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Nonstationary Radiative–Conductive Heat Transfer Problem in a Semitransparent Body with Absolutely Black Inclusions

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Abstract: The paper is devoted to a nonstationary initial–boundary value problem governing complex heat exchange in a convex semitransparent body containing several absolutely black inclusions. The existence and uniqueness of a weak solution to this problem are proven herein. In addition, the stability of solutions with respect to data, a comparison theorem and the results of improving the properties of solutions with an increase in the summability of the data were established. All results are global in terms of time and data.

Keywords: radiative–conductive heat transfer problem; radiative transfer equation; nonlinear initial–boundary value problem; stability of solutions with respect to data; comparison theorem

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1. Introduction

Complex heat transfer problems, in which it is necessary to simultaneously take into account the transfer of energy by thermal radiation and thermal conductivity, arise in various fields of science and industry. The discussion on the properties of complex heat transfer problems and the methods for solving them constitutes an extensive physical literature (see, for example, [1–4]).

Mathematical problems of radiative–conductive heat transfer are nonstandard, interesting and rather complicated. Heat radiation is nonlinearly dependent on temperature, and integro-differential equations or nonlocal boundary conditions are used to describe radiation heat transfer. Various nonlinear nonlocal boundary and initial–boundary value problems arise in this field.

The first mathematical results in this direction were obtained by A.N. Tikhonov [5,6] in the late 1930s. The construction of the mathematical theory of radiative–conductive heat transfer problems was continued for roughly forty years [7–14]. In the early 1990s, many mathematicians were paying attention to such problems. As a result, over the past 30 years, a large number of papers have been devoted to the solvability of complex heat transfer problems (cf. [15–63]). Naturally, the above list is not exhaustive.

To date, the solvability of various statements of complex (radiative–conductive) heat exchange problems in systems consisting either only of radiation-opaque bodies or only of radiation-semitransparent bodies has been studied in sufficient detail. At the same time, the problems of radiative–conductive heat exchange in systems consisting of both radiation-opaque and of radiation-semitransparent bodies remain to date unexplored. This specific area of study, to the best of the author's knowledge, has only been the subject of the following articles: [64–66].

In this paper, the existence and uniqueness of a weak solution to a nonstationary boundary value problem governing radiative–conductive heat transfer in a semitransparent body containing several absolutely black inclusions were proven. All results are global in terms of time and data. The unknown functions u and I physically represent the absolute temperature and radiation intensity. The problem was considered in a gray approximation.

The technique used was developed in [41,53,57]. In the stationary version, this problem was studied in [65].

The paper is organized as follows. In Section 2, the physical sense of the problem is explained. Section 3 is devoted to notations. In Section 4, the boundary value problem for the radiative transfer equation is considered. Section 5 contains the formulation of the main results of the paper. In addition, in this section, the problem is reduced to the equivalent initial–boundary value problem for a nonlinear operator-differential equation (Problem $\mathcal{P}$) with only one unknown function u. In Section 6, a number of important auxiliary assertions are proven. In Section 7, an auxiliary problem $\mathcal{P}^{[n]}$ is introduced and its solvabilty is proven. In Section 8, a priori estimates for weak solutions to Problems $\mathcal{P}$ and $\mathcal{P}^{[n]}$ are established. Section 9 establishes the stability of weak solutions to Problem $\mathcal{P}$ with respect to the data. A comparison theorem, which, in particular, implies the uniqueness of a weak solution, is also proven. Section 10 contains the proof of the existence of a weak solution to the problem $\mathcal{P}$. Finally, Section 11 establishes the validity of the main results of the article.

2. Physical Statement the Problem

Let $\widehat{G}$ be a bounded convex domain in $\mathbb{R}^3$ and $\{G_{b,j}\}_{j=1}^m$ be a system of strictly internal subdomains of the domain $\widehat{G}$. Assume that $\overline{G}_{b,i} \cap \overline{G}_{b,j} = \emptyset$ for all $i \neq j$ and $G_s = \widehat{G} \setminus \bigcup\limits_{j=1}^{m} \overline{G}_{b,j}$ is a domain. We put $G_b = \bigcup\limits_{j=1}^{m} G_{b,j}$ and $G = G_s \cup G_b$.

We assume that each of the domains $G_{b,j}$, $1 \leq j \leq m$ is an absolutely black body and the domain G_s is occupied by a semitransparent optically homogeneous material with a constant absorption coefficient $\varkappa > 0$ and a scattering coefficient $s \geq 0$.

The unknown functions $u(x,t)$ and $I(\omega,x,t)$ physically represent the absolute temperature at point $x \in G$ at moment $t \in (0,T)$ and the intensity of the radiation propagating at point $x \in G_s$ in direction $\omega \in \Omega = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$, respectively. The function u is defined on the set $Q_T = G \times (0,T)$. Its restrictions to the set $Q_{s,T} = G_s \times (0,T)$ and to the set $Q_{b,T} = G_b \times (0,T)$ will be denoted by u_s and u_b, respectively. The function I is defined on the set $D_s \times (0,T)$, where $D_s = \Omega \times G_s$.

To describe the nonstationary process of radiative–conductive heat transfer, a system consisting of two heat equations and radiative transfer equation is used:

$$c_p \frac{\partial u_s}{\partial t} - \operatorname{div}(\lambda(x,u_s)\nabla u_s) + 4\varkappa h(u_s) = \varkappa \int\limits_{\Omega} I \, d\omega + f, \quad (x,t) \in Q_{s,T}, \tag{1}$$

$$c_p \frac{\partial u_b}{\partial t} - \operatorname{div}(\lambda(x,u_b)\nabla u_b) = f, \quad (x,t) \in Q_{b,T}, \tag{2}$$

$$\omega \cdot \nabla I + (\varkappa + s)I = s\mathcal{S}(I) + \frac{\varkappa}{\pi} h(u_s), \quad (\omega,x,t) \in D_s \times (0,T). \tag{3}$$

Here, c_p is the heat capacity coefficient, $\lambda(x,u)$ is the thermal conductivity coefficient, and f is the density of heat sources. The function $h(u) = \sigma_0 |u|^3 u$ for $u > 0$ corresponds to the hemispherical radiation density of an absolutely black body according to the Stefan–Boltzmann law, where $\sigma_0 > 0$ is the Stefan–Boltzmann constant.

Equation (1) describes the heat transfer process in the gray semitransparent medium G_s. The terms $4\varkappa\, h(u_s)$ and $\varkappa \int\limits_{\Omega} I \, d\omega$ in it correspond to the densities of the energies emitted and absorbed in G_s, respectively. Equation (2) describes the heat transfer process in opaque inclusions G_s. Equation (3) describes the transfer of radiation in a radiating, absorbing and scattering medium G_s. The term $\omega \cdot \nabla I = \sum\limits_{i=1}^{3} \omega_i \frac{\partial}{\partial x_i} I$ in (3) denotes the derivative of I along the direction ω. We denote by $\mathcal{S}$ the scattering operator:

$$\mathcal{S}(I)(\omega,x,t) = \frac{1}{4\pi}\int\limits_{\Omega}\theta(\omega'\cdot\omega)I(\omega',x,t)\,d\omega', \quad (\omega,x,t)\in D_s\times(0,T)$$

with the scattering indicatrix possessing the following properties:

$$\theta\in L^1(-1,1), \quad \theta\geq 0, \quad \frac{1}{2}\int\limits_{-1}^{1}\theta(\mu)\,d\mu = 1.$$

We regard $\mathbb{R}^3$ as the Euclidean space of elements $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ equipped with the inner product $x\cdot y = \sum\limits_{i=1}^{3} x_i y_i$. Assume that the domain G_s is Lipschitz. Thus, the domains $G_{b,j}$, $1\leq j\leq m$ are also Lipschitz. We denote by $d\omega$ and $d\sigma(x)$ the measures induced by Lebesgue measure in $\mathbb{R}^3$ on Ω and ∂G_s, respectively.

We also assume that the boundary ∂G_s is piecewise smooth in the following sense. There exists a closed subset $\mathcal{G}\subset\partial G_s$ such that meas $(\mathcal{G};d\sigma) = 0$; moreover, for each point $x\in\partial' G_s = \partial G_s\setminus\mathcal{G}$, there exists a neighborhood of it, in which the boundary ∂G_s is continuously differentiable.

Note that $\partial G_s = \partial\widehat{G}\cup\partial G_b$, where $\partial\widehat{G}\cap\partial G_b = \emptyset$. We put $\partial'\widehat{G} = \partial\widehat{G}\setminus\mathcal{G}$, $\partial' G_b = \partial G_b\setminus\mathcal{G}$. It is clear that the outward normal n to the boundary ∂G_s is defined and continuous on $\partial' G_s$ and the outward normal to the boundary of the set G_b coincides with $-n(x)$ for $x\in\partial' G_b$.

We introduce the sets:

$$\Gamma = \Omega\times\partial' G_s,\ \Gamma^- = \{(\omega,x)\in\Gamma \mid \omega\cdot n(x) < 0\},\ \Gamma^+ = \{(\omega,x)\in\Gamma \mid \omega\cdot n(x) > 0\},$$
$$\widehat{\Gamma}^- = \{(\omega,x)\in\Gamma^- \mid x\in\partial'\widehat{G}\}, \quad \Gamma_b^- = \{(\omega,x)\in\Gamma^- \mid x\in\partial' G_b\},$$
$$\Omega^-(x) = \{\omega\in\Omega \mid \omega\cdot n(x) < 0\}, \quad \Omega^+(x) = \{\omega\in\Omega \mid \omega\cdot n(x) > 0\}, \quad x\in\partial' G_b.$$

Denote by $I|_{\Gamma^\pm}$ the values (traces) of the function I on $\Gamma^\pm$, where $I|_{\Gamma^-}$ and $I|_{\Gamma^+}$ are interpreted as the values of the intensity of radiations entering into G_s and coming out of G_s.

Endow the system (1)–(3) with the boundary conditions:

$$\lambda(x,u_s)\frac{\partial u_s}{\partial n} = 0, \quad (x,t)\in\partial\widehat{Q}_T, \tag{4}$$
$$\lambda(x,u_s)\frac{\partial u_s}{\partial n} + \gamma(tr\,u_s - tr\,u_b) = 0, \quad (x,t)\in\partial Q_{b,T}, \tag{5}$$
$$-\lambda(x,u_b)\frac{\partial u_b}{\partial n} + h(tr\,u_b) + \gamma(tr\,u_b - tr\,u_s) = \mathcal{M}^+(I|_{\Gamma^+}), \quad (x,t)\in\partial Q_{b,T}, \tag{6}$$
$$I|_{\Gamma^-} = \frac{1}{\pi}h(tr\,u_b), \quad (\omega,x,t)\in\Gamma_b^-\times(0,T), \tag{7}$$
$$I|_{\Gamma^-} = J_*, \quad (\omega,x,t)\in\widehat{\Gamma}^-\times(0,T) \tag{8}$$

and the initial condition:

$$u|_{t=0} = u^0, \quad x\in G. \tag{9}$$

Here, $\partial\widehat{Q}_T = \partial\widehat{G}\times(0,T)$ and $\partial Q_{b,T} = \partial G_b\times(0,T)$ are the lateral surfaces of cylinders $\widehat{Q}_T = \widehat{G}\times(0,T)$ and $Q_{b,T} = G_b\times(0,T)$. By tru_s and tru_b , we denote the values (traces) of u_s and u_b on $\partial Q_{b,T}$.

It is assumed that the body $\widehat{G}$ is surrounded in a vacuum. Therefore, on the boundary of the body, the boundary condition (2) is set, which means the absence of heat flux. On the boundary ∂G_b, separating the semitransparent material G_s and absolutely black inclusions G_b, we set two boundary conditions. They account for incoming and outgoing energy flows using a heat transfer mechanism. In addition, it is taken into account that absolutely black inclusions emit energy and absorb the incident radiation on them. Here, γ is the heat transfer coefficient and:

$$\mathcal{M}^+(I|_{\Gamma^+})(x,t) = \int\limits_{\Omega^+(x)} I|_{\Gamma^+}(\omega,x,t)\,\omega\cdot n(x)\,d\omega$$

represents the flux of radiation coming out of G_s and absorbed at ∂G_b. The condition (7) means that on the boundary ∂G_b, the intensity of radiation entering into G_s is equal to the intensity of radiation leaving the set G_b. In (8), J_* denotes the intensity of external radiation incident on $\partial\widehat{G}$.

3. Function Spaces

Throughout the paper, $1 \le p \le \infty$ and p' denotes the conjugate exponent of p, i.e., $1 \le p' \le \infty$ and $1/p + 1/p' = 1$.

Let u be a real number or a real-valued function. We put $u^{[-N,M]} = \max\{-N, \min\{u, M\}\}$, $u^{[M]} = u^{[-M,M]}$, where $-N \le 0 < M$. We also put $[u]_+ = \max\{u, 0\}$ and $[u]_- = \max\{-u, 0\}$.

Let S be a set where the measure $d\mu$ is given. We denote by $L^p(S; d\mu)$ the Lebesgue space of functions f defined on Z that are measurable with respect to the measure $d\mu$ and have the finite norm:

$$\|f\|_{L^p(S;d\mu)} = \begin{cases} \Big(\int\limits_S |f(s)|^p\,d\mu(s)\Big)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}\limits_{s\in S} |f(s)|, & p = \infty. \end{cases}$$

3.1. Spaces of Functions on G, G_s and ∂G_b

We set:

$$L^p(G) = L^p(G; dx), \quad L^p(G_s) = L^p(G_s, dx), \quad L^p(\partial G_b) = L^p(\partial G_b; d\sigma(x)).$$

We introduce the space $L^p(G_s, \partial G_b) = \{(F, J) \in L^p(G_s) \times L^p(\partial G_b)\}$ equipped with the norm:

$$\|(F,J)\|_{L^p(G_s,\partial G_b)} = \begin{cases} \Big(4\pi\varkappa\|F\|^p_{L^p(G_s)} + \pi\|J\|^p_{L^p(\partial G_b)}\Big)^{1/p}, & 1 \le p < \infty, \\ \max\{\|F\|_{L^\infty(G_s)}, \|J\|_{L^\infty(\partial G_b)}\}, & p = \infty. \end{cases}$$

Let functions f, g defined on G or G_s are such that $fg \in L^1(G)$ or $fg \in L^1(G_s)$. In these cases, we use the notations:

$$(f,g)_G = \int\limits_G f(x)g(x)\,dx, \quad (f,g)_{G_s} = \int\limits_{G_s} f(x)g(x)\,dx.$$

Let functions f, g defined on ∂G_b are such that $fg \in L^1(\partial G_b)$. In this case, we use the notation:

$$(f,g)_{\partial G_b} = \int\limits_{\partial G_b} f(x)g(x)\,d\sigma(x).$$

Let u be a function defined on G. We denote by u_s, u_b and $u_{b,j}$, the restrictions of u to G_s, G_b and $G_{b,j}$, $1 \le j \le m$, respectively.

By $W^{1,2}(G)$, we understand the space:

$$W^{1,2}(G) = \{u \in L^2(G) \mid u_s \in W^{1,2}(G_s),\ u_{b,j} \in W^{1,2}(G_{b,j}),\ 1 \le j \le m\}$$

(where $W^{1,2}(G_s)$ and $W^{1,2}(G_{b,j})$ are the classical Sobolev spaces) equipped with the norm:

$$\|u\|_{W^{1,2}(G)} = \Big(\|u_s\|^2_{W^{1,2}(G_s)} + \sum_{j=1}^{m} \|u_{b,j}\|^2_{W^{1,2}(G_{b,j})}\Big)^{1/2}$$

If $u \in W^{1,2}(G)$, then by $tr\, u_s$ and tru_b we denote the traces of the restrictions u_s and u_b on ∂G_b.

We remind the important multiplicative inequalities:

$$\|u\|_{L^p(G)} \leq C_{1,p}(G)\|u\|^{3/2-3/p}_{W^{1,2}(G)}\|u\|^{3/p-1/2}_{L^2(G)} \quad \forall\, u \in W^{1,2}(G), \tag{10}$$

$$\|tr\, u_s\|_{L^q(\partial G_b)} \leq C_{2,q}(G_s)\|u_s\|^{3/2-2/q}_{W^{1,2}(G_s)}\|u_s\|^{2/q-1/2}_{L^2(G_s)} \quad \forall\, u_s \in W^{1,2}(G_s), \tag{11}$$

$$\|tr\, u_b\|_{L^q(\partial G_b)} \leq C_{2,q}(G_b)\|u_b\|^{3/2-2/q}_{W^{1,2}(G_b)}\|u_b\|^{2/q-1/2}_{L^2(G_b)} \quad \forall\, u_b \in W^{1,2}(G_b), \tag{12}$$

which hold for all $p \in [2,6]$, $q \in [2,4]$.

3.2. Spaces of Functions on Q_T and $\partial Q_{b,T}$

We set:

$$L^p(Q_T) = L^p(Q_T; dxdt),\ L^p(Q_{s,T}) = L^p(Q_{s,T}; dxdt),\ L^p(\partial Q_{b,T}) = L^p(\partial Q_{b,T}; d\sigma(x)dt).$$

Note that $L^p(Q_T) = L^p(0,T;L^p(G))$ and $L^p(\partial Q_{b,T}) = L^p(0,T;L^p(\partial G_b))$ for $1 \leq p < \infty$.

We introduce the space $V_2(Q_T) = L^\infty(0,T;L^2(G)) \cap L^2(0,T;W^{1,2}(G))$ equipped with the norm:

$$\|u\|_{V_2(Q_T)} = \|u\|_{L^\infty(0,T;L^2(G))} + \|\nabla u\|_{L^2(Q_T)}$$

and the space $V_2^{1,0}(Q_T) = V_2(Q_T) \cap C([0,T];L^2(G))$.

The inequalities (10)–(12) imply the estimates:

$$\|u\|_{L^{\bar r_1}(0,T;L^{\bar q_1}(G))} \leq C_{1,\bar r_1,\bar q_1}(G,T)\,\|u\|_{V_2(Q_T)} \quad \forall\, u \in V_2(Q_T), \tag{13}$$

$$\|tr\, u_s\|_{L^{\bar r_2}(0,T;L^{\bar q_2}(\partial G_b))} \leq C_{2,\bar r_2,\bar q_2}(G,T)\,\|u\|_{V_2(Q_T)} \quad \forall\, u \in V_2(Q_T), \tag{14}$$

$$\|tr\, u_b\|_{L^{\bar r_2}(0,T;L^{\bar q_2}(\partial G_b))} \leq C_{2,\bar r_2,\bar q_2}(G,T)\,\|u\|_{V_2(Q_T)} \quad \forall\, u \in V_2(Q_T), \tag{15}$$

which hold for all exponents $\bar r_1$, $\bar q_1$, $\bar r_2$, $\bar q_2$, such that:

$$\bar r_1 \in [1,\infty],\ \ \bar q_1 \in [1,6],\ \ \bar r_2 \in [1,4],\ \ \bar q_2 \in [1,4],\quad \frac{2}{\bar r_1} + \frac{3}{\bar q_1} \geq \frac{3}{2},\quad \frac{2}{\bar r_2} + \frac{2}{\bar q_2} \geq \frac{3}{2}. \tag{16}$$

From the estimates (13), (15), it follows that if $u, |u|^{1/2}u \in V_2(Q_T)$, then $u \in L^5(Q_T)$, $tr\, u_b \in L^4(\partial Q_{b,T})$ and the following estimates hold:

$$\|u\|_{L^5(Q_T)} \leq C_3(G,T)\||u|^{1/2}u\|^{2/3}_{V_2(Q_T)}, \tag{17}$$

$$\|tr\, u_b\|_{L^4(\partial Q_{b,T})} \leq C_4(G,T)\||u|^{1/2}u\|^{2/3}_{V_2(Q_T)}. \tag{18}$$

We also draw attention to the following multiplicative inequality, which follows from (11), (12):

$$\|tr\, u_s\|_{L^2(\partial Q_{b,T})} + \|tr\, u_b\|_{L^2(\partial Q_{b,T})} \leq C_5(G,T)\|u\|^{1/2}_{L^2(0,T;W^{1,2}(G))}\|u\|^{1/2}_{L^2(Q_T)}. \tag{19}$$

This inequality, in particular, implies that if $u \in V_2(Q_T)$, the sequence $\{u^k\}_{k=1}^\infty \subset V_2(Q_T)$ is bounded in $V_2(Q_T)$ and $u^k \to u$ in $L^2(Q_T)$ as $k \to \infty$, then $tr\, u_s^k \to tr\, u_s$, $tr\, u_b^k \to tr\, u_b$ in $L^2(\partial Q_{b,T})$.

3.3. Spaces of Functions on D_s and Γ

Remind that:

$$
\begin{aligned}
&D_s = \Omega \times G_s, \quad \Gamma = \Omega \times \partial' G_s, \\
&\Gamma^- = \{(\omega, x) \in \Gamma \mid \omega \cdot n(x) < 0\}, \quad \Gamma^+ = \{(\omega, x) \in \Gamma \mid \omega \cdot n(x) > 0\}, \\
&\widehat{\Gamma}^\pm = \{(\omega, x) \in \Gamma^\pm \mid x \in \partial' \widehat{G}\}, \quad \Gamma_b^\pm = \{(\omega, x) \in \Gamma^\pm \mid x \in \partial' G_b\}.
\end{aligned}
$$

We set: $L^p(D_s) = L^p(D_s; d\omega dx)$. We introduce the following measures on Γ and $\Gamma^\pm$:

$$
\begin{aligned}
&d\Gamma(\omega, x) = d\omega d\sigma(x), \quad (\omega, x) \in \Gamma, \\
&\widehat{d}\Gamma^\pm(\omega, x) = |\omega \cdot n(x)|\, d\omega d\sigma(x), \quad (\omega, x) \in \Gamma^\pm.
\end{aligned}
$$

We set:

$$
\widehat{L}^p(\Gamma^\pm) = L^p(\Gamma^\pm; \widehat{d}\Gamma^\pm),\ \widehat{L}^p(\widehat{\Gamma}^\pm) = L^p(\widehat{\Gamma}^\pm; \widehat{d}\Gamma^\pm),\ \widehat{L}^p(\widehat{\Gamma}_b^\pm) = L^p(\widehat{\Gamma}_b^\pm; \widehat{d}\Gamma^\pm),\ 1 \le p \le \infty.
$$

Note that $\widehat{L}^\infty(\Gamma^\pm) = L^\infty(\Gamma^\pm)$, $\widehat{L}^\infty(\widehat{\Gamma}^\pm) = L^\infty(\widehat{\Gamma}^\pm)$, $\widehat{L}^\infty(\Gamma_b^\pm) = L^\infty(\Gamma_b^\pm)$.

By the weak derivative in direction ω of a function $f \in L^1(D_s)$, we understand a function $z \in L^1(D_s)$, denoted by $z = \omega \cdot \nabla f$ and satisfying the integral identity:

$$
\int_{D_s} [f(\omega, x)\, \omega \cdot \nabla \varphi(x) + z(\omega, x)\varphi(x)]\, \psi(\omega)\, d\omega dx = 0 \ \ \forall\, \varphi \in C_0^\infty(G_s), \ \ \forall\, \psi \in L^\infty(\Omega).
$$

We denote by $\mathcal{W}^p(D_s)$ the Banach space of functions $f \in L^p(D_s)$ possessing the weak derivative $\omega \cdot \nabla f \in L^p(D_s)$ and equipped with the norm:

$$
\|f\|_{\mathcal{W}^p(D_s)} = \begin{cases} \left(\|f\|^p_{L^p(D_s)} + \|\,\omega \cdot \nabla f\|^p_{L^p(D_s)}\right)^{1/p}, & 1 \le p < \infty, \\ \max\{\|f\|_{L^\infty(D_s)}, \|\,\omega \cdot \nabla f\|_{L^\infty(D_s)}\}, & p = \infty. \end{cases}
$$

We will denote by $f|_{\Gamma^-}$ and $f|_{\Gamma^+}$ the traces of the function $f \in \mathcal{W}^p(D_s)$ on Γ^- and Γ^+, respectively. It is known that $f|_{\Gamma^\pm} \in L^p_{loc}(\Gamma^\pm)$. Moreover, if $f \in \mathcal{W}^p(D_s)$ and $f|_{\Gamma^-} \in \widehat{L}^p(\Gamma^-)$, then $f|_{\Gamma^+} \in \widehat{L}^p(\Gamma^+)$.

We refer to [67–69] for more detailed information about the properties of functions $f \in \mathcal{W}^p(D_s)$ and their traces $f|_{\Gamma^\pm}$.

4. Boundary Value Problem for Radiative Transfer Equation

For almost all $t \in (0, T)$, the unknown function $I(t)$, involved in the problem (1)–(9), is a solution to the following subproblem:

$$
\omega \cdot \nabla I + (\varkappa + s)I = s\mathcal{S}(I) + \varkappa F, \quad (\omega, x) \in D_s, \tag{20}
$$

$$
I|_{\Gamma^-} = J_b, \quad (\omega, x) \in \Gamma_b^-, \tag{21}
$$

$$
I|_{\Gamma^-} = J_*, \quad (\omega, x) \in \widehat{\Gamma}^-. \tag{22}
$$

where $F = \dfrac{1}{\pi} h(tr\, u_s(t))$, $J_b = \dfrac{1}{\pi} h(tr\, u_b(t))$.

We formulate some results on the properties of the problem (20)–(22) which follow from [67,69].

Let $(F, J_b, J_*) \in L^1(G_s, \partial G_b) \times \widehat{L}^1(\widehat{\Gamma}^-)$. By a solution to the problem (20)–(22), we mean a function $I \in \mathcal{W}^1(D_s)$ that satisfies Equation (20) almost everywhere on D_s, condition (21) almost everywhere on Γ_b^-, and condition (22) almost everywhere on $\widehat{\Gamma}^-$.

Theorem 1. *If $(F, J_b, J_*) \in L^p(G_s, \partial G_b) \times \widehat{L}^p(\widehat{\Gamma}^-)$, where $1 \le p \le \infty$, then a solution to the problem (20)–(22) exists and is unique. Moreover, $I \in \mathcal{W}^p(D_s)$ and for $1 \le p < \infty$ the following estimates hold:*

$$\varkappa \|I\|^p_{L^p(D_s)} + \|I|_{\Gamma^+}\|^p_{\widehat{L}^p(\Gamma^+)} \le \|(F, J_b)\|^p_{L^p(G_s, \partial G_b)} + \|J_*\|^p_{\widehat{L}^p(\widehat{\Gamma}^-)}, \tag{23}$$

$$\varkappa \|\,\omega \cdot \nabla I\|^p_{L^p(D_s)} \le 2^p(\varkappa + s)^p \left(\|(F, J_b)\|^p_{L^p(G_s, \partial G_b)} + \|J_*\|^p_{\widehat{L}^p(\widehat{\Gamma}^-)} \right) \tag{24}$$

and for $p = \infty$, the following estimates hold:

$$\|I\|_{L^\infty(D_s)} \le \max\Big\{ \|(F, J_b)\|_{L^\infty(G_s, \partial G_b)}, \|J_*\|_{L^\infty(\widehat{\Gamma}^-)} \Big\}, \tag{25}$$

$$\|\,\omega \cdot \nabla I\|_{L^p(D_s)} \le 2(\varkappa + s) \max\Big\{ \|(F, J_b)\|_{L^\infty(G_s, \partial G_b)}, \|J_*\|_{L^\infty(\widehat{\Gamma}^-)} \Big\}. \tag{26}$$

In addition:

$$\varkappa \|[I]_\pm\|_{L^1(D_s)} + \|[I]_\pm|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} \le \|([F]_\pm, [J_b]_\pm)\|_{L^1(G_s, \partial G_b)} + \|[J_*]_\pm\|_{\widehat{L}^1(\widehat{\Gamma}^-)} \tag{27}$$

and as a consequence, if $F \ge 0$, $J_b \ge 0$, $J_* \ge 0$, then $I \ge 0$.

We denote by $\mathcal{A}$ the resolving operator for the problem:

$$\begin{aligned}
&\omega \cdot \nabla I_s + (\varkappa + s) I_s = s\mathcal{S}(I_s) + \varkappa F, \quad (\omega, x) \in D_s, \\
&I_s|_{\Gamma^-} = J_b, \quad (\omega, x) \in \Gamma_b^-, \\
&I_s|_{\Gamma^-} = 0, \quad (\omega, x) \in \widehat{\Gamma}^-.
\end{aligned}$$

It with a pair $(F, J_b) \in L^p(G_s, \partial G_b)$, $1 \le p \le \infty$ associates the solution $I_s = \mathcal{A}[F, J_b] \in \mathcal{W}^p(D_s)$. This operator is linear and continuous.

We denote by $\widehat{\mathcal{A}}$ the resolving operator for the problem:

$$\begin{aligned}
&\omega \cdot \nabla I_* + (\varkappa + s) I_* = s\mathcal{S}(I_*), \quad (\omega, x) \in D_s, \\
&I_*|_{\Gamma^-} = 0, \quad (\omega, x) \in \Gamma_b^-, \\
&I_*|_{\Gamma^-} = J_*, \quad (\omega, x) \in \widehat{\Gamma}^-.
\end{aligned}$$

It with a function $J_* \in \widehat{L}^p(\widehat{\Gamma}^-)$, $1 \le p \le \infty$ associates the solution $I_* = \widehat{\mathcal{A}}[J_*] \in \mathcal{W}^p(D_s)$. This operator is also linear and continuous.

We introduce the operators $\langle \mathcal{A} \rangle_\Omega : L^p(G_s, \partial G_b) \to L^p(G_s)$, $\langle \widehat{\mathcal{A}} \rangle_\Omega : \widehat{L}^p(\widehat{\Gamma}^-) \to L^p(G_s)$ and $\mathcal{B} : L^p(G_s, \partial G_b) \to L^p(\partial G_b)$, $\widehat{\mathcal{B}} : \widehat{L}^p(\widehat{\Gamma}^-) \to L^p(\partial G_b)$ by the formulas:

$$\langle \mathcal{A} \rangle_\Omega [F, J_b](x) = \frac{1}{4\pi} \int\limits_\Omega \mathcal{A}[F, J_b](\omega, x)\, d\omega, \quad \langle \widehat{\mathcal{A}} \rangle_\Omega [J_*](x) = \frac{1}{4\pi} \int\limits_\Omega \widehat{\mathcal{A}}[J_*](\omega, x)\, d\omega;$$

$$\mathcal{B}[F, J_b](x) = \frac{1}{\pi} \mathcal{M}^+(\mathcal{A}[F, J_b]|_{\Gamma^+})(x), \quad \widehat{\mathcal{B}}[J_*](x) = \frac{1}{\pi} \mathcal{M}^+(\widehat{\mathcal{A}}[J_*]|_{\Gamma^+})(x),$$

where:

$$\mathcal{M}^+(\psi)(x) = \int\limits_{\Omega^+(x)} \psi(\omega, x)\, \omega \cdot n(x)\, d\omega, \quad x \in \partial G_b, \quad \psi \in \widehat{L}^1(\Gamma_b^+).$$

These operators are linear and continuous. Their continuity follows from the estimates (23), (25) and the estimate:

$$\|\mathcal{M}^+(\psi)\|_{L^p(\partial G_b)} \le \pi^{1/p'} \|\psi\|_{\widehat{L}^p(\Gamma_b^+)} \quad \forall\, \psi \in \widehat{L}^p(\Gamma_b^+).$$

We introduce the characteristic functions 1_{D_s}, 1_{G_s}, $1_{\partial G_b}$ and $\widehat{1}^-$ of sets D_s, G_s, ∂G_b and $\widehat{\Gamma}^-$. Note that $I = 1_{D_s}$ is the solution to the problem (20)–(22) with $(F, J_b, \widehat{J}) = (1_{G_s}, 1_{\partial G_b}, \widehat{1}^-)$. Consequently, $\mathcal{A}[1_{G_s}, 1_{\partial G_b}] + \widehat{\mathcal{A}}[\widehat{1}^-] = 1_{D_s}$ and:

$$\langle \mathcal{A} \rangle_\Omega [1_{G_s}, 1_{\partial G_j}] + \widehat{\alpha} = 1_{G_s}, \quad \mathcal{B}[1_{G_s}, 1_{\partial G_b}] + \widehat{\beta} = 1_{\partial G_b}, \tag{28}$$

where $\widehat{\alpha} = \langle \widehat{\mathcal{A}} \rangle_\Omega [\widehat{1}^-]$ for $x \in G_s$ and $\widehat{\beta} = \widehat{\mathcal{B}}[\widehat{1}^-]$ for $x \in \partial G_b$. Since $\widehat{\mathcal{A}}[\widehat{1}^-] \geq 0$ then $\widehat{\alpha} \geq 0$, $\widehat{\beta} \geq 0$.

It follows from (25) that:

$$\| \langle \mathcal{A} \rangle_\Omega \|_{L^\infty(G_s, \partial G_b) \to L^\infty(G_s)} \leq 1, \quad \| \mathcal{B} \|_{L^\infty(G_s, \partial G_b) \to L^\infty(\partial G_b)} \leq 1. \tag{29}$$

We also introduce the operators $\mathcal{C} : L^p(G_s, \partial G_b) \to L^p(G_s)$, $\mathcal{D} : L^p(G_s, \partial G_b) \to L^p(\partial G_b)$ by the formulas:

$$\mathcal{C}[F, J] = F - \langle \mathcal{A} \rangle_\Omega [F, J], \quad \mathcal{D}[F, J] = J - \mathcal{B}[F, J].$$

It follows from (29) that:

$$\| \mathcal{C} \|_{L^\infty(G_s, \partial G_b) \to L^\infty(G_s)} \leq 2, \quad \| \mathcal{D} \|_{L^\infty(G_s, \partial G_b) \to L^\infty(\partial G_b)} \leq 2. \tag{30}$$

We draw attention to the following equality proven in: [65]

$$4\varkappa(\mathcal{C}[F, J], F^*)_{G_s} + (\mathcal{D}[F, J], J^*)_{\partial G_b} = 4\varkappa(F, \mathcal{C}[F^*, J^*])_{G_s} + (J, \mathcal{D}[F^*, J^*])_{\partial G_b}. \tag{31}$$

It holds for all $(F, J) \in L^p(G_s, \partial G_b)$ $(F^*, J^*) \in L^{p'}(G_s, \partial G_b)$, $1 \leq p \leq \infty$.

5. Main Results: Reducing the Problem (1)–(9) to Problem $\mathcal{P}$

5.1. Formulation of the Main Results

In what follows, it is assumed that the following conditions on the data are satisfied.

(A_1) The function $\lambda(x, u)$ is defined on $G \times \mathbb{R}$, and for any $u \in \mathbb{R}$, it is measurable with respect to x. Furthermore:

$$0 < \lambda_{\min} \leq \lambda(x, u) \leq \lambda_{\max} \quad \forall (x, u) \in G \times \mathbb{R}, \tag{32}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are constants.

In addition, the following holder condition holds:

$$|\lambda(x, u + v) - \lambda(x, u)| \leq L|v|^{1/2} \quad \forall (x, u) \in G \times \mathbb{R}, \quad \forall v \in [-1, 1], \tag{33}$$

where L is a constant.

(A_2) $c_p \in L^\infty(G)$, $0 < \underline{c}_p \leq c_p(x) \leq \overline{c}_p$ for $x \in G$, where $\underline{c}_p$ and $\overline{c}_p$ are constants.

(A_3) $\gamma \in L^\infty(\partial G_b)$, $\gamma \geq 0$.

(A_4) $u^0 \in L^3(G)$, $f \in L^r(0, T; L^q(G))$, $J_* \in L^{r_*}(0, T; \widehat{L}^{q_*}(\widehat{\Gamma}^-))$, where:

$$r \in [1, \infty], \ q \in [9/7, \infty], \ r_* \in [3/2, \infty], \ q_* \in [3/2, \infty], \quad \frac{2}{r} + \frac{3}{q} \leq 3, \quad \frac{2}{r_*} + \frac{2}{q_*} \leq 2.$$

We introduce the spaces:

$$\mathcal{V}(Q_T) = \{u \in V_2^{1,0}(Q_T) \cap L^4(Q_{s,T}) \mid tr\, u_b \in L^4(\partial Q_{b,T})\},$$
$$V = W^{1,2}(G) \cap L^\infty(G), \quad C_*^\infty[0, T] = \{\eta \in C^\infty[0, T] \mid \eta(T) = 0\}.$$

By a weak solution to the problem (1)–(9), we mean a pair of functions $(u, I) \in \mathcal{V}(Q_T) \times L^1(0, T; \mathcal{W}^1(D_s))$ such that:

(1) The following identity holds:

$$-\int_0^T (c_p u(t), v)_G \frac{d\eta}{dt}(t)\,dt + \int_0^T a(u(t), v)\,\eta(t)\,dt + \int_0^T b(u(t), I(t), v)\,\eta(t)\,dt$$
$$= (c_p u^0, v)_G \cdot \eta(0) + \int_0^T (f(t), v)_G\,\eta(t)\,dt\,dt \quad \forall v \in V, \quad \forall \eta \in C_*^\infty[0,T], \tag{34}$$

where:

$$\begin{aligned}
a(u,v) &= a_0(u,v) + a_1(u,v),\\
a_0(u,v) &= (\lambda(\cdot,u)\nabla u, \nabla v)_G = \int_G \lambda(x,u)\nabla u \cdot \nabla v\,dx,\\
a_1(u,v) &= (\gamma[tr\,u_s - tr\,u_b], tr\,v_s - tr\,v_b)_{\partial G_b},\\
b(u,I,v) &= \Big(4\varkappa h(u_s) - \varkappa \int_\Omega I\,d\omega, v_s\Big)_{G_s} + \Big(h(tr\,u_b) - \mathcal{M}^+(I|_{\Gamma^+}), tr\,v_b\Big)_{\partial G_b}.
\end{aligned}$$

Here and below, by $u_s(t), v_s$ and $u_b(t), v_b$ we denote the restrictions of $u(t), v$ to G_s and G_b, respectively, by $tr\,u_s(t), tr\,u_b(t), tr\,v_s, tr\,v_b$ we denote the traces of the functions $u_s(t), u_b(t), v_s, v_b$ on ∂G_b.

(2) For almost all $t \in (0,T)$, the function $I(t)$ satisfies Equation (3) almost everywhere on D_s and the conditions (7), (8) almost everywhere on Γ_b^-, $\widehat{\Gamma}^-$, respectively. meaning that:

$$I(t) = I_s(t) + I_*(t), \tag{35}$$

where $I_s(t) = \dfrac{1}{\pi}\mathcal{A}[h(u_s(t)), h(tr\,u_b(t))]$, $I_*(t) = \widehat{\mathcal{A}}\,[J_*(t)]$ for almost all $t \in (0,T)$.

Remark 1. *The fulfillment of the identity (34) is equivalent to the fact that $(c_p u, v)_G \in W^{1,1}(0,T)$ for all $v \in V$; moreover:*

$$\begin{aligned}
&\frac{d}{dt}(c_p u, v)_G + a(u,v) + b(u,I,v) = (f,v)_G \quad \text{for almost all } t \in (0,T),\\
&(c_p u(t), v)_G \to (c_p u^0, v)_G \quad \text{as } t \to 0.
\end{aligned}$$

Remark 2. *It follows from Theorem 1 that $I_* \in L^{r_*}(0,T;\mathcal{W}^{q_*}(D_s))$ and:*

$$\|I_*\|_{L^{r_*}(0,T;\mathcal{W}^{q_*}(D_s))} \le \varkappa^{1/q_*}[1 + 2(\varkappa + s)]\,\|J_*\|_{L^{r_*}(0,T;\widehat{L}^{q_*}(\widehat{\Gamma}^-))}. \tag{36}$$

In addition, for almost all $t \in (0,T)$, the following estimates hold:

$$\Big(\varkappa\,\|I_*(t)\|^{q_*}_{L^{q_*}(D_s)} + \|I_*(t)|_{\Gamma^+}\|^{q_*}_{\widehat{L}^{q_*}(\Gamma^+)}\Big)^{1/q_*} \le \|J_*(t)\|_{\widehat{L}^{q_*}(\widehat{\Gamma}^-)}, \quad 1 \le q_* < \infty, \tag{37}$$
$$\max\{\|I_*(t)\|_{L^\infty(D_s)}, \|I_*(t)|_{\Gamma^+}\|_{L^\infty(\Gamma^+)}\} \le \|J_*(t)\|_{L^\infty(\widehat{\Gamma}^-)}, \quad q_* = \infty. \tag{38}$$

In what follows, the following notations are used:

$$\begin{aligned}
&\|(f, J_*)\|_{r,q,r_*,q_*} = \|f\|_{L^r(0,T;L^q(G))} + \|J_*\|_{L^{r_*}(0,T;\widehat{L}^{q_*}(\widehat{\Gamma}^-))},\\
&\|(u^0, f, J_*)\|_{p,r,q,r_*,q_*} = \|u^0\|_{L^p(G)} + \|f\|_{L^r(0,T;L^q(G))} + \|J_*\|_{L^{r_*}(0,T;\widehat{L}^{q_*}(\widehat{\Gamma}^-))}.
\end{aligned}$$

The main results of this paper are the following theorems.

Theorem 2. *A weak solution to the problem (1)–(9) exists and is unique.*

Theorem 3. *Let (u^1, I^1) and (u^2, I^2) be two weak solutions to the problem (1)–(9) with $(u^{0,1}, f^1, J_*^1)$ and $(u^{0,2}, f^2, J_*^2)$ instead of (u^0, f, J_*). Then, the following estimates hold:*

$$\|c_p \Delta u\|_{C([0,T];L^1(G))} \le \|c_p \Delta u^0\|_{L^1(G)} + \|\Delta f\|_{L^1(Q_T)} + \|\Delta J_*\|_{L^1(0,T;\widehat{L}^1(\widehat{\Gamma}^-))}, \quad (39)$$

$$\|c_p [\Delta u]_\pm\|_{C([0,T];L^1(G))} \le \|c_p [\Delta u^0]_\pm\|_{L^1(G)} + \|[\Delta f]_\pm\|_{L^1(Q_T)} + \|[\Delta J_*]_\pm\|_{L^1(0,T;\widehat{L}^1(\widehat{\Gamma}^-))}, \quad (40)$$

where $\Delta u = u^1 - u^2$, $\Delta u^0 = u^{0,1} - u^{0,2}$, $\Delta f = f^1 - f^2$, $\Delta J_* = J_*^1 - J_*^2$.

Theorem 4 (Comparison theorem). *Let (u^1, I^1) and (u^2, I^2) be two weak solutions to the problem (1)–(9), with $(u^{0,1}, f^1, J_*^1)$ and $(u^{0,2}, f^2, J_*^2)$ instead of (u^0, f, J_*). If $u^{0,1} \le u^{0,2}$, $f^1 \le f^2$ and $J_*^1 \le J_*^2$, then $u^1 \le u^2$ and $I^1 \le I^2$.*

Note that the uniqueness of the solution to the problem (1)–(9) and Theorem 4 are direct consequences of Theorem 3.

Consider that u and I are interpreted as the absolute temperature and the radiation intensity. Therefore, it is important to show that u and I are nonnegative under some natural assumptions on the data. It is clear that $(u, I) = (0, 0)$ is a solution to the problem (1)–(9) with $u^0 = 0$, $f = 0$ and $J_* = 0$. Thus, Theorem 4 implies the following result.

Corollary 1. *Let (u, I) be a weak solution to the problem (1)–(9). If $u^0 \ge 0$, $f \ge 0$ and $J_* \ge 0$, then $u \ge 0$ and $I \ge 0$.*

The following three theorems show that an increase in summability exponents of f and g leads to improved properties of a weak solution.

Theorem 5. *Let (u, I) be a weak solution to the problem (1)–(9). If:*

$$u^0 \in L^p(G), \quad f \in L^r(0,T; L^q(G)), \quad J_* \in L^{r_*}(0,T; \widehat{L}^{q_*}(\widehat{\Gamma}^-)), \quad (41)$$

$$p \in [3, \infty),\ r \in [1, \infty],\ q \in \left[\frac{3}{2 + 1/p}, \infty\right],\ r_* \in \left[\frac{2}{1 + 1/p}, \infty\right],\ q_* \in \left[\frac{2}{1 + 1/p}, \infty\right], \quad (42)$$

$$2/r + 3/q \le 2 + 3/p, \quad 2/r_* + 2/q_* \le 1 + 3/p, \quad (43)$$

then $|u|^{\gamma-1}u \in V_2(Q_T)$ for all $\gamma \in [1, p/2]$, $I_s \in L^{r_s}(0,T; \mathcal{W}^{q_s}(D))$ for all $r_s \in [1, p/2]$, $q_s \in [1, p/2]$ such that $1/r_s + 1/q_s \ge 6/p$ and:

$$\| \, |u|^{\gamma-1}u\|^{1/\gamma}_{V_2(Q_T)} \le C\|(u^0, f, g)\|_{p,r,q,r_*,q_*}, \quad (44)$$

$$\|I_s\|_{L^{r_s}(0,T;\mathcal{W}^{q_s}(D_s))} \le C\|(u^0, f, g)\|^4_{p,r,q,r_*,q_*}. \quad (45)$$

In addition, $u \in C([0,T]; L^s(G))$ for all $s \in [1, p)$.

By C (with or without indices), we denote various positive constants that may depend on G, T, $\underline{c}_p$, $\overline{c}_p$, $\lambda_{\min}$, $\lambda_{\max}$, σ_0, $\varkappa$, s and p, r, q, r_*, q_*.

Theorem 6. *Let (u, I) be a weak solution to the problem (1)–(9). If:*

$$e^{\beta_0 |u^0|} \in L^1(G) \text{ for some } \beta_0 > 0, \ \ f \in L^r(0,T; L^q(G)), \ \ J_* \in L^{r_*}(0,T; \widehat{L}^{q_*}(\widehat{\Gamma}^-)), \quad (46)$$

$$r \in [1, \infty], \ \ q \in [3/2, \infty], \ \ r_* \in [2, \infty), \ \ q_* \in [2, \infty), \quad \frac{2}{r} + \frac{3}{q} \le 2, \ \ \frac{2}{r_*} + \frac{2}{q_*} \le 1, \quad (47)$$

then there exists a constant $\beta \in (0, \beta_0/2]$ such that $e^{\beta|u|} \in V_2(Q_T)$ and:

$$\|e^{\beta|u|}\|_{V_2(Q_T)} \le C(\|e^{\beta|u^0|}\|_{L^2(G)} + 1), \quad (48)$$

where C depends on $\|(f, J_*)\|_{r,q,r_*,q_*}$, but does not depend on u^0.

In addition, $u \in C([0,T];L^s(G))$ for all $s \in [1,\infty)$ and $I_s \in L^{r_s}(0,T;\mathcal{W}^{q_s}(D_s))$ for all $r_s, q_s \in [1,\infty)$.

Theorem 7. *Let (u,I) be a weak solution to the problem (1)–(9). If:*

$$u^0 \in L^\infty(G), \quad f \in L^r(0,T;L^q(G)), \quad J_* \in L^{r_*}(0,T;\widehat{L}^{q_*}(\widehat{\Gamma}^-)), \tag{49}$$

$$r \in (1,\infty], \; q \in (3/2,\infty], \; r_* \in (2,\infty], \; q_* \in (2,\infty], \quad \frac{2}{r}+\frac{3}{q} < 2, \; \frac{2}{r_*}+\frac{2}{q_*} < 1, \tag{50}$$

then $u \in L^\infty(Q_T)$ and $I_s(t) \in \mathcal{W}^\infty(D_s)$ for almost all $t \in (0,T)$. Moreover:

$$\|u\|_{L^\infty(Q_T)} \le C\|(u^0, f, J_*)\|_{\infty,r,q,r_*,q_*}, \tag{51}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|I(t)\|_{\mathcal{W}^\infty(D_s)} \le C\|(u^0, f, g)\|^4_{\infty,r,q,r_*,q_*}. \tag{52}$$

5.2. Reducing the Problem (1)–(9) to Problem $\mathcal{P}$

Let (u,I) be a weak solution to the problem (1)–(9) and $I_* = \mathcal{A}[J_*]$. We put:

$$f_* = \varkappa \int_\Omega I_*\, d\omega = 4\pi\varkappa \langle \widehat{\mathcal{A}} \rangle_\Omega [J_*] \quad \text{for } (x,t) \in Q_{s,T},$$

$$g_* = \mathcal{M}^+(I_*|_{\Gamma^+}) = \pi \widehat{\mathcal{B}}[J_*] \;\text{ for } (x,t) \in \partial Q_{b,T}.$$

It follows from (37), (38) that $f_* \in L^{r_*}(0,T;L^{q_*}(G_s))$, $g_* \in L^{r_*}(0,T;L^{q_*}(\partial G_b))$ and

$$\|f_*\|_{L^{r_*}(0,T;L^{q_*}(G_s))} + \|g_*\|_{L^{r_*}(0,T;L^{q_*}(\partial G_b))} \le ((4\pi\varkappa)^{1/q_*'} + \pi^{1/q_*'})\|J_*\|_{L^{r_*}(0,T;\widehat{L}^{q_*}(\widehat{\Gamma}^-))}. \tag{53}$$

It follows from (35) that:

$$\begin{aligned}
4\varkappa h(u_s) - \varkappa \int_\Omega I\, d\omega &= 4\varkappa h(u_s) - 4\varkappa \langle \mathcal{A} \rangle_\Omega [h(u_s), h(tr\, u_b)] - f_* \\
&= 4\varkappa \mathcal{C}[h(u_s), h(tr\, u_b)] - f_*, \\
h(tr\, u_b) - \mathcal{M}^+(I|_{\Gamma^+}) &= h(tr\, u_b) - \frac{1}{\pi}\mathcal{M}^+(\mathcal{A}[h(u_s), h(tr\, u_b)]|_{\Gamma_b^+}) - g_* \\
&= h(tr\, u_b) - \mathcal{B}[h(u_s), h(tr\, u_b)] - g_* = \mathcal{D}[h(u_s), h(tr\, u_b)] - g_*.
\end{aligned}$$

Using these formulas, we exclude the function I from the problem (1)–(9) and arrive at the problem:

$$\begin{aligned}
& c_p \frac{\partial u_s}{\partial t} - \operatorname{div}(\lambda(x,u_s)\nabla u_s) + 4\varkappa \mathcal{C}[h(u_s), h(tr\, u_b)] = f + f_*, \quad (x,t) \in Q_{s,T}, \\
& c_p \frac{\partial u_b}{\partial t} - \operatorname{div}(\lambda(x,u_b)\nabla u_b) = f, \quad (x,t) \in Q_{b,T}, \\
& \quad \lambda(x,u_s)\frac{\partial u_s}{\partial n} = 0, \quad (x,t) \in \partial \widehat{Q}_T, \\
& \quad \lambda(x,u_s)\frac{\partial u_s}{\partial n} + \gamma(tr\, u_s - tr\, u_b) = 0, \quad (x,t) \in \partial Q_{b,T}, \\
& -\lambda(x,u_b)\frac{\partial u_b}{\partial n} + \mathcal{D}[h(u_s), h(tr\, u_b)] + \gamma(tr\, u_b - tr\, u_s) = g_*, \quad (x,t) \in \partial Q_{b,T}, \\
& u|_{t=0} = u^0, \quad x \in G,
\end{aligned}$$

in which only one function u is unknown. This problem will be called Problem $\mathcal{P}$.

Remind that $\mathcal{V}(Q_T) = \{u \in V_2^{1,0}(Q_T) \cap L^4(Q_{s,T}) \mid tru_b \in L^4(\partial Q_{b,T})\}$. Therefore, it follows from $u \in \mathcal{V}(Q_T)$ and the boundedness of the operators $\mathcal{C} : L^1(G_s, \partial G_b) \to$

$L^1(G_s)$, $\mathcal{D} : L^1(G_s, \partial G_b) \to L^1(\partial G_b)$ that $\mathcal{C}[h(u_s), h(tr\, u_b)] \in L^1(Q_{s,T})$, $\mathcal{D}[h(u_s), h(tr\, u_b)] \in L^1(\partial Q_{b,T})$.

By a weak solution to Problem $\mathcal{P}$, we mean a function $u \in \mathcal{V}(Q_T)$ satisfying the identity:

$$-\int_0^T (c_p u(t), v)_G \frac{d\eta}{dt}(t)\, dt + \int_0^T A(u(t), v)\, \eta(t)\, dt$$
$$= (c_p u^0, v)_G \cdot \eta(0) + \int_0^T \langle \mathcal{F}(t), v \rangle\, \eta(t)\, dt \quad \forall v \in V, \ \ \forall \eta \in C_*^\infty[0, T], \tag{54}$$

where:

$$\begin{aligned} A(u, v) &= a(u, v) + b(u, v), \\ b(u, v) &= (4\varkappa \mathcal{C}[h(u_s)), h(tr\, u_b)], v_s)_{G_s} + (\mathcal{D}[h(u_s), h(tr\, u_b)], tr\, v_b)_{\partial G_b}, \\ \langle \mathcal{F}(t), v \rangle &= (f(t), v)_G + (f_*(t), v_s)_{G_s} + (g_*(t), tr\, v_b)_{\partial G_b}. \end{aligned} \tag{55}$$

Remark 3. *Due to the equality (31) instead of the formula (55), it is possible to use an alternative formula:*

$$b(u, v) = (h(u_s), 4\varkappa \mathcal{C}[v_s, tr\, v_b])_{G_s} + (h(tr\, u_b), \mathcal{D}[v_s, tr\, v_b])_{\partial G_b}. \tag{56}$$

Remark 4. *It is easy to see that if $(u, I) \in \mathcal{V}(Q_T) \times L^1(0, T; \mathcal{W}^1(D_s))$ is a weak solution to the problem (1)–(9), then u is a weak solution to Problem $\mathcal{P}$. On the other hand, if $u \in \mathcal{V}(Q_T)$ is a weak solution to Problem $\mathcal{P}$, then defining I by the formula (35) for almost all $t \in (0, T)$, we obtain the pair $(u, I) \in \mathcal{V}(Q_T) \times L^1(0, T; \mathcal{W}^1(D_s))$ that is a weak solution to the problem (1)–(9). The fact that $I \in L^1(0, T; \mathcal{W}^1(D_s))$ follows from the continuity of the operator $\mathcal{A} : L^1(G_s, \partial G_b) \to \mathcal{W}^1(D_s)$ and properties $h(u_s) \in L^1(Q_{s,T})$, $h(tr\, u_b) \in L^1(\partial Q_{b,T})$ that the function $u \in \mathcal{V}(Q_T)$ possesses.*

6. Auxiliaries

6.1. Forms $\widehat{D}(U_i, U_{Ii}, V_i, V_{Ii})$, $D(U_i, U_{Ii}, V_i, V_{Ii})$ and Some of Their Properties

We set:

$$\begin{aligned} \widehat{d}(u_I, u_{II}, v_I, v_{II}) &= (4\varkappa \mathcal{C}[u_I, u_{II}], v_I)_{G_s} + (\mathcal{D}[u_I, u_{II}], v_{II})_{\partial G_b}, \\ d(u_I, u_{II}, v_I, v_{II}) &= (4\varkappa \langle \mathcal{A} \rangle_\Omega [1_{G_s}, 1_{\partial G_b}] u_I, v_I)_{G_s} - (4\varkappa \langle \mathcal{A} \rangle_\Omega [u_I, u_{II}], v_I)_{G_s} \\ &\quad + \big(\mathcal{B}[1_{G_s}, 1_{\partial G_b}] u_{II}, v_{II}\big)_{\partial G_b} - \big(\mathcal{B}[u_I, u_{II}], v_{II}\big)_{\partial G_b}, \end{aligned}$$

where $(u_I, u_{II}, v_I, v_{II}) \in L^1(G_s, \partial G_b) \times L^\infty(G_s, \partial G_b)$ or $(u_I, u_{II}, v_I, v_{II}) \in L^2(G_s, \partial G_b) \times L^2(G_s, \partial G_b)$.

It follows from (28) that:

$$\widehat{d}(u_I, u_{II}, v_I, v_{II}) = d(u_I, u_{II}, v_I, v_{II}) + \big(4\varkappa\, \widehat{\alpha}\, u_I, v_I\big)_{G_s} + (\widehat{\beta}\, u_{II}, v_{II})_{\partial G_b}. \tag{57}$$

The following three statements are proven in [65].

Lemma 1. *Assume that $(u_I, u_{II}) \in L^1(G_s, \partial G_b)$, $(v_I, v_{II}) \in L^\infty(G_s, \partial G_b)$, $\{(u_I^N, u_{II}^N)\}_{N=1}^\infty \subset L^1(G_s, \partial G_b)$, $\{(v_I^N, v_{II}^N)\}_{N=1}^\infty \subset L^\infty(G_s, \partial G_b)$. Assume also that $(u_I^N, u_{II}^N) \to (u_I, u_{II})$ in $L^1(G_s, \partial G_b)$, $v_I^N \to v_I$ almost everywere on G_s, $v_{II}^N \to v_{II}$ almost everywhere on ∂G_b as $N \to \infty$ and $\sup\limits_{N \geq 1} \|(v_I^N, v_{II}^N)\|_{L^\infty(G_s, \partial G_b)} < \infty$. Then:*

$$\lim_{N \to \infty} \widehat{d}(u_I^N, u_{II}^N, v_I^N, v_{II}^N) = \widehat{d}(u_I, u_{II}, v_I, v_{II}).$$

Lemma 2. *Assume that* $(u_I, u_{II}) \in L^2(G_s, \partial G_b)$, $\{(u_I^N, u_{II}^N)\}_{N=1}^{\infty} \subset L^2(G_s, \partial G_b)$ *and* $(u_I^N, u_{II}^N) \to (u_I, u_{II})$ *in* $L^2(G_s, \partial G_b)$ *as* $N \to \infty$. *Then:*

$$\lim_{N\to\infty} \widehat{d}(u_I^N, u_{II}^N, u_I^N, u_{II}^N) = \widehat{d}(u_I, u_{II}, u_I, u_{II}).$$

Let $\{E_{si}\}_{i=1}^N$ be a system of measurable pairwise disjoint subsets of G_s, such that $G_s = \bigcup_{i=1}^N E_{si}$ and let $\{S_{b\ell}\}_{\ell=1}^N$ be a system of measurable pairwise disjoint subsets of ∂G_b, such that $\partial G_b = \bigcup_{\ell=1}^N S_{b\ell}$. We denote by 1_{si} and $1_{b\ell}$ the characteristic functions of sets E_{si} and $S_{b\ell}$, respectively. We set:

$$\alpha_{ki} = \big(4\varkappa\langle\mathcal{A}\rangle_\Omega[1_{sk}, 0], 1_{si}\big)_{G_s},\ \beta_{\ell k} = \big(\mathcal{B}[0, 1_{s\ell}], 1_{sk}\big)_{\partial G_b},\ \delta_{\ell i} = \big(4\varkappa\langle\mathcal{A}\rangle_\Omega[0, 1_{b\ell}], 1_{si}\big)_{G_s}.$$

Note that $\alpha_{ki} \geq 0$, $\beta_{\ell k} \geq 0$, $\delta_{\ell i} \geq 0$.

Lemma 3. *Let:*

$$u_I^N(x) = \sum_{i=1}^N u_{si} 1_{si}(x), \quad u_{II}^N(x) = \sum_{\ell=1}^N u_{b\ell} 1_{b\ell}(x) \tag{58}$$

be simple functions defined on G_s, ∂G_b, *respectively, and let:*

$$v_I^N(x) = \sum_{i=1}^N v_{si} 1_{si}(x), \quad v_{II}^N(x) = \sum_{\ell=1}^N v_{b\ell} 1_{b\ell}(x)$$

be other simple functions defined on G_s, ∂G_b, *respectively. Then:*

$$\begin{aligned} d(u_I^N, u_{II}^N, v_I^N, v_{II}^N) &= \frac{1}{2}\sum_{i=1}^N\sum_{k=1}^N \alpha_{ki}[u_{si} - u_{sk}][v_{si} - v_{sk}] \\ &+ \frac{1}{2}\sum_{\ell=1}^N\sum_{k=1}^N \beta_{k\ell}[u_{b\ell} - u_{bk}][v_{b\ell} - v_{bk}] + \sum_{i=1}^N\sum_{\ell=1}^N \delta_{\ell i}[u_{si} - u_{b\ell}][v_{si} - v_{b\ell}]. \end{aligned}$$

6.2. Forms $b(u, v)$, $b^{[n]}(u, v)$ and Some of Their Properties

Consider that:

$$b(u, v) = (4\varkappa\mathcal{C}[h(u_s), h(tr\, u_b)], v_s)_{G_s} + (\mathcal{D}[h(u_s), h(tr\, u_b)], tr\, v_b)_{\partial G_b}, \qquad (u, v) \in W^{1,2}(G) \times V.$$

We also set:

$$b^{[n]}(u, v) = (4\varkappa\mathcal{C}[h^{[n]}(u_s), h^{[n]}(tr\, u_b)], v_s)_{G_s} + (\mathcal{D}[h^{[n]}(u_s), h^{[n]}(tr\, u_b)], tr\, v_b)_{\partial G_b}, \qquad (u, v) \in W^{1,2}(G) \times W^{1,2}(G), \tag{59}$$

where $h^{[n]} = \min\{\max\{h(u), -n\}, n\}$.

Note that:

$$\begin{aligned} b(u, v) &= \widehat{d}(h(u_s), h(tr\, u_b), v_s, tr\, v_b) = d(h(u_s), h(tr\, u_b), v_s, tr\, v_b) \\ &+ \big(4\varkappa\,\widehat{\alpha}\, h(u_s), v_s\big)_{G_s} + (\widehat{\beta}\, h(tr\, u_b), tr\, v_s)_{\partial G_b}, \\ b^{[n]}(u, v) &= \widehat{d}(h^{[n]}(u_s), h^{[n]}(tr\, u_b), v_s, tr\, v_b) = d(h^{[n]}(u_s), h^{[n]}(tr\, u_b), v_s, tr\, v_b) \\ &+ \big(4\varkappa\,\widehat{\alpha}\, h^{[n]}(u_s), v_s\big)_{G_s} + (\widehat{\beta}\, h^{[n]}(tr\, u_b), tr\, v_s)_{\partial G_b}. \end{aligned}$$

Lemma 4. *For all $n \geq 1$, the following inequality holds:*

$$b^{[n]}(u,u) \geq 0 \quad \forall\, u \in W^{1,2}(G). \tag{60}$$

Proof of Lemma 4. We construct sequences of simple functions $\{u_I^N\}_{N=1}^{\infty}$ and $\{u_{II}^N\}_{N=1}^{\infty}$ of the forms (58) such that:

$$|u_I^N(x)| \leq |u_s(x)| \quad \text{and} \quad \lim_{N\to\infty} u_I^N(x) = u_s(x) \quad \text{for almost all} \quad x \in G_s,$$
$$|u_{II}^N(x)| \leq |tr\, u_b(x)| \quad \text{and} \quad \lim_{N\to\infty} u_{II}^N(x) = tr\, u_b(x) \quad \text{for almost all} \quad x \in \partial G_b.$$

It follows from Lemma 3 and the monotonicity of the function h that:

$$\begin{aligned}
&\widehat{d}(h^{[n]}(u_I^N), h^{[n]}(u_{II}^N), u_I^N, u_{II}^N) = \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} \alpha_{ki}[h^{[n]}(u_{si}^N) - h^{[n]}(u_{sk}^N)][u_{si}^N - u_{sk}^N] \\
&+ \frac{1}{2}\sum_{\ell=1}^{N}\sum_{k=1}^{N} \beta_{k\ell}[h^{[n]}(u_{b\ell}^N) - h^{[n]}(u_{bk}^N)][u_{b\ell}^N - u_{bk}^N] \\
&+ \sum_{i=1}^{N}\sum_{\ell=1}^{N} \delta_{\ell i}[h^{[n]}(u_{si}^N) - h^{[n]}(u_{b\ell}^N)][u_{si}^N - u_{b\ell}^N] \\
&+ (4\varkappa\,\widehat{\alpha}\, h^{[n]}(u_I^N), u_I^N)_{G_s} + (\widehat{\beta}\, h^{[n]}(u_{II}^N), u_{II}^N)_{\partial G_b} \geq 0.
\end{aligned} \tag{61}$$

It is clear that $(h^{[n]}(u_I^N), h^{[n]}(u_{II}^N)) \to (h^{[n]}(u_s), h^{[n]}(tr\, u_b))$ in $L^2(G_s, \partial G_b)$ and $(u_I^N, u_{II}^N) \to (u_s, tr\, u_b)$ in $L^2(G_s, \partial G_b)$. Therefore, by Lemma 2:

$$\lim_{N\to\infty} \widehat{d}(h^{[n]}(u_I^N), h^{[n]}(u_{II}^N), u_I^N, u_{II}^N) \to \widehat{d}(h^{[n]}(u_s), h^{[n]}(tr\, u_b), u_s, tr\, u_b) = b^{[n]}(u,u).$$

Passing in the inequality (61) to the limit as $N \to \infty$, we arrive at the inequality (60). □

Lemma 5. *Assume that $w \in C(\mathbb{R})$, w be a non-decreasing function such that $w(0) = 0$. Then, for all $n \geq 1$, $M > 0$, the following inequalities hold:*

$$b(u, w(u^{[M]})) \geq 0 \quad \forall\, u \in W^{1,2}(G), \tag{62}$$
$$b^{[n]}(u, w(u^{[M]})) \geq 0 \quad \forall\, u \in W^{1,2}(G). \tag{63}$$

Proof of Lemma 5. Let $\{u_I^N\}_{N=1}^{\infty}$ and $\{u_{II}^N\}_{N=1}^{\infty}$ be the same sequences of simple functions as in the proof of the previous lemma.

It follows from Lemma 3 and the monotonicity of the function h that:

$$\begin{aligned}
&\widehat{d}\big(h(u_I^N), h(u_{II}^N), w((u_I^N)^{[M]}), w((u_{II}^N)^{[M]})\big) \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} \alpha_{ki}\left[h(u_{si}^N) - h(u_{sk}^N)\right]\left[w((u_{si}^N)^{[M]}) - w((u_{sk}^N)^{[M]})\right] \\
&+ \frac{1}{2}\sum_{\ell=1}^{N}\sum_{k=1}^{N} \beta_{k\ell}\left[h(u_{b\ell}^N) - h(u_{bk}^N)\right]\left[w((u_{b\ell}^N)^{[M]}) - w((u_{bk}^N)^{[M]})\right] \\
&+ \sum_{i=1}^{N}\sum_{\ell=1}^{N} \delta_{\ell i}[h(u_{si}^N) - h(u_{b\ell}^N)][w((u_{si}^N)^{[M]}) - w((u_{b\ell}^N)^{[M]})] \\
&+ (4\varkappa\,\widehat{\alpha}\, h(u_I^N), w((u_I^N)^{[M]}))_{G_s} + (\widehat{\beta}\, h(u_{II}^N), w((u_{II}^N)^{[M]}))_{\partial G_b} \geq 0.
\end{aligned} \tag{64}$$

It is clear that $(h(u_I^N), h(u_{II}^N)) \to (h(u_s), h(tr\, u_b))$ in $L^1(G_s, \partial G_b)$ as $N \to \infty$; $w((u_I^N)^{[M]}) \to w(u_s^{[M]})$ almost everywhere on G_s, $w((u_{II}^N)^{[M]}) \to w(tr\, u_b^{[M]})$ almost everywhere on ∂G_b as $N \to \infty$; in addition, $\|w((u_I^N)^{[M]})\|_{L^\infty(G_s)} \leq C_M$, $\|w((u_{II}^N)^{[M]})\|_{L^\infty(\partial G_b)} \leq C_M$, where $C_M = \max\{w(M), |w(-M)|\}$. Therefore, by Lemma 1:

$$\lim_{N\to\infty} \widehat{d}\big(h(u_I^N), h(u_{II}^N), w((u_I^N)^{[M]}), w((u_{II}^N)^{[M]})\big)$$
$$= \widehat{d}\big(h(u_s), h(tr\, u_b), w(u_s^{[M]}), w(tr\, u_b^{[M]})\big) = b\big(u, w(u^{[M]})\big).$$

Passing in (64) to the limit as $N \to \infty$, we arrive at the inequality (62). The proof of the inequality (63) is quite the same. $\square$

7. An Auxiliary Problem $\mathcal{P}^{[N]}$ and Its Solvability

Consider an auxiliary Problem $\mathcal{P}^{[n]}$, which differs from Problem $\mathcal{P}$ only in that in its formulation, the function $h(u)$ is replaced by $h^{[n]} = \min\{\max\{h(u), -n\}, n\}$, where $1 \leq n$ is a natural parameter.

By a weak solution to Problem $\mathcal{P}^{[n]}$, we mean a function $u \in V_2^{1,0}(Q_T)$ satisfying the identity:

$$-\int_0^T (c_p u(t), v)_G \frac{d}{dt}\eta(t)\, dt + \int_0^T A^{[n]}(u(t), v)\, \eta(t)\, dt$$
$$= (c_p u^0, v)_G \cdot \eta(0) + \int_0^T \langle \mathcal{F}(t), v\rangle\, \eta(t)\, dt \quad \forall\, v \in W^{1,2}(G),\ \ \forall\, \eta \in C_*^\infty[0,T], \tag{65}$$

where $A^{[n]}(u,v) = a(u,v) + b^{[n]}(u,v)$ and $b^{[n]}(u,v)$ is given by the formula (59).

Theorem 8. *Assume that $u^0 \in L^2(G)$, $f \in L^r(0,T; L^q(G))$, $J_* \in L^{r_*}(0,T; L^{q_*}(\widehat{\Gamma}^-))$, where $r \in (1,\infty]$, $q \in [6/5,\infty]$, $r_* \in [4/3,\infty]$, $q_* \in [4/3,\infty]$, $2/r + 3/q \leq 7/2$, $2/r_* + 2/q_* \leq 5/2$. Then, a weak solution to Problem $\mathcal{P}^{[n]}$ exists.*

Proof of Theorem 8. Let $\{e_\ell\}_{\ell=1}^\infty$ be a basis in $W^{1,2}(G)$ that is orthonormal in $L^2(G)$ with weight c_p.

We set $V_k = \mathrm{span}\,\{e_1, \ldots, e_k\}$, $k \geq 1$ and will seek an approximate solution to Problem $\mathcal{P}^{[n]}$ in the form $u^{(k)}(t) = \sum_{\ell=1}^{k} d_\ell^{(k)}(t) e_\ell$, determining the coefficients $d_\ell^{(k)}$ from the Galerkin method:

$$\Big(c_p \frac{d}{dt} u^{(k)}(t), v\Big)_G + A^{[n]}(u^{(k)}(t), v) = \langle \mathcal{F}(t), v\rangle, \quad t \in (0,T) \quad \forall\, v \in V_k, \tag{66}$$
$$u^{(k)}(0) = u^{0,k} = \sum_{\ell=1}^{k} (c_p u^0, e_\ell)_G\, e_\ell.$$

Note that $u^{0,k} \to u^0$ in $L^2(G)$ as $k \to \infty$, moreover $\|c_p^{1/2} u^{0,k}\|_{L^2(G)} \leq \|c_p^{1/2} u^0\|_{L^2(G)}$.

The Caratheodory theorem implies the existence of a time-local solution $u^{(k)}$. It is defined on the whole interval $(0,T)$ by virtue of the global to time a priori estimate:

$$\|u^{(k)}\|_{V_2(Q_T)} \leq C_1 \|(u^0, f, J_*)\|_{2,r,q,r_*,q_*}. \tag{67}$$

To obtain this estimate, we substitute $v = u^{(k)}(t)$ in (66) and use the inequalities:

$$\lambda_{\min} \|\nabla u^{(k)}\|^2_{L^2(G)} \leq a(u^{(k)}, u^{(k)}), \quad 0 \leq b^{[n]}(u^{(k)}, u^{(k)}),$$

which follow from the condition (32) and Lemma 4, and arrive at the inequality:

$$\frac{1}{2}\frac{d}{dt}\|c_p^{1/2}u^{(k)}(t)\|^2_{L^2(G)} + \lambda_{\min}\|\nabla u^{(k)}(t)\|^2_{L^2(G)} \le \langle \mathcal{F}(t), u^{(k)}(t)\rangle, \quad t \in (0,T).$$

Integrating it, we deduce the inequality:

$$\begin{aligned}
&\frac{1}{2}\|c_p^{1/2}u^{(k)}(t)\|^2_{L^2(G)} + \lambda_{\min}\|\nabla u^{(k)}\|^2_{L^2(Q_t)} \\
&\le \frac{1}{2}\|c_p^{1/2}u^{0,k}\|^2_{L^2(G)} + \int_0^t \langle \mathcal{F}(t'), u^{(k)}(t')\rangle\, dt' \\
&\le \frac{1}{2}\|c_p^{1/2}u^{0}\|^2_{L^2(G)} + \|f\|_{L^r(0,T;L^q(G))}\|u^{(k)}\|_{L^{r'}(0,T;L^{q'}(G))} \\
&+ \|f_*\|_{L^{r_*}(0,T;L^{q_*}(G)}\|u_s^{(k)}\|_{L^{r'_*}(0,T;L^{q'_*}(G_s))} \\
&+ \|g_*\|_{L^{r_*}(0,T;L^{q_*}(\partial G_b))}\|tr\, u_b^{(k)}\|_{L^{r'_*}(0,T;L^{q'_*}(\partial G_b))}, \quad t \in (0,T)
\end{aligned}$$

Applying the inequality (13) with r', q' and r'_*, q'_* instead of $\bar{r}_1$, $\bar{q}_1$, the inequality (14) with r'_*, q'_* instead of $\bar{r}_2$, $\bar{q}_2$ and using the inequality (53), we arrive at the estimate (67).

Let us derive one more estimate. Since $\|(h^{[n]}(u_s^{(k)}), h^{[n]}(tr\, u_b^{(k)}))\|_{L^\infty(G_s,\partial G_b)} \le n$, then it follows from (30) that:

$$\|\mathcal{C}[h^{[n]}(u_s^{(k)}), h^{[n]}(tr\, u_b^{(k)})]\|_{L^\infty(G_s)} \le 2n, \quad \|\mathcal{D}[h^{[n]}(u_s^{(k)}), h^{[n]}(tr\, u_b^{(k)})]\|_{L^\infty(\partial G_b)} \le 2n.$$

Consequently:

$$|b^{[n]}(u^{(k)}, v)| \le 2n\big(4\varkappa\|v_s\|_{L^1(G_s)} + \|tr\, v_b\|_{L^1(\partial G_b)}\big).$$

Integrating (66) over $(t, t+\tau)$, where $0 < \tau < T$, and taking into account the estimate (67), we have:

$$\begin{aligned}
&(c_p\Delta^{(\tau)}u^{(k)}(t), v)_G = \int_t^{t+\tau} \big[-A^{[n]}(u^{(k)}(t'), v) + \langle \mathcal{F}(t'), v\rangle\big]\, dt' \\
&\le \tau^{1/2}\lambda_{\max}\|\nabla u^{(k)}\|_{L^2(Q_T)}\|\nabla v\|_{L^2(G)} \\
&+ \tau^{1/2}\|\gamma\|_{L^\infty(\partial G_b)}\|tr\, u_b^{(k)} - tr\, u_s^{(k)}\|_{L^2(\partial Q_{b,T})}\|tr\, v_b - tr\, v_s\|_{L_2(\partial G_b)} \\
&+ 2\tau n\big(4\varkappa\|v_s\|_{L^1(G_s)} + \|tr\, v_b\|_{L^1(\partial G_b)}\big) + \tau^{1-1/r}\|f\|_{L^r(0,T;L^q(G))}\|v\|_{L^{q'}(G)} \\
&+ \tau^{1-1/r_*}\|f_*\|_{L^{r_*}(0,T;L^{q_*}(G))}\|v_s\|_{L^{q'_*}(G)} + \tau^{1-1/r_*}\|g_*\|_{L^{r_*}(0,T;L^{q_*}(\partial G_b))}\|tr\, v_b\|_{L^{q'_*}(\partial G_b)},
\end{aligned}$$

where $\Delta^{(\tau)}u^{(k)}(t) = u^{(k)}(t+\tau) - u^{(k)}(t)$.

Taking $v = \Delta^{(\tau)}u^{(k)}(t)$, integrating the resulting inequality over t from 0 to $T-\tau$ and using the inequalities (53), (67), we obtain:

$$\begin{aligned}
&\underline{c}_p\|\Delta^{(\tau)}u^{(k)}\|^2_{L^2(Q_{T-\tau})} \\
&\le 2\tau^{1/2}T^{1/2}\Big[\lambda_{\max}\|\nabla u^{(k)}\|^2_{L^2(Q_T)} + \|\gamma\|_{L^\infty(\partial G_b)}\|tr\, u_b^{(k)} - tr\, u_s^{(k)}\|^2_{L^2(\partial Q_{b,T})}\Big] \\
&+ 4\tau n\big(4\varkappa\|u_s^{(k)}\|_{L^1(Q_{s,T})} + \|tr\, u_b^{(k)}\|_{L^1(\partial Q_{b,T})}\big) + 2\tau^{1-1/r}\|f\|_{L^r(0,T;L^q(G))}\|u^{(k)}\|_{L^1(0,T;L^{q'}(G))} \\
&+ 2\tau^{1-1/r_*}\|f_*\|_{L^{r_*}(0,T;L^{q_*}(G_s))}\|u_s^{(k)}\|_{L^1(0,T;L^{q'_*}(G_s))} \\
&+ 2\tau^{1-1/r_*}|g\|_{L^{r_*}(0,T;L^{q_*}(\partial G_b))}\|tr\, u_b^{(k)}\|_{L^1(0,T;L^{q'_*}(\partial G_b))}\Big] \\
&\le \tau^\nu C_2\big(\|(u^0, f, J_*)\|_{2,r,g,r_*,q_*} + n\big)^2,
\end{aligned}$$

where $\nu = \min\{1/2, 1-1/r, 1-1/r_*\}$. Thus:

$$\|\Delta^{(\tau)} u^{(k)}\|_{L^2(Q_{T-\tau})} \le C_3 \tau^{\nu/2} \big(\|(u^0, f, J_*)\|_{2,r,g,r_*,q_*} + n\big). \tag{68}$$

It follows from (67) that there exist a function $u \in V_2(Q_T)$ and the subsequence $\{u^{(k_\ell)}\}_{\ell=1}^{\infty}$ such that $u^{(k_\ell)} \to u$ weakly in $L^2(0,T;W^{1,2}(G))$ and weakly stars in $L^\infty(0,T;L^2(G))$ as $k_\ell \to \infty$.

By virtue of the Riesz precompactness criterion for $L^2(Q_T)$, the estimates (67) and (68) allow us to select a subsequence, such that $u^{(k_\ell)} \to u$ strongly in $L^2(Q_T)$ and almost everywhere on Q_T.

It is clear that $\lambda(\cdot, u^{(k_\ell)})\nabla u^{(k_\ell)} \to \lambda(\cdot, u)\nabla u$ weakly in $L^2(Q_T)$ and therefore $a_0(u^{(k_\ell)}, v) \to a_0(u, v)$ weakly in $L^1(0,T)$ for all $v \in W^{1,2}(G)$.

From the estimate (19) applied to $u^{(k_\ell)} - u$, it follows that $tr\, u_s^{(k_\ell)} \to tr\, u_s$ and $tr\, u_b^{(k_\ell)} \to tr\, u_b$ in $L^2(\partial Q_{b,T})$ as $\ell \to \infty$. So $a_1(u^{(k_\ell)}, v) \to a_1(u, v)$ in $L^1(0,T)$.

It is also easy to see that $h^{[n]}(u_s^{(k_\ell)}) \to h^{[n]}(u_s)$ in $L^1(Q_{s,T})$ and $h^{[n]}(tr\, u_b^{(k_\ell)}) \to h^{[n]}(tr\, u_b)$ in $L^1(\partial Q_{b,T})$. Using the formula (59), we have:

$$\begin{aligned} b^{[n]}(u^{(k_\ell)}, v) &= 4\varkappa\big(h^{[n]}(u_s^{(k_\ell)}), \mathcal{C}[v_s, tr\, v_b]\big)_{G_s} + \big(h^{[n]}(tr\, u_b^{(k_\ell)}), \mathcal{D}[v_s, tr\, v_b]\big)_{\partial G_b} \\ &\to 4\varkappa(h^{[n]}(u_s), \mathcal{C}[v_s, tr\, v_b])_{G_s} + (h^{[n]}(tr\, u_b), \mathcal{D}[v_s, tr\, v_b])_{\partial G_b} = b^{[n]}(u, v) \end{aligned}$$

in $L^1(0,T)$ for all $v \in W^{1,2}(G)$.

Multiplying (66) on $\eta(t)$, where $\eta \in C_*^\infty[0,T]$, and integrating the result over t from 0 to T, we have:

$$\begin{aligned} &-\int_0^T (c_p u^{(k)}(t), v)_G \frac{d}{dt}\eta(t)\, dt + \int_0^T A^{[n]}(u^{(k)}(t), v)\, \eta(t)\, dt \\ &= (c_p u^{0,k}, v)_G \cdot \eta(0) + \int_0^T \langle \mathcal{F}(t), v\rangle\, \eta(t)\, dt. \end{aligned}$$

Passing to the limit at $k = k_\ell \to \infty$, we establish the validity of the identity (65) for an arbitrary function $v \in \bigcup_{k=1}^{\infty} V_k$. Since the set $\bigcup_{k=1}^{\infty} V_k$ is dense everywhere in $W^{1,2}(G)$, then the identity (65) holds for all $v \in W^{1,2}(G)$.

Since the function u satisfies this identity, it follows (see, for example [41], Lemma 4.1) that $u \in C([0,T]; L^2(G))$. Thus, $u \in V_2^{1,0}(Q_T)$. □

8. Estimates for Weak Solutions to Problems $\mathcal{P}$ and $\mathcal{P}^{[N]}$

We need the following statement, following from [41], in Lemma 4.4.

Lemma 6. *Assume that a function $u \in V_2^{1,0}(Q_T)$ satisfies the identity:*

$$-\int_0^T (c_p u(t), v)_G \frac{d}{dt}\eta(t)\, dt = \int_0^T \langle \widehat{\mathcal{F}}(t), v\rangle\, \eta(t)\, dt \quad \forall\, v \in V, \ \ \forall\, \eta \in C_0^\infty[0,T], \tag{69}$$

where:

$$\begin{aligned} &\langle \widehat{\mathcal{F}}(t), v\rangle = (F(t), \nabla v)_G + (\widehat{f}(t), v)_G + (g_s(t), tr\, v_s)_{\partial G_b} + (g_b(t), tr\, v_b)_{\partial G_b}, \\ &F \in L^2(0,T; (L^2(G))^3), \quad \widehat{f} \in L^1(Q_T), \quad g_s, g_b \in L^1(\partial Q_{b,T}). \end{aligned}$$

Assume also that $w \in C^1(\mathbb{R})$, $w' \geq 0$, $w(0) = 0$ *and* $W^{(M)}(u) = \int_0^u w(s^{[M]})\, ds$, *where* $M > 0$.

Then:

$$\|c_p W^{(M)}(u(t))\|_{L^1(G)} = \|c_p W^{(M)}(u(0))\|_{L^1(G)} + \int_0^t \langle \widehat{\mathcal{F}}(t), w(u^{[M]}(t'))\rangle\, dt', \quad t \in [0,T], \tag{70}$$

$$\|c_p W^{(M)}([u]_+(t))\|_{L^1(G)} = \|c_p W^{(M)}([u]_+(0))\|_{L^1(G)} + \int_0^t \langle \widehat{\mathcal{F}}(t), w(u^{[0,M]}(t'))\rangle\, dt', \quad t \in [0,T]. \tag{71}$$

Lemma 7. *Let u be a weak solution to Problem $\mathcal{P}$ or to Problem $\mathcal{P}^{[n]}$.*

Suppose that $U \in C^1(\mathbb{R})$, $U' \geq 0$, $w(u) = \int_0^u (U'(s))^2\, ds$, $W(u) = \int_0^u w(s)\, ds$. *Then:*

$$\|c_p W(u^{[M]}(t))\|_{L^1(G)} + \lambda_{\min}\|\nabla U(u^{[M]})\|^2_{L^2(Q_t)} \leq \|c_p W(u^0)\|_{L^1(G)} + \int_0^t \langle \mathcal{F}(t'), w(u^{[M]}(t'))\rangle\, dt', \quad t \in [0,T]. \tag{72}$$

Proof of Lemma 7. A weak solution to Problem $\mathcal{P}$ satisfies the identity (69) with:

$$\langle \widetilde{\mathcal{F}}(t), v\rangle = -A(u(t), v) + \langle \mathcal{F}(t), v\rangle.$$

Using Lemma 6, we arrive at the equality:

$$\|c_p W^{(M)}(u(t))\|_{L^1(G)} + \int_0^t A\big(u(t'), w(u^{[M]}(t'))\big)\, dt' = \|c_p W^{(M)}(u^0)\|_{L^1(G)} + \int_0^t \langle \mathcal{F}(t), w(u^{[M]}(t'))\rangle\, dt'. \tag{73}$$

Note that:

$$\begin{aligned} a_0(u, w(u^{[M]})) &= (\lambda(\cdot,u)\nabla u, \nabla w(u^{[M]}))_G = (\lambda(\cdot,u)\nabla u, [U'(u^{[M]})]^2 \nabla u^{[M]})_G \\ &= (\lambda(\cdot,u)\nabla U(u^{[M]}), \nabla U(u^{[M]}))_G \geq \lambda_{\min}\|\nabla U(u^{[M]})\|^2_{L^2(G)}, \\ a_1(u, w(u^{[M]})) &= (\gamma[tr\, u_b(t) - tr\, u_s(t)], w(tr\, u_b^{[M]}) - w(tr\, u_s^{[M]}))_{\partial G_b} \geq 0. \end{aligned}$$

Using these inequalities, the inequality (62) and the estimates:

$$W(u^{[M]}) \leq W^{(M)}(u) \leq W(u),$$

we arrive from (73) at (72).

The inequality (72) for the weak solution to Problem $\mathcal{P}^{[n]}$ is established in the same way. The only difference is that the inequality (63) is used instead of the inequality (62). □

Theorem 9. *Let u be a weak solution to Problem $\mathcal{P}$ or to Problem $\mathcal{P}^{[n]}$. If the assumptions (41)–(43) are satisfied, then $|u|^{\gamma-1}u \in V_2(Q_T)$ for all $\gamma \in [1, p/2]$; moreover, the estimate (44) holds. In addition, $u \in C([0,T]; L^s(G))$ for all $s \in [1,p)$.*

Proof of Theorem 9. Let $\gamma \in [1, p/2]$. We set:

$$U_\gamma(u) = |u|^{\gamma-1}u, \quad w_\gamma(u) = \int_0^u (U'_\gamma(s))^2\,ds = \frac{\gamma^2}{2\gamma-1}|u|^{2\gamma-2}u,$$
$$W_\gamma(u) = \int_0^u w_\gamma(s)\,ds = \frac{\gamma}{2(2\gamma-1)}|u|^{2\gamma}.$$

Since:

$$\frac{1}{4}[U_\gamma(u)]^2 \le W_\gamma(u) \le \frac{1}{2}[U_\gamma(u)]^2, \quad |w_\gamma(u)| \le \gamma |U_\gamma(u)|^{2-1/\gamma},$$

the inequality (72) implies the inequality:

$$\begin{aligned}
&\frac{1}{4}\|U_\gamma(u^{[M]}(t))\|^2_{L^2(G)} + \lambda_{\min}\|\nabla U_\gamma(u^{[M]})\|^2_{L^2(Q_t)} \\
&\le \frac{1}{2}\|U_\gamma(u^0)\|^2_{L^2(G)} + \gamma\|f\|_{L^r(0,T;L^q(G))}\|U_\gamma(u^{[M]})\|^{2-1/\gamma}_{L^{\bar r}(0,T;L^{\bar q}(G))} \\
&+\gamma\|f_*\|_{L^{r_*}(0,T;L^{q_*}(G_s))}\|U_\gamma(u_s^{[M]})\|^{2-1/\gamma}_{L^{\bar r_*}(0,T;L^{\bar q_*}(G_s))} \\
&+\gamma\|g_*\|_{L^{r_*}(0,T;L^{q_*}(\partial G_b))}\|U_\gamma(tr\, u_b^{[M]})\|^{2-1/\gamma}_{L^{\bar r_*}(0,T;L^{\bar q_*}(\partial G_b))},
\end{aligned}$$

where $\bar r = (2-1/\gamma)r'$, $\bar q = (2-1/\gamma)q'$, $\bar r_* = (2-1/\gamma)r'_*$, $\bar q_* = (2-1/\gamma)q'_*$.

As a consequence, we have:

$$\begin{aligned}
&\|U_\gamma(u^{[M]})\|^2_{V_2(Q_T)} \le C_1\Big[\|u^0\|^{2\gamma}_{L^{2\gamma}(G)} + \gamma\|f\|_{L^r(0,T;L^q(G))}\|U_\gamma(u^{[M]})\|^{2-1/\gamma}_{L^{\bar r}(0,T;L^{\bar q}(G))} \\
&+\gamma\|J_*\|_{L^{r_*}(0,T;\widehat{L}^{q_*}(\widehat{\Gamma}^-))}\Big(\|U_\gamma(u_s^{[M]})\|^{2-1/\gamma}_{L^{\bar r_*}(0,T;L^{\bar q_*}(G_s))} + \|U_\gamma(tr\, u_b^{[M]})\|^{2-1/\gamma}_{L^{\bar r_*}(0,T;L^{\bar q_*}(\partial G_b))}\Big)\Big].
\end{aligned} \tag{74}$$

Using (13) and (14), we derive from (74) the estimate:

$$\||u^{[M]}|^{\gamma-1}u^{[M]}\|^{1/\gamma}_{V_2(Q_T)} \le C\|(u^0, f, J_*)\|_{p,r,q,r_*,q_*} \tag{75}$$

with a constant C that does not depend on M. Since $u^{[M]} \to u$ in $L^2(0,T;W^{1,2}(G))$ and $u^{[M]}(t) \to u(t)$ in $L^2(G)$ for all $t \in [0,T]$ as $M \to \infty$, the estimate (75) implies that $|u|^{\gamma-1}u \in V_2(Q_T)$ and the estimate (44) holds.

Since $|u|^{p/2-1}u \in V_2(Q_T)$ then $u \in L^\infty(0,T;L^p(G))$. Taking into account that $u \in C([0,T];L^2(G))$, we come to the conclusion that $u \in C([0,T];L^s(G))$ for all $s \in [1,p)$. □

Theorem 10. *Let u be a weak solution to Problem $\mathcal{P}$ or to Problem $\mathcal{P}^{[n]}$. If the assumptions (49), (50) are satisfied, then $u \in L^\infty(Q_T)$ and the estimate (51) holds.*

Proof of Theorem 10. We put $A = \|(u^0, f, J_*)\|_{\infty,r,q,r_*,q_*}$. If $A = 0$, then it follows from (44) that $u = 0$ and the estimate (51) holds.

Let $A > 0$. We divide both sides of the inequality (74) by $A^{2\gamma}$. Taking into account that $\gamma \ge 1$ and (74) holds for all $M > 0$, we obtain:

$$\begin{aligned}
\|U_\gamma(\overline{u}^{[M]})\|^2_{V_2(Q_T)} &\le C_1\gamma\Big[\operatorname{meas} G + \|U_\gamma(\overline{u}^{[M]})\|^{2-1/\gamma}_{L^{\bar r}(0,T;L^{\bar q}(G))} \\
&+ \|U_\gamma(\overline{u}^{[M]})\|^{2-1/\gamma}_{L^{\bar r_*}(0,T;L^{\bar q_*}(G_s))} + \|U_\gamma(tr\,\overline{u}_b^{[M]})\|^{2-1/\gamma}_{L^{\bar r_*}(0,T;L^{\bar q_*}(\partial G_b))}\Big] \\
&\le C_2\gamma\Big[\|U_\gamma(\overline{u}^{[M]})\|^2_{L^{2r'}(0,T;L^{2q'}(G))} + \|U_\gamma(\overline{u}^{[M]})\|^2_{L^{2r'_*}(0,T;L^{2q'_*}(G))} \\
&+ \|U_\gamma(tr\,\overline{u}_b^{[M]})\|^2_{L^{2r'_*}(0,T;L^{2q'_*}(\partial G_b))} + 1\Big],
\end{aligned}$$

where $\overline{u} = u/A$.

By condition (50), we can assume that $2/r + 3/q \le 2 - 3\delta$, $\;2/r_* + 2/q_* \le 1 - 3\delta$ with some $\delta \in (0,1)$.

Setting $\gamma = \gamma_k = (1+\delta)^k$, $k \ge 0$, we obtain the inequality:

$$\|U_{\gamma_k}(\overline{u}^{[M]})\|^2_{V_2(Q_T)} \le C_2(1+\delta)^k \Big[\|U_{\gamma_{k-1}}(\overline{u}^{[M]})\|^{2(1+\delta)}_{L^{\widetilde{r}}(0,T;L^{\widetilde{q}}(G))}$$
$$+ \|U_{\gamma_{k-1}}(\overline{u}^{[M]})\|^{2(1+\delta)}_{L^{\widetilde{r}_*}(0,T;L^{\widetilde{q}_*}(G))} + \|U_{\gamma_{k-1}}(tr\,\overline{u}_b^{[M]})\|^{2(1+\delta)}_{L^{\widetilde{r}_*}(0,T;L^{\widetilde{q}_*}(\partial\widehat{G}))} + 1\Big], \; k \ge 1,$$

where $\widetilde{r} = 2(1+\delta)r'$, $\widetilde{q} = 2(1+\delta)q'$, $\widetilde{r}_* = 2(1+\delta)r'_*$, $\widetilde{q}_* = 2(1+\delta)q'_*$. It is easy to check that $\widetilde{r}$, $\widetilde{q}$ and $\widetilde{r}_*$, $\widetilde{q}_*$ satisfy (16) in the role of $\overline{r}_1$, $\overline{q}_1$ and $\widetilde{r}_*$, $\widetilde{q}_*$ satisfy (16) in the role of $\overline{r}_2$, $\overline{q}_2$. Using (13) and (15), we arrive at the inequality:

$$\| U_{\gamma_k}(\overline{u}^{[M]})\|^2_{V_2(Q_T)} + 1 \le C_3(1+\delta)^k \Big[\| U_{\gamma_{k-1}}(\overline{u}^{[M]})\|^{2(1+\delta)}_{V_2(Q_T)} + 1\Big], \quad k \ge 1,$$

which implies the inequality:

$$d_k \le C_3^{1/\gamma_k}(1+\delta)^{k/\gamma_k} d_{k-1}, \quad k \ge 1,$$

where $d_k = \Big(\| \, |\overline{u}^{[M]}|^{\gamma_k}\|^2_{V_2(Q_T)} + 1\Big)^{1/\gamma_k}$.

Iterating these inequalities, we find:

$$d_k \le C_4 d_0 = C_4(\|\overline{u}^{[M]}\|^2_{V_2(Q_T)} + 1) \le C_4(\|\overline{u}\|^2_{V_2(Q_T)} + 1), \quad k \ge 1.$$

Thus:

$$\|\overline{u}^{[M]}\|^2_{L^{2\gamma_k}(Q_T)} \le T^{1/\gamma_k} \| \, |\overline{u}^{[M]})|^{\gamma_k - 1}\overline{u}^{[M]}\|^{2/\gamma_k}_{V_2(Q_T)}$$
$$\le T^{1/\gamma_k} y_k \le T^{1/\gamma_k} C_4(\|\overline{u}\|^2_{V_2(Q_T)} + 1), \quad k \ge 1.$$

The limit passage as $k \to \infty$ leads to the estimate:

$$\|\overline{u}^{[M]}\|^2_{L^\infty(Q_T)} \le C_4(\|\overline{u}\|^2_{V_2(Q_T)} + 1) \quad \forall\, M \ge 1,$$

which implies that $u \in L^\infty(Q_T)$ and:

$$\|u\|^2_{L^\infty(Q_T)} \le C_6(\|u\|^2_{V_2(Q_T)} + A^2).$$

Taking into account the estimate (44) with $\gamma = 1$, we obtain the estimate (51). $\square$

Theorem 11. *Let u be a weak solution to Problem $\mathcal{P}$. If the assumptions (46), (47) are satisfied, then there exists a constant $\beta \in (0, \beta_0/2)$ such that $e^{\beta|u|} \in V_2(Q_T)$, the estimate (48) holds and $u \in C([0,T]; L^s(G)$ for all $s \in [1,\infty)$.*

Proof of Theorem 11. Let:

$$U_\beta(u) = [e^{\beta|u|} - 1]\,\mathrm{sgn}\,u, \quad w_\beta(u) = \int_0^u (U'_\beta(s))^2\,ds = \frac{\beta}{2}[e^{2\beta|u|} - 1]\,\mathrm{sgn}\,u,$$
$$W_\beta(u) = \int_0^u w_\beta(s)\,ds = \frac{1}{4}[e^{2\beta|u|} - 2\beta u],$$

where $\beta > 0$. Since:

$$\frac{1}{4}(U_\beta(u))^2 \le W_\beta(u) \le \frac{1}{2}(U_\beta(u))^2, \quad |w_\beta(u)| \le \beta[U_\beta(u)^2 + 1],$$

inequality (72) implies the inequality

$$\begin{aligned}
&\|U_\beta(u^{[M]}(t))\|^2_{L^2(G)} + \|\nabla U_\beta(u^{[M]})\|^2_{L^2(Q_t)} \le C_1\|U_\beta(u^{[M]}(0))\|^2_{L^2(G)} \\
&+ C_1\beta\Big[\|f\|_{L^r(0,T;L^q(G))}\big(\|U_\beta(u^{[M]})\|^2_{L^{\bar r_1}(0,T;L^{\bar q_1}(G))} + 1\big) \\
&+ \|f_*\|_{L^{r_*}(0,T;L^{q_*}(G))}\big(\|U_\beta(u^{[M]})\|^2_{L^{\bar r_*}(0,T;L^{\bar q_*}(G))} + 1\big) \\
&+ \|g_*\|_{L^{r_*}(0,T;L^{q_*}(\partial\widehat{G}))}\big(\|U_\beta(tr\, u_b^{[M]})\|_{L^{\bar r_*}(0,T;L^{\bar q_*}(\partial\widehat{G}))} + 1\big)\Big], \quad t \in (0,T],
\end{aligned}$$

where $\bar r_1 = 2r'$, $\bar q_1 = 2q'$, $\bar r_* = 2r'_*$, $\bar q_* = 2q'_*$.

Using (13) and (14), we derive the estimate:

$$\|U_\beta(u^{[M]})\|^2_{V_2(Q_T)} \le 2C_1\|U_\beta(u^0)\|^2_{L^2(G)} + \beta C_{f,J_*}\Big(\|U_\beta(u^{[M]})\|^2_{V_2(Q_T)} + 1\Big),$$

where $C_{f,J_*} = C_2\big(\|f\|_{L^r(0,T;L^q(G))} + \|J_*\|_{L^{r_*}(0,T;\widehat{L}^{q_*}(\widehat{\Gamma}^-))}\big)$.

Taking $\beta = \min\{\beta_0, C^{-1}_{f,J_*}\}/2$, we obtain the estimate:

$$\|e^{\beta|u^{[M]}|}\|_{V_2(Q_T)} \le C\Big(\|e^{\beta|u^0|}\|_{L^1(G)} + 1\Big). \tag{76}$$

Since $u^{[M]} \to u$ in $L^2(0,T;W^{1,2}(G))$ and $u^{[M]}(t) \to u(t)$ in $L^2(G)$ for all $t \in [0,T]$ as $M \to \infty$, the estimate (76) implies that $e^{\beta|u|} \in V_2(Q_T)$ and the estimate (48) holds.

Note that the conditions of Theorem 9 are satisfied for all $p \in [3,\infty)$. Therefore, $u \in C([0,T];L^s(G))$ for all $s \in [1,\infty)$. □

9. Stability and Uniqueness of Weak Solutions to Problem $\mathcal{P}$: Comparison Theorem

The proof given in this section uses some ideas of the method [70] proposed for proving comparison theorems for quasilinear elliptic equations. Special modifications of this method for some nonstationary radiative–conductive heat transfer problems were used in [41,53,57].

The following theorem concerns the stability of weak solutions to Problem $\mathcal{P}$ with respect to data.

Theorem 12. *Let u^1 and u^2 be two weak solutions to Problem $\mathcal{P}$ with $(u^{0,1}, f^1, J^1_*)$ and $(u^{0,2}, f^2, J^2_*)$ instead of (u^0, f, J_*). Then, the estimates (39), (40) hold.*

Proof of Theorem 12. We put $\Delta u = u^1 - u^2$, $\Delta u^0 = u^{0,1} - u^{0,2}$ $\Delta f = f^1 - f^2$, $\Delta J_* = J^1_* - J^2_*$.

Let $0 < t \le T$, $0 < \delta < 1$, δ is a parameter. We introduce the sets:

$$\begin{aligned}
&Q_t^+ = \{(x,t') \in Q_t \mid \Delta u(x,t') > 0\}, \quad Q_t^- = \{(x,t') \in Q_t \mid \Delta u(x,t') \le 0\}, \\
&Q_t^\delta = \{(x,t') \in Q_t \mid \Delta u(x,t') \ge \delta\}, \quad Q_t^{(0,\delta)} = \{(x,t') \in Q_t \mid 0 < \Delta u(x,t') < \delta\}, \\
&Q^+_{s,t} = Q_{s,t} \cap Q_t^+, \quad Q^-_{s,t} = Q_{s,t} \cap Q_t^-, \quad Q^\delta_{s,t} = Q_{s,t} \cap Q_t^\delta, \quad Q^{(0,\delta)}_{s,t} = Q_{s,t} \cap Q_t^{(0,\delta)}.
\end{aligned}$$

We introduce the function $v^\delta = \delta^{-1}(\Delta u)^{[0,\delta]} = \min\{\delta^{-1}[\Delta u]_+, 1\}$. It is clear that $0 \le v^\delta \le 1$; moreover $v^\delta(x,t) = 0$ for $(x,t) \in Q_T^-$, $v^\delta(x,t) = 1$ for $(x,t) \in Q_T^\delta$ and $\lim\limits_{\delta \to 0} v^\delta(x,t) = 1$ for $(x,t) \in Q_T^+$.

Subtracting from each other the identities (34) corresponding to the definitions of the weak solutions u^1 and u^2 leads to the identity:

$$-\int_0^T (c_p \Delta u(t), v)_G \frac{d}{dt}\eta(t)\,dt + \int_0^T [a(u^1(t), v) - a(u^2(t), v)]\eta(t)\,dt$$
$$+\int_0^T [b(u^1(t), v) - b(u^2(t), v)]\eta(t)\,dt = (c_p \Delta u^0, v)_G \cdot \eta(0) + \int_0^T (\Delta f(t), v)_G\, \eta(t)\,dt$$
$$+\int_0^T (\Delta f_*(t), v_s)_{G_s}\, \eta(t)\,dt + \int_0^T (\Delta g_*(t), tr\, v_b)_{\partial G_b} \eta(t)\,dt \quad \forall\, v \in V,\ \forall\, \eta \in C_*^\infty[0,T], \tag{77}$$

where $\Delta f_* = f_*^1 - f_*^2 = 4\varkappa \langle \widehat{\mathcal{A}} \rangle_\Omega [\Delta J_*]$, $\Delta g_* = g_*^1 - g_*^2 = \pi \widehat{\mathcal{B}}[\Delta J_*]$.

Using Lemma 6 with $w(u) = \delta^{-1} u$, $M = \delta$, $W^{(\delta)}(u) = \delta^{-1} \int_0^u s^{[\delta]}\, ds$ and taking into account that:

$$W^{(\delta)}([\Delta u^0]_+) \le [\Delta u^0]_+, \quad 0 \le w((\Delta u)^{[0,\delta]}) = v^\delta \le 1,$$
$$\|[\Delta f_*]_+\|_{L^1(Q_t)} + \|[\Delta g_*]_+\|_{L^1(\partial Q_{b,t})} \le \|[\Delta J_*]\|_{L^1(0,T;\widehat{L}^1(\widehat{\Gamma}^-))} \tag{78}$$

(the inequality (78) follows from (27)), we have:

$$\|c_p W^{(\delta)}([\Delta u]_+(t))\|_{L^1(G)} + (\lambda(\cdot, u^1)\nabla u^1 - \lambda(\cdot, u^2)\nabla u^2, \nabla v^\delta)_{L^2(Q_t)}$$
$$+ \int_0^t [a_1(u_1(t'), v^\delta(t')) - a_1(u_2(t'), v^\delta(t'))]\, dt'$$
$$+ \int_0^t [b(u^1(t'), v^\delta(t')) - b(u^2(t'), v^\delta(t'))] dt' = \|c_p W^{(\delta)}([\Delta u^0]_+)\|_{L^1(G)}$$
$$+ \int_0^t \left[(\Delta f(t'), v^\delta(t'))_G + (\Delta f_*(t'), v_s^\delta(t'))_{G_s} + (\Delta g_*(t'), tr\, v_b^\delta(t'))_{\partial G_b} \right] dt'$$
$$\le \|c_p [\Delta u^0]_+\|_{L^1(G)} + \|[\Delta f]_+\|_{L^1(Q_t)} + \|[\Delta J_*]_+\|_{L^1(0,t;\widehat{L}^1(\widehat{\Gamma}^-))} \quad \forall\, t \in (0,T]. \tag{79}$$

Using the fact that $\nabla v^\delta = \delta^{-1}\nabla(u^1 - u^2)$ almost everywhere on $Q_T^{(0,\delta)}$ and $\nabla v^\delta = 0$ almost everywhere on $Q_T^- \cup Q_T^\delta$ and taking into account assumptions (32), (33), we find that:

$$(\lambda(\cdot, u^1)\nabla u^1 - \lambda(\cdot, u^2)\nabla u^2, \nabla v^\delta)_{L^2(Q_t)}$$
$$= \delta(\lambda(\cdot, u^1)\nabla v^\delta, \nabla v^\delta)_{L^2(Q_t)} + ([\lambda(\cdot, u^1) - \lambda(\cdot, u^2)]\nabla u^2, \nabla v^\delta)_{L^2(Q_t^{(0,\delta)})}$$
$$\ge \delta\, \lambda_{\min} \|\nabla v^\delta\|^2_{L^2(Q_t)} - L\delta^{1/2} \|\nabla u^2\|_{L^2(Q_t^{(0,\delta)})} \|\nabla v^\delta\|_{L^2(Q_t)}$$
$$\ge -\frac{L^2}{4\lambda_{\min}} \|\nabla u^2\|^2_{L^2(Q_t^{(0,\delta)})}. \tag{80}$$

We note also that:

$$a_1(u^1, v^\delta) - a_1(u^2, v^\delta) = (\gamma[tr\, \Delta u_s - tr\, \Delta u_b], tr\, v_s^\delta - tr\, v_b^\delta)_{\partial G_b} \ge 0. \tag{81}$$

We set $\Delta h(u_s) = h(u_s^1) - h(u_s^2)$ for $(x,t) \in Q_{s,T}$, $\Delta h(tr\, u_b) = h(tr\, u_b^1) - h(tr\, u_b^2)$ for $(x,t) \in \partial Q_{b,T}$ and introduce the sets:

$$\partial^+ Q_{b,T} = \{(x,t) \in \partial Q_{b,T} \mid tr\, \Delta u_b(x,t) > 0\},$$
$$\partial^- Q_{b,T} = \{(x,t) \in \partial Q_{b,T} \mid tr\, \Delta u_b(x,t) \le 0\},$$
$$\partial^\delta Q_{b,T} = \{(x,t) \in \partial Q_{b,T} \mid tr\, \Delta u_b(x,t) \ge \delta\},$$
$$\partial^{(0,\delta)} Q_{b,T} = \{(x,t) \in \partial Q_{b,T} \mid 0 < tr\, \Delta u_b(x,t) < \delta\}.$$

From the formula (56), it follows that:

$$b(u^1, v^\delta) - b(u^2, v^\delta) = (4\varkappa \Delta h(u_s), \mathcal{C}[v_s^\delta, tr\, v_b^\delta])_{G_s} + (\Delta h(tr\, u_b), \mathcal{D}[v_s^\delta, tr\, v_b^\delta])_{\partial G_b}.$$

Noticing that:

$$\mathcal{C}[v_s^\delta, tr\, v_b^\delta] = v_s^\delta - \langle \mathcal{A} \rangle_\Omega [v_s^\delta, tr\, v_b^\delta] = -\langle \mathcal{A} \rangle_\Omega [v_s^\delta, tr\, v_b^\delta] \le 0, \quad (x,t) \in Q_{s,T}^-,$$
$$\mathcal{C}[v_s^\delta, tr\, v_b^\delta] = v_s^\delta - \langle \mathcal{A} \rangle_\Omega [v_s^\delta, tr\, v_b^\delta] \ge v_s^\delta - 1, \quad (x,t) \in Q_{s,T}^+,$$
$$\mathcal{D}[v_s^\delta, tr\, v_b^\delta] = tr\, v_b^\delta - \mathcal{B}[v_s^\delta, tr\, v_b^\delta] = -\mathcal{B}[v_s^\delta, tr\, v_b^\delta] \le 0, \quad (x,t) \in \partial^- Q_{b,T},$$
$$\mathcal{D}[v_s^\delta, tr\, v_b^\delta] = tr\, v_b^\delta - \mathcal{B}[v_s^\delta, tr\, v_b^\delta] \ge tr\, v_b^\delta - 1, \quad (x,t) \in \partial^+ Q_{b,T},$$

we find that:

$$\begin{aligned}
&\int_0^t [b(u^1(t'), v^\delta(t')) - b(u^2(t'), v^\delta(t'))]\, dt' \\
&= (4\varkappa \Delta h(u_s), \mathcal{C}[v_s^\delta, tr\, v_b^\delta])_{Q_{s,t}^-} + (4\varkappa(\Delta h(u_s), \mathcal{C}[v_s^\delta, tr\, v_b^\delta])_{Q_{s,t}^+} \\
&+ (\Delta h(tr\, u_b), \mathcal{D}[v_s^\delta, tr\, v_b^\delta])_{\partial Q_{b,t}^-} + (\Delta h(tr\, u_b), \mathcal{D}[v_s^\delta, tr\, v_b^\delta])_{\partial^+ Q_{b,t}} \\
&\ge (4\varkappa \Delta h(u_s), v_s^\delta - 1)_{Q_{s,t}^+} + (\Delta h(tr\, u_b), tr\, v_b^\delta - 1)_{\partial^+ Q_{b,t}} \\
&= (4\varkappa \Delta h(u_s), v_s^\delta - 1)_{Q_{s,t}^{(0,\delta)}} + (\Delta h(tr\, u_b), tr\, v_b^\delta - 1)_{\partial^{(0,\delta)} Q_{b,t}} \\
&\ge -4\varkappa \|\Delta h(u_s)\|_{L^1(Q_{s,t}^{(0,\delta)})} - \|\Delta h(tr\, u_b)\|_{L^1(\partial^{(0,\delta)} Q_{b,t})}.
\end{aligned} \tag{82}$$

It follows from the inequality (79) and the estimates (80)–(82) that:

$$\begin{aligned}
&\|c_p W^{(\delta)}([\Delta u]_+(t))\|_{L^1(G)} \\
&\le \frac{L^2}{4\lambda_{\min}} \|\nabla u^2\|^2_{L^2(Q_T^{(0,\delta)})} + 4\varkappa \|\Delta h(u_s)\|_{L^1(Q_{s,T}^{(0,\delta)})} + \|\Delta h(tr\, u_b)\|_{L^1(\partial^{(0,\delta)} Q_{b,T})} \\
&+ \|c_p [\Delta u^0]_+\|_{L^1(G)} + \|([\Delta f]_+, [\Delta J_*]_+)\|_{1,1,1,1} \quad \forall\, t \in (0,T].
\end{aligned} \tag{83}$$

We pass to the limit as $\delta \to 0$ in this inequality. Since:

$$\left| W^{(\delta)}([\Delta u]_+) - [\Delta u]_+ \right| \le \delta,$$

then:

$$\|c_p W^{(\delta)}([\Delta u]_+(t))\|_{L^1(G)} \to \|c_p [\Delta u]_+(t)\|_{L^1(G)}.$$

The first three terms on the right hand side of (83) tend to zero as $\delta \to 0$, since:

$$\text{meas}\,(Q_T^{(0,\delta)}; dxdt) \to 0, \ \text{meas}\,(Q_{s,T}^{(0,\delta)}; dxdt) \to 0, \ \text{meas}\,(\partial^{(0,\delta)} Q_{b,T}; d\sigma(x) dt) \to 0.$$

Thus, (83) implies the inequality:

$$\|c_p [\Delta u]_+(t)\|_{L^1(G)} \le \|c_p [\Delta u^0]_+\|_{L^1(G)} + \|([\Delta f]_+, [\Delta J_*]_+)\|_{1,1,1,1} \quad \forall\, t \in (0,T]. \tag{84}$$

The following inequality can be established in the same way:

$$\|c_p[\Delta u]_-(t)\|_{L^1(G)} \le \|c_p[\Delta u^0]_-\|_{L^1(G)} + \|([\Delta f]_-, [\Delta J_*]_-)\|_{1,1,1,1} \quad \forall\, t \in (0,T]. \tag{85}$$

Adding (84) and (85), we obtain the inequality:

$$\|c_p \Delta u(t)\|_{L^1(G)} \le \|c_p \Delta u^0\|_{L^1(G)} + \|(\Delta f, \Delta J_*)\|_{1,1,1,1} \quad \forall\, t \in (0,T]. \tag{86}$$

The inequalities (84), (86) imply the estimates (39), (40). □

Corollary 2 (Comparison theorem)**.** *If $u^{0,1} \le u^{0,2}$, $f^1 \le f^2$ and $J_*^1 \le J_*^2$, then $u^1 \le u^2$.*

Corollary 3 (Uniqueness theorem)**.** *If a weak solution to Problem $\mathcal{P}$ exists, then it is unique.*

10. Solvability of Problem $\mathcal{P}$

Theorem 13. *A weak solution to Problem $\mathcal{P}$ exists and is unique.*

Proof of Theorem 13. Firstly, we suppose that assumptions (49), (50) hold. By Theorems 8 and 10, for all $n > 0$, there exists a function $u \in V_2^{1,0}(Q_T) \cap L^\infty(Q_T) \subset \mathcal{V}(Q_T)$, which is a weak solution to Problem $\mathcal{P}^{[n]}$ and satisfies the estimate:

$$\|u\|_{L^\infty(Q_T)} \le M_\infty = C\|(u^0, f, J_*)\|_{\infty,r,q,r_*,q_*},$$

where M_∞ does not depend on n. By this estimate, $h^{[n]}(u) = h(u)$ for $n > h(M_\infty)$. Therefore, a weak solution to Problem $\mathcal{P}^{[n]}$ with $n > h(M_\infty)$ is simultaneously a weak solution to Problem $\mathcal{P}$.

Now, we prove the existence of a solution without additional assumptions (49), (50). Let N,M be natural numbers. Since $(u^0)^{[-N,M]} \in L^\infty(G)$, $f^{[-N,M]} \in L^\infty(Q_T) \subset L^\infty(0,T;$ $L^2(G))$, $J_*^{[-N,M]} \in L^\infty(\widehat{\Gamma}^- \times (0,T)) \subset L^\infty(0,T;\widehat{L}^3(\widehat{\Gamma}^-))$ then, by the first part of the proof, Problem $\mathcal{P}$ with $(u^0)^{[-N,M]}$, $f^{[-N,M]}$ and $J_*^{[-N,M]}$ in the role of u^0, f, and J_* has a weak solution $u^{(-N,M)}$ such that:

$$\begin{aligned}
&-\int_0^T (c_p u^{(-N,M)}(t), v)_G \frac{d}{dt}\eta(t)\,dt + \int_0^T A(u^{(-N,M)}(t), v)\,\eta(t)\,dt \\
&= (c_p(u^0)^{[-N,M]}, v)_G \cdot \eta(0) + \int_0^T (f^{[-N,M]}(t), v)_G\,\eta(t)\,dt \\
&+ \int_0^T \big[(f_*^{[-N,M]}, v_s)_{G_s} + (g_*^{(-N,M)}(t), tr\, v_b)_{\partial G_b}\big]\,\eta(t)\,dt \quad \forall\, v \in V,\ \forall\, \eta \in C_*^\infty[0,T].
\end{aligned} \tag{87}$$

Here, $f_*^{(-N,M)} = 4\pi\varkappa\,\langle \widehat{\mathcal{A}}\,\rangle_\Omega [J_*^{[-N,M]}]$, $g_*^{(-N,M)} = \pi\widehat{\mathcal{B}}[J_*^{[-N,M]}]$.
Note that:

$$\|(u^0)^{[-N,M]}, f^{[-N,M]}, J_*^{[-N,M]})\|_{p,r,q,r_*,q_*} \le C_0 = \|(u^0, f, J_*)\|_{p,r,q,r_*,q_*}.$$

So, by Theorem 9, the following uniform parameters N and M estimates hold:

$$\|u^{(-N,M)}\|_{V_2(Q_T)} \le C\|(u^0)^{[-N,M]}, f^{[-N,M]}, J_*^{[-N,M]})\|_{p,r,q,r_*,q_*} \le C_1 = CC_0, \tag{88}$$

$$\||u^{[-N,M]}|^{1/2} u^{[-N,M]}\|^{2/3}_{V_2(Q_T)} \le C\|(u^0)^{[-N,M]}, f^{[-N,M]}, J_*^{[-N,M]})\|_{p,r,q,r_*,q_*} \le C_1. \tag{89}$$

The estimate (89) and the inequalities (17), (18) imply the estimates

$$\|u^{(-N,M)}\|_{L^\infty(0,T;L^3(G))} \leq C_1, \tag{90}$$

$$\|u^{(-N,M)}\|_{L^5(Q_T)} \leq C_2, \tag{91}$$

$$\|tr\, u_b^{(-N,M)}\|_{L^4(\partial Q_{b,T})} \leq C_3. \tag{92}$$

We fix N. Since:

$$(u^0)^{[-N,M]} \leq (u^0)^{[-N,M+1]}, \quad f^{[-N,M]} \leq f^{[-N,M+1]}, \quad J_*^{[-N,M]} \leq J_*^{[-N,M+1]},$$

then by Corollary 2, the sequence $\{u^{(-N,M)}\}_{M=1}^{\infty}$ is non-decreasing with respect to M. Therefore, from the estimate (91), by virtue of Levi's monotone convergence theorem, there exists a function $u^{(-N)} \in L^5(Q_T)$ such that $u^{(-N,M)} \to u^{(-N)}$ in $L^5(Q_T)$ and almost everywhere on Q_T as $M \to \infty$. From (88), it follows that $u^{(-N)} \in V_2(Q_T)$, $u^{(-N,M)} \to u^{(-N)}$ weakly in $L^2(0,T;W^{1,2}(G))$ and weakly stars in $L^\infty(0,T;L^3(G))$ as $M \to \infty$. From the multiplicative inequality (19), it follows that $tr\, u_s^{(-N,M)} \to tr\, u_s^{(-N)}$, $tr\, u_b^{(-N,M)} \to tr\, u_b^{(-N)}$ in $L^2(\partial Q_{b,T})$.

Since the sequence $\{tr\, u_b^{(-N,M)}\}_{M=1}^{\infty}$ does not decrease with respect to M, then it follows from the estimate (92) and Levi's monotone convergence theorem that $tr\, u_b^{(-N)} \in L^4(\partial Q_{b,T})$ and $tr\, u_b^{(-N,M)} \to tr\, u_b^{(-N)}$ in $L^4(\partial Q_{b,T})$. So $h(tr\, u_b^{(-N,M)}) \to h(tr\, u_b^{(-N)})$ in $L^1(\partial Q_{b,T})$.

Passage to the limit as $M \to \infty$ in (88), (90)–(92) leads to the inequalities:

$$\|u^{(-N)}\|_{V_2(Q_T)} \leq C_1, \quad \|u^{(-N)}\|_{L^\infty(0,T;L^3(G))} \leq C_1, \tag{93}$$

$$\|u^{(-N)}\|_{L^5(Q_T)} \leq C_2, \quad \|tr\, u_b^{(-N)}\|_{L^4(\partial Q_{b,T})} \leq C_2. \tag{94}$$

Let $v \in V$. Since $\nabla u^{(-N,M)} \to \nabla u^{(-N)}$ weakly in $L^2(Q_T)$, $\lambda(\cdot, u^{(-N,M)}) \to \lambda(\cdot, u^{(-N)})$ almost everywhere on Q_T, then $a_0(u^{(-N,M)}, v) \to a_0(u^{(-N)}, v)$ weakly in $L^1(0,T)$ as $M \to \infty$. It is also clear that $a_1(u^{(-N,M)}, v) \to a_1(u^{(-N)}, v)$ in $L^1(0,T)$. Thus, $a(u^{(-N,M)}, v) \to a(u^{(-N)}, v)$ weakly in $L^1(0,T)$ as $M \to \infty$.

Since $h(u^{(-N,M)}) \to h(u^{(-N)})$ in $L^1(Q_T)$ and $h(tr\, u_b^{(-N,M)}) \to h(tr\, u_b^{(-N)})$ in $L^1(\partial Q_{b,T})$, then:

$$(h(u^{(-N,M)}), \mathcal{C}[v_s, tr\, v_b])_{G_s} \to (h(u^{(-N)}), \mathcal{C}[v_s, tr\, v_b])_{G_s} \quad \text{in} \quad L^1(0,T),$$
$$(h(tr\, u_b^{(-N,M)}), \mathcal{D}[v_s, tr\, v_b])_{\partial G_b} \to (h(tr\, u_b^{(-N)}), \mathcal{D}[v_s, tr\, v_b])_{\partial G_b} \quad \text{in} \quad L^1(0,T).$$

Thus, $b(u^{(-N,M)}, v) \to b(u^{(-N)}, v)$ in $L^1(0,T)$.

Passing to the limit as $M \to \infty$ in the identity (87), we arrive at the identity:

$$- \int_0^T (c_p u^{(-N)}(t), v)_G \frac{d}{dt}\eta(t)\, dt + \int_0^T A(u^{(-N)}(t), v)\, \eta(t)\, dt$$

$$= (c_p (u^0)^{[-N,\infty]}, v)_G \cdot \eta(0) + \int_0^T (f^{[-N,\infty]}(t), v)_G\, \eta(t)\, dt + \int_0^T (f_*^{(-N,\infty)}(t), v)_{G_s}\, \eta(t)\, dt \tag{95}$$

$$+ (g_*^{(-N,\infty)}(t), v)_{\partial G_b}\, \eta(t)\, dt \quad \forall\, v \in V, \ \ \forall\, \eta \in C_*^\infty[0,T], \tag{96}$$

where $(u^0)^{[-N,\infty]} = \max\{u^0, -N\}$, $f^{[-N,\infty]} = \max\{f, -N\}$, $J_*^{[-N,\infty]} = \max\{J_*, -N\}$, $f_*^{(-N,\infty)} = 4\pi\varkappa\, \langle \widehat{\mathcal{A}}\, \rangle_\Omega [J_*^{[-N,\infty]}]$, $g_*^{(-N,\infty)} = \pi \widehat{\mathcal{B}}[J_*^{[-N,\infty]}]$.

Theorem 12 implies the estimate:

$$\underline{c}_p\|u^{(-N,M_1)} - u^{(-N,M_2)}\|_{C([0,T];L^1(G))} \le \overline{c}_p\|(u^0)^{[-N,M_1]} - (u^0)^{[-N,M_2]}\|_{L^1(G)}$$
$$+ \|(f^{[-N,M_1]} - f^{[-N,M_2]}, J_*^{[-N,M_1]} - J_*^{[-N,M_2]})\|_{1,1,1,1} \quad \forall\, M_1 \ge 1,\ M_2 \ge 1.$$

This estimate means that $\{u^{(-N,M)}\}_{M=1}^{\infty}$ is a Cauchy sequence in $C([0,T];L^1(G))$. Hence, $u^{(-N)} \in C([0,T];L^1(G))$. Taking into account that $u^{(-N)} \in L^\infty(0,T;L^3(G))$, we have $u^{(-N)} \in C([0,T];L^2(G))$.

Thus, the function $u^{(-N)}$ is a weak solution to Problem $\mathcal{P}$ corresponding to the data $(u^0)^{[-N,\infty]}$, $f^{[-N,\infty]}$ and $J_*^{[-N,\infty]}$ in the role of u^0, f and J_*.

Since:

$$(u^0)^{[-(N+1),\infty]} \le (u^0)^{[-N,\infty]}, \quad f^{[-(N+1),\infty]} \le f^{[-N,\infty]}, \quad J_*^{[-(N+1),\infty]} \le g^{[-N,\infty]},$$

then by virtue of Corollary 2, the sequence $\{u^{(-N)}\}_{N=1}^{\infty}$ is non-increasing. Therefore, from the estimates (93), (94) it follows that there exists a function $u \in V_2(Q_T) \cap L^5(Q_T)$ such that $u^{(-N)} \to u$ weakly in $L^2(0,T;W^{1,2}(G))$, weakly stars in $L^\infty(0,T;L^3(G))$, strongly in $L^5(Q_T)$ and almost everywhere on Q_T as $N \to \infty$. In addition, $tr\, u_s^{(-N)} \to tr\, u_s$ in $L^2(\partial Q_{b,T})$ and $tr\, u_b^{(-N)} \to tr\, u_b$ in $L^4(\partial Q_{b,T})$. As a consequence, $h(u^{(-N)}) \to h(u)$ in $L^1(Q_T)$, $h(tr\, u_b^{(-N)}) \to h(\, u_b)$ in $L^1(\partial \widehat{Q}_{b,T})$.

Therefore, $a(u^{(-N)}, v) \to a(u,v)$ weakly in $L^1(0,T)$ and $b(u^{(-N)}, v) \to b(u,v)$ in $L^1(0,T)$ for all $v \in V$. Passage to the limit as $N \to \infty$ in the identity (96) gives the identity (34).

Theorem 12 implies the estimate:

$$\underline{c}_p\|u^{(-N_1)} - u^{(-N_2)}\|_{C([0,T];L^1(G))} \le \overline{c}_p\|(u^0)^{[-N_1,\infty]} - (u^0)^{[-N_2,\infty]}\|_{L^1(G)}$$
$$+ \|(f^{[-N_1,\infty]} - f^{[-N_2,\infty]}, J_*^{[-N_1,\infty]} - J_*^{[-N_2,\infty]})\|_{1,1,1,1} \quad \forall\, N_1 \ge 1,\ N_2 \ge 1.$$

This inequality means that $\{u^{(-N)}\}_{N=1}^{\infty}$ is a Cauchy sequence in $C([0,T];L^1(G))$. Hence, $u \in C([0,T];L^1(G))$. Taking into account that $u \in L^\infty(0,T;L^3(G))$, we have $u \in C([0,T];L^2(G))$.

We proved the existence of a weak solution to Problem $\mathcal{P}$. Its uniqueness follows from Corollary 3. □

11. Justification of the Main Results

Proof of Theorem 2. Note that (see Remark 4) the pair $(u, I) \in \mathcal{V}(Q_T) \times L^1(0,T;\mathcal{W}^1(D))$ is a weak solution to the problem (1)–(9) if and only if $u \in \mathcal{V}(Q_T)$ is a weak solution to problem $\mathcal{P}$ and I is expressed by the formula:

$$I = \frac{1}{\pi}\mathcal{A}[h(u_s), h(tr\, u_b)] + \widehat{\mathcal{A}}\,[J_*]. \tag{97}$$

Therefore, the existence and uniqueness of a weak solution to the problem (1)–(9) follow directly from Theorem 13. □

Proof of Theorem 4. Assume that the conditions of Theorem 4 be satisfied.

By Corallary 2 we have $u^1 \le u^2$. Thus, $h(u_s^1) \le h(u_s^2)$, $h(tr\, u_b^1) \le h(tr\, u_b^2)$ and

$$I^1 = \frac{1}{\pi}\mathcal{A}[h(u_s^1), h(tr\, u_b^1)] + \widehat{\mathcal{A}}\,[J_*^1] \le \frac{1}{\pi}\mathcal{A}[h(u_s^2), h(tr\, u_b^2)] + \widehat{\mathcal{A}}\,[J_*^2] = I^2.$$

□

Proof of Theorem 5. Assume that the conditions of Theorem 5 are satisfied.

By Theorem 9, $|u|^{\gamma-1}u \in V_2(Q_T)$ for all $\gamma \in [1, p/2]$, the estimate (44) is valid and $u \in C([0,T]; L^s(G))$ for all $s \in [1, p)$.

From $|u|^{p/2-1}u \in V_2(Q_T)$, it follows (see (13), (14)) that $h(u_s) \in L^{r_s}(0,T; L^{q_s}(G_s))$, $h(tr\, u_b) \in L^{r_s}(0,T; L^{q_s}(\partial G_b))$ for all $r_s \in [1, p/2]$, $q_s \in [1, p/2]$ such that $1/r_s + 1/q_s \geq 6/p$. In addition, the following estimate holds:

$$\|h(u_s)\|_{L^{r_s}(0,T;L^{q_s}(G_s))} + \|h(tr\, u_b)\|_{L^{r_s}(0,T;L^{q_s}(\partial G_b))} \leq C_1 \| |u|^{p/2-1}u\|^{8/p}_{V_2(Q_T)}.$$

From this estimate, the boundedness of the operator $\mathcal{A} : L^{q_s}(G_s, \partial G_b) \to \mathcal{W}^{q_s}(D_s)$ and the estimate (44) with $\gamma = p/2$, it follows that $I_s \in L^{r_s}(0,T; \mathcal{W}^{q_s}(D_s))$ and:

$$\|I_s\|_{L^{r_s}(0,T;\mathcal{W}^{q_s}(D_s))} \leq C_2 \| |u|^{p/2-1}u\|^{8/p}_{V_2(Q_T)} \leq C\|(u^0, f, J_*)\|^4_{p,r,q,r_*,q_*}.$$

□

Proof of Theorem 6. Assume that the conditions of Theorem 6 are satisfied.

By Theorem 11, there exists a constant $\beta \in (0, \beta_0/2)$ such that $e^{\beta|u|} \in V_2(Q_T)$ and the estimate (48) holds.

Note that the conditions of Theorem 5 are satisfied for all $p \in [3, \infty)$. Thus, $I_s \in L^{r_s}(0,T; \mathcal{W}^{q_s}(D_s))$ for all $r_s \in [1, \infty)$, $q_s \in [1, \infty)$. □

Proof of Theorem 7. Assume that the conditions of Theorem 7 are satisfied.

By Theorem 10, $u \in L^\infty(Q_T)$ and the estimate (51) holds. Consequently, $(h(u_s(t)), h(tr\, u_b(t))) \in L^\infty(G_s, \partial G_b)$ for almost all $t \in (0,T)$ and:

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in(0,T)} \|(h(u_s(t)), h(tr\, u_b(t))\|_{L^\infty(G_s,\partial G_b)} \\ &\qquad \leq \sigma_0 \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|^4_{L^\infty(G)} \leq C_1\|(u^0, f, J_*)\|^4_{\infty,r,q,r_*,q_*}. \end{aligned} \tag{98}$$

Therefore, $I_s(t) = \dfrac{1}{\pi}\mathcal{A}[h(u_s(t)), h(tr\, u_b(t))] \in \mathcal{W}^\infty(D_s)$ for almost all $t \in (0,T)$ and by virtue of the estimate (98):

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|I_s(t)\|_{\mathcal{W}^\infty(D_s)} \leq C\|(u^0, f, J_*)\|^4_{\infty,r,q,r_*,q_*},$$

where $C = C_1 \dfrac{1}{\pi}\|\mathcal{A}\|_{L^\infty(G_s,\partial G_b)\to\mathcal{W}^\infty(D_s)}$. □

12. Conclusions

In this paper, the author continues to construct a mathematical theory of complex heat transfer problems.

A nonstationary initial–boundary value problem governing a radiative–conductive heat transfer in a convex semitransparent body with an absolutely black inclusions was considered. To describe the process, a system consisting of two heat equations and the integro-differential radiative transfer equation was used. This system is supplied by boundary conditions, which describe the energy exchange between semitransparent body, external media and opaque inclusions.

The unique solvability of this problem was proven. In addition, the stability of solutions with respect to the data was proven, which established a comparison theorem. Besides, results on improving the properties of solutions with an increase in the summability of the data were established. All results are global in terms of time and data.

The considered mathematical model of radiative–conductive heat transfer contains a number of simplifying assumptions. One should consider the process of heat transfer in a system of bodies, and not in one convex body. In a more complex model, it should be taken into account that the properties of the semitransparent medium and the radiation intensity depend on the radiation frequency. In addition, inclusions may not be completely black, but gray or even "colored". The author expects to study the more complex corresponding models in the near future.

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