M-hazy field.

The paper is organized as follows: Section 2 consists of fundamental notions about completely residuated lattices, field and vector spaces, M-hazy groups, M-hazy rings, and M-fuzzifying convex spaces. In Section 3, the concept of M-hazy vector space is defined and obtained its fundamental properties. In Section 4, the concept of M-hazy subspaces is introduced, and it has been shown that all the M-hazy subspaces of M-hazy vector space form a convex structure. In Section 5, the linear transformation of M-hazy vector spaces is introduced. Finally, M-fuzzifying convex spaces are induced by M-hazy subspaces of M-hazy vector spaces. Section 6 concludes the paper.

2. Preliminaries

This section contains the fundamental definitions about completely residuated lattices, fields, vector spaces, vector subspaces, M-hazy groups, M-hazy rings, and M-fuzzifying convexity.

All through this paper, $(M, \vee, \wedge, \diamond, \rightarrow, \perp, \top)$ represents a completely residuated lattice, and $\leq$ denotes the partial order of M. Assume P is a nonempty set and $K \subseteq M$, then $\bigvee K$ denotes the least upper bound of K and $\bigwedge K$ the greatest lower bound of K. 2^P (resp., M^P) denotes the collection of all subsets (resp., M-subsets) on P. A family $\{A_i \mid i \in \omega\}$ is up-directed provided for each $A_1, A_2 \in \{A_i \mid i \in \omega\}$ that there exists a third element $A_3 \in \{A_i \mid i \in \omega\}$ such that $A_1 \subseteq A_3$ and $A_2 \subseteq A_3$ are denoted by: $\{A_i \mid i \in \omega\} \overset{dir}{\subseteq} 2^P$.

Definition 1 ([55]). *Assume that $\diamond : M \times M \longrightarrow M$ is a function. $\diamond$ is defined to be a triangular norm (for short, t-norm) on M, if the following conditions holds:*

(1) $u \diamond v = v \diamond u$,
(2) $(u \diamond v) \diamond w = u \diamond (v \diamond w)$,
(3) $u \leq w, v \leq x$ *implies* $u \diamond v \leq w \diamond x$,
(4) $u \diamond 1 = u$ *for all* $u \in M$.

Definition 2 ([55]). *Assume that $\rightarrow: M \times M \longrightarrow M$ is a function, and $\diamond$ is a t-norm in M. Then, $\rightarrow$ is defined to be the residuum of $\diamond$, if, for all $u, v, w \in M$,*

$$u \leq v \rightarrow w \Leftrightarrow u \diamond v \leq w.$$

Definition 3 ([56]). *Assume that $(M, \vee, \wedge, \perp, \top)$ is a bounded lattice, where $\perp$ represents the least element, $\top$ the greatest element, $\diamond$ is a t-norm on M, and $\rightarrow$ denotes the residuum of $\diamond$. Then, $(M, \vee, \wedge, \diamond, \rightarrow, \perp, \top)$ is said to be a residuated lattice.*

A residuated lattice is defined to be a completely residuated lattice if the primary lattice is complete. In addition, we define $u \leftrightarrow v = (u \rightarrow v) \wedge (v \rightarrow u)$. The proposition below shows properties of the implication operation.

Proposition 1 ([57,58]). *Assume $(M, \vee, \wedge, \diamond, \rightarrow, \perp, \top)$ is a completely residuated lattice. Then, for every $u, v, w \in M$, $\{u_i\}_{i \in I}$, $\{v_i\}_{i \in I} \subseteq M$, the below statements are valid:*

(1) $u \rightarrow v = \bigvee \{w \in M \mid u \diamond w \leq v\}$.
(2) $v \leq u \rightarrow v$, $\top \rightarrow u = u$.
(3) $u \rightarrow (\bigwedge_{i \in I} v_i) = \bigwedge_{i \in I} (u \rightarrow v_i)$.
(4) $(\bigvee_{i \in I} u_i) \rightarrow v = \bigwedge_{i \in I} (u_i \rightarrow v)$.

Definition 4 ([11]). *Assume that $* : P \times P \longrightarrow M^P$ is a function; then, $*$ is defined to be an M-hazy operation on P, if the conditions given below hold:*

(MH1) $\forall u, v \in P$, *we have* $\bigvee_{p \in P} (u * v)(p) \neq \perp$.

(MH2) $\forall u, v, p, q \in P$, $(u * v)(p) \diamond (u * v)(q) \neq \perp \Rightarrow p = q$.

Definition 5 ([11]). *Assume $* : P \times P \longrightarrow M^P$ is an M-hazy operation on a nonempty set P. Then, $(P, *)$ is defined to be an M-hazy group (in short, MHG) if the following conditions hold:*

(MG1) $\forall u, v, w, p, q \in P$, $(u * v)(p) \diamond (v * w)(q) \leq \bigwedge_{r \in P} ((p * w)(r) \leftrightarrow (u * q)(r))$, *i.e., the M-hazy associative law holds.*

(MG2) *An element $o \in P$ is said to be the left identity element of P, if $o * u = u_\top$ for all $u \in P$.*

(MG3) *An element $v \in P$ is said to be the left inverse of u, if for each $u \in P$, $v * u = o_\top$, and is denoted by u^{-1}.*

$(P, *)$ is defined to be an abelian MHG if the following condition holds:

(MG4) *$u * v = v * u$ for all $u, v \in P$.*

Definition 6 ([12]). *Assume $+ : R \times R \longrightarrow M^R$ and $\bullet : R \times R \longrightarrow M^R$ are the M-hazy addition operation and M-hazy multiplication operation on R, respectively. Then, $(R, +, \bullet)$ is defined to be an M-hazy ring (in short, MHR) if the below conditions hold:*

(MHR1) *$(R, +)$ is an abelian MHG.*

(MHR2) *$(R, \bullet)$ is an M-hazy semigroup.*

(MHR3) $\forall u, v, w, p, q, r \in R, (u \bullet v)(p) \diamond (v + w)(q) \diamond (u \bullet w)(r) \leq \bigwedge_{s \in R} ((u \bullet q)(s) \leftrightarrow (p + r)(s))$.

We now give the definition of M-fuzzifying convex space, and we refer to Vel [32] for all of the background on the convexity theory that may be required.

Definition 7 ([34]). *A function $\mathscr{S} : 2^P \to M$ is said to be an M-fuzzifying convexity on a nonempty set P if the below conditions hold:*

(1) $\mathscr{S}(\varnothing) = \mathscr{S}(P) = \top$;

(2) *If $\{D_i | i \in \Omega\} \subseteq 2^P$ is nonempty, then* $\mathscr{S}(\bigcap_{i \in \Omega} D_i) \geq \bigwedge_{i \in \Omega} \mathscr{S}(D_i)$;

(3) *If $\{D_i | i \in \Omega\} \overset{dir}{\subseteq} 2^P$, then* $\mathscr{S}(\bigcup_{i \in \Omega} D_i) \geq \bigwedge_{i \in \Omega} \mathscr{S}(D_i)$.

Then, $(P, \mathscr{S})$ is said to be an M-fuzzifying convex space.

A function $\pi : (P, \mathscr{S}_P) \longrightarrow (Q, \mathscr{S}_Q)$ is defined as M-fuzzifying convexity preserving (M-CP, in short) given that $\mathscr{S}_P(\pi^{\leftarrow}(B)) \geq \mathscr{S}_Q(B)$ for each $B \in 2^Q$; π is called M-fuzzifying convex-to-convex (M-CC, in short) provided that $\mathscr{S}_Q(\pi^{\rightarrow}(A)) \geq \mathscr{S}_P(A)$ for each $A \in 2^P$.

Definition 8 ([22]). *A field is a set F with two operations, called addition and multiplication, which satisfy the following conditions:*

(F1) $\forall u, v \in F, u + v = v + u$,

(F2) $\forall u, v, w \in F, (u + v) + w = u + (v + w)$,

(F3) *F contains an element 0 such that* $0 + u = u, \forall u \in F$,

(F4) *For each $u \in F$, there is an element $-u \in F$ such that* $u + (-u) = 0$,

(F5) $\forall u, v \in F \implies u \cdot v \in F$,

(F6) $\forall u, v \in F, u \cdot v = v \cdot u$,

(F7) $\forall u, v, w \in F, (u \cdot v) \cdot w = u \cdot (v \cdot w)$,

(F8) *F contains an element $1 \neq 0$ and $\forall u \in F$ such that* $1 \cdot u = u$,

(F9) *For each $0 \neq u \in F$, there is an element $u^{-1} \in F$ such that* $u \cdot u^{-1} = 1$,

(F10) $\forall u, v, w \in F, u \cdot (v + w) = u \cdot w + v \cdot w$.

Definition 9 ([22]). *A vector space is a nonempty set V over a field F, whose objects are called vectors equipped with two operations, called addition and scalar multiplication: for any two vectors u, v in V and a scalar a in F defined by the mappings $+ : V \times V \longrightarrow V$ and $\cdot : F \times V \longrightarrow V$, the following conditions are satisfied:*

(V1) $\forall u, v, w \in V, (u+v)+w = u+(v+w)$,

(V2) *There is a vector 0, called the zero vector, such that* $u+0=u$,

(V3) *For any vector* u, *there is a vector* $-u$ *such that* $u+(-u)=0$,

(V4) $\forall u, v \in V, u+v = v+u$,

(V5) $\forall u, v \in V, \forall a \in F, a \cdot (u+v) = a \cdot u + a \cdot v$,

(V6) $\forall a, b \in F, \forall u \in V, (a+b) \cdot u = a \cdot u + b \cdot u$,

(V7) $\forall a, b \in F, \forall u \in V, a \cdot (b \cdot u) = (a \cdot b) \cdot u$,

(V8) $\forall 1 \in F, \forall u \in V, 1 \cdot u = u$.

3. M-Hazy Vector Spaces

In this section, we introduce the concept of M-hazy vector spaces over the M-hazy field.

We first introduce the concept of M-hazy field and give its properties, which are necessary to present the concept of M-hazy vector space.

Definition 10. *Assume* $+ : F \times F \longrightarrow M^F$ *and* $\bullet : F \times F \longrightarrow M^F$ *are the M-hazy addition operation and M-hazy multiplication operation on F, respectively. Then, the 3-tuple* $(F, +, \bullet)$ *is defined to be an M-hazy field (in short, MHF) if the below conditions hold:*

(MHF1) $(F, +)$ *is an abelian MHG.*

(MHF2) $(F, \bullet)$ *is an abelian MHG.*

(MHF3) $\forall u, v, w, p, q, r \in F, (u \bullet v)(p) \diamond (v+w)(q) \diamond (u \bullet w)(r) \leq \bigwedge_{s \in F} ((u \bullet q)(s) \leftrightarrow (p+r)(s))$.

Proposition 2. *Assume that* $(F, +, \bullet)$ *is an M-hazy field, and o and e are the additive and multiplicative identity elements of F, respectively. Then,* $\forall u \in F$;

(1) $o + u = u + o = u_\top$.

(2) $e \bullet u = u \bullet e = u_\top$.

Proof. The proof is similar to the proof of Proposition 3.8 in [11] so it is omitted. □

Proposition 3. *In an M-hazy field* $(F, +, \bullet)$, *the left additive inverse* $-u$ *of u is also the right additive inverse of u in* $(F, +, \bullet)$. *In addition, the left multiplicative inverse* u^{-1} *of u is also the right multiplicative inverse of u in* $(F, +, \bullet)$. *That is, the following conditions hold:*

(1) $(-u) + u = u + (-u) = o_\top$.

(2) $u^{-1} \bullet u = u \bullet u^{-1} = e_\top$.

Proof. The proof is similar to the proof of Proposition 3.7 in [11] so it is omitted. □

Example 1. *Assume that* $F = \{o, e, u, v\}$ *is a set and assume* $(M, \diamond) = ([0,1], \wedge)$. *The mappings* $+ : F \times \times F \longrightarrow [0,1]^F$ *and* $\bullet : F \times F \longrightarrow [0,1]^F$ *are defined by the following tables:*

(a) Values of the $[0,1]$-hazy operation $+$.

$x+y$ (x \ y)	o	e	u	v
o	o_1	e_1	u_1	v_1
e	e_1	o_1	$v_{0.2}$	$u_{0.2}$
u	u_1	$v_{0.2}$	o_1	e_1
v	v_1	$u_{0.2}$	e_1	o_1

(b) Values of the $[0,1]$-hazy operation $\bullet$.

$x \bullet y$ (x \ y)	o	e	u	v
o	o_1	o_1	o_1	o_1
e	o_1	e_1	$u_{0.3}$	$v_{0.4}$
u	o_1	$u_{0.3}$	$v_{0.4}$	e_1
v	o_1	$v_{0.4}$	e_1	$u_{0.3}$

It is easy to verify (MHF1), (MHF2), and (MHF3) analogous to Example 3.3 in [12].

Proposition 4. *Assume that $(F, +, \bullet)$ is an M-hazy field; then, the following equations hold:*

(1) $$(u+v)(w) = ((-u)+w)(v) = (w+(-v))(u) = (v+(-w))(-u) = ((-w)+u)(-v) = ((-v)+(-u))(-w).$$

(2) $$(u \bullet v)(w) = (u^{-1} \bullet w)(v) = (w \bullet v^{-1})(u) = (v \bullet w^{-1})(u^{-1}) = (w^{-1} \bullet u)(v^{-1}) = (v^{-1} \bullet u^{-1})(w^{-1}).$$

Proof. The proof is similar to the proof of Proposition 3.11 and Corollary 3.12 in [11] so it is omitted. □

We now present the concept of M-hazy vector space.

Definition 11. *Assume $(F, +, \bullet)$ is an M-hazy field and $(V, \oplus)$ is an abelian M-hazy group. We define an M-hazy vector space over F as a quadruple $(V, \oplus, \circ, F)$, where $\circ$ is a mapping $\circ : F \times V \longrightarrow M^V$ such that the following conditions hold:*

(MHV1) $\forall u, v, p, q, r \in V$ *and* $a \in F$, $(a \circ u)(p) \diamond (u \oplus v)(q) \diamond (a \circ v)(r) \leq \bigwedge_{s \in V}((a \circ q)(s) \leftrightarrow (p \oplus r)(s))$.

(MHV2) $\forall u, p, q \in V$ *and* $a, b \in F$, $(a \circ u)(p) \diamond (a+b)(q) \diamond (b \circ u)(r) \leq \bigwedge_{s \in V}((q \circ u)(s) \leftrightarrow (p \oplus r)(s))$.

(MHV3) $\forall u, q \in V$, *and* $\forall a, b, c \in F$, $(a \bullet b)(c) \diamond (b \circ u)(q) \leq \bigwedge_{r \in V}((c \circ u)(r) \leftrightarrow (a \circ q)(r))$.

(MHV4) $\forall u \in V$ *and* $e \in F$, $e \circ u = e_\top$.

Proposition 5. *Assume $\oplus : V \times V \longrightarrow M^V$ and $\circ : F \times V \longrightarrow M^V$ are the M-hazy operations under addition and under scalar multiplication on V, respectively; then, the following statements are equivalent for all $u, v \in V$, and $\forall a \in F$.*

(MHV1) $\forall u, v, p, q, r \in V$, *and* $\forall a \in F$,

$$(a \circ u)(p) \diamond (u \oplus v)(q) \diamond (a \circ v)(r) \leq \bigwedge_{s \in V}((a \circ q)(s) \leftrightarrow (p \oplus r)(s)).$$

(MHV1′) $\forall u, v, p, q, r, s \in V$, *and* $\forall a \in F$,

$$(a \circ u)(p) \diamond (u \oplus v)(q) \diamond (a \circ v)(r) \diamond (a \circ q)(s) \leq (p \oplus r)(s)$$

and

$$(a \circ u)(p) \diamond (u \oplus v)(q) \diamond (a \circ v)(r) \diamond (p \oplus r)(s) \leq (a \circ q)(s).$$

(MHV1″)

$$If a \circ u = p_\lambda, u \oplus v = q_\mu, a \circ v = r_\nu,$$

then $(a \circ q) \diamond \lambda \diamond \mu \diamond \nu \leq (p \oplus r)$ *and* $(p \oplus r) \diamond \lambda \diamond \mu \diamond \nu \leq (a \circ q)$.

(MHV1‴) *If* $a \circ u = p_\lambda$, $u \oplus v = q_\mu$, $a \circ v = r_\nu$, $a \circ q = t_\alpha$, $p \oplus r = u_\beta$, *then*

$$t = u, \lambda \diamond \mu \diamond \nu \diamond \alpha \leq \beta \text{ and } \lambda \diamond \mu \diamond \nu \diamond \beta \leq \alpha.$$

Proof. (MHV1) ⇒ (MHV1′) The proof is simple so it is omitted.
(MHV1′) ⇒ (MHV1″) $\forall a \in F$ and $\forall q \in V$, we have

$$\begin{aligned} & ((a \circ q) \diamond \lambda \diamond \mu \diamond \nu)(s) \\ = & (a \circ q)(s) \diamond \lambda \diamond \mu \diamond \nu \\ = & (a \circ q)(s) \diamond (a \circ u)(p) \diamond (u \oplus v)(q) \diamond (a \circ v)(r) \\ \leq & (p \oplus r)(s), \end{aligned}$$

that is, $(a \circ q) \diamond \lambda \diamond \mu \diamond \nu \leq p \oplus r$. A similar argument shows the other inequality.

(MHV1$''$) $\Rightarrow$ (MHV1$'''$) We need to only verify $t = u$. According to (MHV1$''$), we have $t_\alpha \diamond \lambda \diamond \mu \diamond \nu \leq u_\beta$, that is, $t_{\alpha\diamond\lambda\diamond\mu\diamond\nu} \leq u_\beta$, whence by (MH2), $t = u$.

(MHV1$'''$) $\Rightarrow$ (MHV1) $\forall p, q, r, s \in P$ and $\forall a \in F$, we have

$$\begin{aligned} & (a \circ u)(p) \diamond (u \oplus v)(q) \diamond (a \circ v)(r) \\ = \; & \lambda \diamond \mu \diamond \nu \\ \leq \; & (\alpha \to \beta) \wedge (\beta \to \alpha) \\ = \; & \alpha \leftrightarrow \beta = (a \circ q)(t) \leftrightarrow (p \oplus r)(u) \end{aligned}$$

and, by (MH2), we can complete the proof. □

Example 2. *(1) The Euclidean space R^n is an M-hazy vector space under the addition and scalar multiplication.*

(2) The set P_n of all polynomials of degree less than or equal to n is an M-hazy vector space under the addition and scalar multiplication of polynomials.

(3) The set $M(m, n)$ of all $m \times n$ matrices is an M-hazy vector space under the addition and scalar multiplication of matrices.

In the following discussion, we assume that the operation $\diamond$ in the completely residuated lattice M is $\wedge$; that is, the lattice valued environment M is degenerated to complete Heyting algebra. We also assume that the smallest element $\perp$ is prime in M.

Theorem 1. *Assume that $(V, \oplus, \circ, F)$ is an M-hazy vector space over an M-hazy field $(F, +, \bullet)$. Then, $\forall u \in V$ and $\forall a \in F$:*

(1) $(a \circ o)(o) \neq \perp$.

(2) $(o \circ u)(o) \neq \perp$.

(3) If $(a \circ u)(o) \neq \perp$, then $a = o$ or $u = o$.

Proof. We only prove (1). Assume $(a \circ o)(o) = \perp$. By (MHV1) $\forall a \in F$ and $o \in V$, we have

$$(a \circ o)(p) \diamond (o \oplus o)(o) \diamond (a \circ o)(r) \leq \bigwedge_{s \in V} ((a \circ o)(s) \leftrightarrow (p \oplus r)(s)).$$

When $r = s = p$, we have,

$$(a \circ o)(p) \diamond (a \circ o)(p) \diamond (a \circ o)(p) \leq (p \oplus p)(p)).$$

If $p \neq o$, then $(p \oplus p)(p) = (p \oplus (-p))(p) = \perp$ by the condition (1) of Proposition 4, which is a contradiction. Hence, $(a \circ o)(o) \neq \perp$. □

4. *M*-Hazy Subspaces

In this section, we introduce the concept of M-hazy subspaces of M-hazy vector space.

Definition 12. *Assume that $(V, \oplus, \circ, F)$ is an M-hazy vector space over an M-hazy field $(F, +, \bullet)$. A nonempty subset W of V is called an M-hazy subspace of V if W itself is an M-hazy vector space over F.*

Theorem 2. *Assume W is a nonempty subset of an M-hazy vector space $(V, \oplus, \circ, F)$ over an M-hazy field $(F, +, \bullet)$; then, $(W, \oplus, \circ, F)$ is an M-hazy subspace of $(V, \oplus, \circ, F)$ over $(F, +, \bullet)$ if and only if the following conditions hold:*

(1) $\forall u, v \in W$, we have $\bigvee_{p \in W} (u \oplus v)(p) \neq \perp$,

(2) $\forall u \in W$ and $\forall a \in F$, we have $\bigvee_{p \in W} (a \circ u)(p) \neq \perp$,

(3) $\forall u \in W$, we have $-u \in W$.

Proof. The proof is simple and omitted. □

Theorem 3. *Assume W is a nonempty subset of an M-hazy vector space $(V, \oplus, \circ, F)$ over an M-hazy field $(F, +, \bullet)$; then, $(W, \oplus, \circ, F)$ is an M-hazy subspace of $(V, \oplus, \circ, F)$ over $(F, +, \bullet)$ if and only if the following conditions hold:*

(1) $\forall u, v \in W$, *we have* $\bigvee_{p \in W} (u \oplus (-v))(p) \neq \perp$,

(2) $\forall u \in W$ *and* $\forall a \in F$, *we have* $\bigvee_{p \in W} (a \circ u)(p) \neq \perp$.

Proof. The proof is similar to the proof of Theorem 4.4 in [12] so it is omitted. □

Theorem 4. *Assume W is a nonempty subset of an M-hazy vector space $(V, \oplus, \circ, F)$ over an M-hazy field $(F, +, \bullet)$; then, $(W, \oplus, \circ, F)$ is an M-hazy subspace of $(V, \oplus, \circ, F)$ over $(F, +, \bullet)$ if and only if $\forall u, v, p, q, r \in W$ and $\forall a, b \in F$, we have*

$$\bigvee_{r \in W} ((a \circ u) \oplus (b \circ v))(r) \neq \perp.$$

Proof. Assume that W is an M-hazy subspace of V over an M-hazy field F. Suppose that

$$\bigvee_{r \in W} ((a \circ u) \oplus (b \circ v))(r) = \perp.$$

On the other hand, $\bigvee_{p \in W} (a \circ u)(p) \neq \perp$ and $\bigvee_{q \in W} (b \circ v)(q) \neq \perp$ by Theorem 3. Hence, $\bigvee_{r \in W} (p \oplus q)(r) \neq \perp$, which is a contradiction. Hence,

$$\bigvee_{r \in W} ((a \circ u) \oplus (b \circ v))(r) \neq \perp.$$

Conversely, suppose $\bigvee_{r \in W} ((a \circ u) \oplus (b \circ v))(r) \neq \perp$. Since W is a nonempty subset of V and we know that (MH1) and (MH2) holds in V, thus it holds in W. Hence, $\forall u, v \in W$ and $e \in F$, we have $\bigvee_{u \in W} (e \circ u)(u) \neq \perp$ and $\bigvee_{v \in W} (e \circ v)(v) \neq \perp$. Hence, $\bigvee_{r \in W} (u \oplus v)(r) \neq \perp$. In addition, $\forall b \in F$ and $o \in W$, we have $(b \circ o)(o) \neq \perp$ by statement (1) of Theorem 1. Since $\forall a \in F$ and $\forall u \in W$, we have $\bigvee_{p \in W} (a \circ u)(p) \neq \perp$; thus, $\bigvee_{r \in W} (p \oplus o)(r) \neq \perp$. Hence, $\bigvee_{r \in W} (a \circ u)(r) \neq \perp$ by (MG2). Now, we already know that (MG3) holds in V, so it holds in W. Hence, $\forall u \in W$, we have $-u \in W$. Hence, W is an M-hazy subspace of V over F. □

Proposition 6. *The intersection of a family of M-hazy subspace of an M-hazy vector space $(V, \oplus, \circ, F)$ over an M-hazy field $(F, +, \bullet)$ is an M-hazy subspace of $(V, +, \circ, F)$.*

Proof. Assume Λ is an index set and W_i is an M-hazy subspace of $(V, \oplus, \circ, F)$. Assume $K = \bigcap_{i \in \Lambda} W_i$.

(1) Since $o \in W_i$ for each $i \in \Lambda$ and K is a nonempty subset of V, we have $o \in K$.

(2) For every $u, v \in K, a \in F$ and for every $i \in \Lambda$, we obtain $u, v \in W_i$. Since W is an M-hazy subspace of V, we obtain $\bigvee_{p \in W_i} (u \oplus (-v))(p) \neq \perp$ and $\bigvee_{p \in W_i} (a \circ u)(p) \neq \perp$ by Theorem 3. Since $\bigvee_{p \in W_i} (a \circ u)(p) \neq \perp$. This implies that there exists $x_{au} \in W_i$ such that $(a \circ u)(x_{au}) \neq \perp$. This implies, for all $i \in \Lambda$, $x_{au} \in W_i$. Thus, we can obtain $x_{au} \in \bigcap_{i = \Lambda} W_i$.

Hence, $\bigvee_{p\in\bigcap_{i\in\Lambda} W_i}(a\circ u)(p) \geq (a\circ u)(x_{au}) \neq \bot$. Similarly, $\bigvee_{p\in\bigcap_{i\in\Lambda} W_i}(u\oplus(-v))(p) \neq \bot$. Hence, $K = \bigcap_{i=\Lambda} W_i$ is an M-hazy subspace of $(V,\oplus,\circ,F)$. □

Proposition 7. *The union of a nonempty up-directed family of M-hazy subspace of M-hazy vector space $(V,\oplus,\circ,F)$ over an M-hazy field $(F,+,\bullet)$ is an M-hazy subspace of $(V,\oplus,\circ,F)$. In particular, the union of a nonempty chain of M-hazy subspace of M-hazy vector space $(V,\oplus,\circ,F)$ is an M-hazy subspace of $(V,\oplus,\circ,F)$.*

Proof. Assume Λ is an index set and W_i is an M-hazy subspace of $(V,\oplus,\circ,F)$, where $\{W_i \mid i\in\Lambda\}$ is an up-directed subfamily of 2^V. Let $N = \bigcup_{i\in\Lambda} W_i$.

(1) Clearly, N is a nonempty subset of V.

(2) For every $u,v\in N$ and $a\in F$, there exists $i,j\in\Lambda$ such that $u\in W_i$ and $v\in W_j$. Since N is an up-directed family, then there exists $m\in\Lambda$ such that $W_i\subseteq W_m$; this implies that $u,v\in W_m$. As $(W,\oplus,\circ,F)$ is an M-hazy subspace of $(V,\oplus,\circ,F)$, we obtain $\bigvee_{p\in W_m}(u\oplus(-v))(p)\neq\bot$ and $\bigvee_{p\in W_m}(a\circ u)(p)\neq\bot$ by Theorem 3. Hence, $\bigvee_{p\in N}(u\oplus(-v))(p)\neq\bot$ and $\bigvee_{p\in N}(a\circ u)(p)\neq\bot$. Hence, $\bigcup_{i\in\Lambda} W_i$ is an M-hazy subspace of $(V,\oplus,\circ,F)$. □

Based on the above results, we can draw an important and interesting conclusion; that is, we have the following result.

Proposition 8. *All of the M-hazy subspaces of M-hazy vector space and the empty set form a convex structure.*

5. Linear Transformation of *M*-Hazy Vector Spaces

In this section, we introduce the linear transformation of M-hazy vector spaces. We have also shown that M-fuzzifying convex spaces are induced by M-hazy subspace of M-hazy vector space.

Definition 13. *Assume that $(V,\oplus,\circ,F)$ and $(W,\boxplus,\odot,F)$ are two M-hazy vector spaces over an M-hazy field $(F,+,\bullet)$. Then, the mapping $\tau: V\longrightarrow W$ is called a linear transformation if the following conditions hold:*

(1) $\forall u,v\in V, \tau_M^{\rightarrow}(u\oplus v) = (\tau(u)\boxplus\tau(v))$,

(2) $\forall u\in V, \forall a\in F, \tau_M^{\rightarrow}(a\circ u) = (a\odot\tau(u))$.

Definition 14. *Assume V and W are two M-hazy vector spaces over an M-hazy field F, $\tau: V\longrightarrow W$ is a linear transformation, and o' is the additive identity element of W. Then, the kernel of τ,* $\mathrm{Ker}\tau$ *is determined by*

$$\mathrm{Ker}\tau = \tau^{\leftarrow}(\{o'\}) = \{p\in V \mid \tau(p) = o'\}.$$

Example 3. *(1) Assume that $(V,\oplus,\circ,F)$ is an M-hazy vector space over an M-hazy field $(F,+,\bullet)$, the set $\{o\}$ and the whole M-hazy vector space V are M-hazy subspaces of V; they are called the trivial M-hazy subspaces of V.*

(2) Assume R^n and R^m are the Euclidean spaces and $\tau: R^n\longrightarrow R^m$ is a linear transformation. The image

$$\tau^{\rightarrow}(p) = \{\tau(p) : p\in R^n\}$$

of τ is an M-hazy subspace of R^m, and the inverse image

$$\tau^{\leftarrow}(\{o'\}) = \{p\in R^n \mid \tau(p) = o'\}$$

is an M-hazy subspace of R^m.

Proposition 9. *Assume the mapping $\tau : V \longrightarrow W$ is a linear transformation, $(V, \oplus, \circ, F)$ and $(W, \boxplus, \odot, F)$ are two M-hazy vector spaces over an M-hazy field $(F, +, \bullet)$. Then, the following statements are valid:*

(1) If J is an M-hazy subspace of V, then $\tau^{\rightarrow}(J)$ is an M-hazy subspace of W.
(2) If K is an M-hazy subspace of W, then $\tau^{\leftarrow}(K)$ is an M-hazy subspace of V containing $\ker\tau$.

Proof. (1) For all $u, v \in J$, we have $\bigvee_{p\in J} (u \oplus (-v))(p) \neq \perp$. Then,

$$\begin{aligned}
& \bigvee_{\tau(p)\in\tau^{\rightarrow}(J)} (\tau(u) \boxplus (-\tau(v)))(\tau(p)) \\
= & \bigvee_{\tau(p)\in\tau^{\rightarrow}(J)} (\tau(u) \boxplus \tau(-v))(\tau(p)) \\
= & \bigvee_{\tau(p)\in\tau^{\rightarrow}(J)} \tau_M^{\rightarrow}(u \oplus (-v))(\tau(p)) \\
= & \bigvee_{\tau(p)\in\tau^{\rightarrow}(J)} \bigvee_{\tau(x)=\tau(p)} (u \oplus (-v))(x) \\
= & \bigvee_{\tau(p)\in\tau^{\rightarrow}(J)} (u \oplus (-v))(p) \\
\geq & \bigvee_{p\in J} (u \oplus (-v))(p) \\
\neq & \perp.
\end{aligned}$$

Similarly,

$$\bigvee_{\tau(p)\in\tau^{\rightarrow}(J)} (a \odot \tau(u))(\tau(p)) \geq \bigvee_{p\in J} (a \circ u)(p) \neq \perp.$$

Then, by Theorem 3, it follows that $\tau^{\rightarrow}(J)$ is an M-hazy subspace of W.

(2) For all $u, v \in \tau^{\leftarrow}(K)$, we have $\tau(u), \tau(v) \in K$. We find that $\bigvee_{\tau(p)\in K} (\tau(u) \boxplus (-\tau(v)))(\tau(p)) \neq \perp$ and $\bigvee_{\tau(p)\in K} (a \odot \tau(u))(\tau(p)) \neq \perp$, since K is an M-hazy subspace of W. Furthermore,

$$\begin{aligned}
& \bigvee_{\tau(p)\in K} (\tau(u) \boxplus (-\tau(v)))(\tau(p)) \\
= & \bigvee_{\tau(p)\in K} (\tau(u) \boxplus \tau(-v))(\tau(p)) \\
= & \bigvee_{\tau(p)\in K} \tau_M^{\rightarrow}(u \oplus (-v))(\tau(p)) \\
= & \bigvee_{\tau(p)\in K} \bigvee_{\tau(x)=\tau(p)} (u \oplus (-v))(x) \\
= & \bigvee_{\tau(p)\in K} (u \oplus (-v))(p) \\
= & \bigvee_{p\in\tau^{\leftarrow}(K)} (u \oplus (-v))(p) \\
& \neq \perp.
\end{aligned}$$

Similarly,

$$\bigvee_{\tau(p)\in K} (a \odot \tau(u))(\tau(p)) = \bigvee_{p\in\tau^{\leftarrow}(K)} (a \circ u)(p) \neq \perp.$$

Consequently, $\tau^{\leftarrow}(K)$ is an M-hazy subspace of V.

Now, assume $p \in \ker\tau$. Since K is an M-hazy subspace of W, then $\tau(p) = o' \in K$, and so $p \in \tau^{\leftarrow}(K)$. Hence, $\ker\tau \subseteq \tau^{\leftarrow}(K)$. □

Proposition 10. *Assume V and W are two M-hazy vector spaces over an M-hazy field F and $\tau : V \longrightarrow W$ is a linear transformation. Then,* Kerτ *is an M-hazy subspace of V.*

Proof. It is easy to see that $\{o'\}$ is an M-hazy subspace of W. Then, by Proposition 9, we have that Kerτ is an M-hazy subspace of V. □

The theorems below give an approach to induce M-fuzzifying convex spaces using M-hazy subspace of M-hazy vector space.

Theorem 5. *Assume $(V, \oplus, \circ, F)$ is an M-hazy vector space over an M-hazy field $(F, +, \bullet)$ and define $\mathscr{S} : 2^V \to M$ as follows:*

$$\forall A \in 2^V,\ \mathscr{S}(A) = \bigwedge_{p\in V}\Big(\big((\bigvee_{u,v\in A}(u \oplus (-v))(p)) \to A(p)\big) \wedge \big((\bigvee_{a\in F, u\in A}(a\circ u)(p)) \to A(p)\big)\Big).$$

Then, $(V, \mathscr{S})$ is an M-fuzzifying convex space.

Proof. The proof is similar to the proof of Theorem 7 in [13] so it is omitted. □

Theorem 6. *Assume that the mapping $\tau : V \longrightarrow W$ is a linear transformation, $(V, \oplus, \circ, F)$ and $(W, \boxplus, \odot, F)$ are two M-hazy vector spaces over an M-hazy field $(F, +, \bullet)$; then, $\tau : (V, \mathscr{S}_V) \longrightarrow (W, \mathscr{S}_W)$ is an M-CP mapping.*

Proof. The proof is similar to the proof of Theorem 8 in [13] so it is omitted. □

Theorem 7. *Assume the mapping $\tau : V \longrightarrow W$ is a linear transformation, $(V, \oplus, \circ, F)$ and $(W, \boxplus, \odot, F)$ are two M-hazy vector spaces over an M-hazy field $(F, +, \bullet)$; then, $\tau : (V, \mathscr{S}_V) \longrightarrow (W, \mathscr{S}_W)$ is an M-CC mapping.*

Proof. Since the mapping $\tau : V \longrightarrow W$ is a linear transformation if the following conditions hold:

(1) $\forall u, v \in V, \tau_M^{\rightarrow}(u \oplus v) = (\tau(u) \boxplus \tau(v))$,
(2) $\forall u \in V, \forall a \in F, \tau_M^{\rightarrow}(a \circ u) = (a \odot \tau(u))$.

Then, for all $A \in 2^V$, we have

$$\begin{aligned}
&\mathscr{S}_W(\tau^{\rightarrow}(A)) = \bigwedge_{p\in W}\Big(\big((\bigvee_{\tau(u),\tau(v)\in\tau^{\rightarrow}(A)}(\tau(u)\boxplus\tau(-v))(\tau(p))\big) \to (\tau^{\rightarrow}(A))(\tau(p))\big)\\
&\wedge\big((\bigvee_{a\in F,\tau(u)\in\tau^{\rightarrow}(A)}(a\odot\tau(u))(\tau(p))) \to (\tau^{\rightarrow}(A))(\tau(p))\big)\Big)\\
&\geq \bigwedge_{p\in W}\Big(\big((\bigvee_{\tau(u),\tau(v)\in\tau^{\rightarrow}(A)}(\tau_M^{\rightarrow}\cdot\tau_M^{\leftarrow}(\tau(u)\boxplus\tau(-v)))(\tau(p))\big) \to (\tau^{\rightarrow}(A))(\tau(p))\big)\\
&\wedge\big((\bigvee_{a\in F,\tau(u)\in\tau^{\rightarrow}(A)}(\tau_M^{\rightarrow}\cdot\tau_M^{\leftarrow}(a\odot\tau(u)))(\tau(p))) \to (\tau^{\rightarrow}(A))(\tau(p))\big)\Big)\\
&= \bigwedge_{p\in W}\Big(\big((\bigvee_{\tau(u),\tau(v)\in\tau^{\rightarrow}(A)}(\tau_M^{\rightarrow}(u\oplus(-v)))(\tau(p))\big) \to (\tau^{\rightarrow}(A))(\tau(p))\big)\\
&\wedge\big((\bigvee_{a\in F,\tau(u)\in\tau^{\rightarrow}(A)}(\tau_M^{\rightarrow}(a\circ u))(\tau(p))) \to (\tau^{\rightarrow}(A))(\tau(p))\big)\Big)\\
&= \bigwedge_{x\in V}\Big(\big((\bigvee_{\tau(p)\in\tau_M^{\rightarrow}(A)}\ \bigvee_{\tau(x)=\tau(p)}(u\oplus(-v)))(x)\big) \to (A)(x)\big)\\
&\wedge\big((\bigvee_{a\in F,\tau(p)\in\tau_M^{\rightarrow}(A)}\ \bigvee_{\tau(x)=\tau(p)}((a\circ u))(x)) \to (A)(x)\big)\Big)\\
&\geq \bigwedge_{p\in V}\Big(\big((\bigvee_{u,-v\in A}(u\oplus(-v))(p)) \to (A)(p)\big)\\
&\wedge\big((\bigvee_{a\in F,u\in A}(a\odot u)(p)) \to (A)(p)\big)\Big)\\
&= \mathscr{S}_V(A).
\end{aligned}$$

This implies that $\tau : (V, \mathscr{S}_V) \longrightarrow (W, \mathscr{S}_W)$ is an M-CC mapping. □

6. Conclusions

Liu and Shi [11] introduced M-hazy groups by using the M-hazy binary operation. Mehmood et al. [12,13] extended this idea by defining M-hazy rings and obtained its induced fuzzifying convexities. By getting the motivations of these new proposed concepts through M-hazy operations, we proposed a new generalization of vector spaces over field based on M-hazy binary operation, which is denoted as M-hazy vector spaces over M-hazy field. In addition, by using the completely residuated lattice valed logic, some important properties of M-hazy field, M-hazy vector space, and M-hazy subspace are introduced. Based on these properties, it is shown that an M-hazy subspace of M-hazy vector space forms a convex structure. In addition, the linear transformation of M-hazy vector space is defined and proves its important results. The convexity of the M fuzzy set on the set P is the M fuzzy set on the power set with certain properties. Therefore, to a certain extent, each subset of P can be regarded as a convex set. Finally, considering the importance of this fact, a method is given that uses the M-hazy subspace of M-hazy vector space to induce the M-fuzzifying convex space.

The possible directions for the future work could be M-hazy modules, bases, dimensions of M-hazy vector spaces, M-hazy topological vector spaces, and other fuzzy algebra.

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