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New Asymptotic Properties of Positive Solutions of Delay Differential Equations and Their Application

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Abstract: In this study, new asymptotic properties of positive solutions of the even-order delay differential equation with the noncanonical operator are established. The new properties are of an iterative nature, which allows it to be applied several times. Moreover, we use these properties to obtain new criteria for the oscillation of the solutions of the studied equation using the principles of comparison.

Keywords: delay differential equation; even-order; asymptotic properties; oscillation; noncanonical case

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1. Introduction

Our interest in this work revolves around the study of the asymptotic behavior of positive solutions of the delay differential equation (DDE):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a\cdot\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\psi\right)\right)+q\cdot(\psi\circ g)=0,\ t\geq t_0, \tag{1}$$

under the following hypotheses:

Hypothesis 1 (H1). *$n\geq 4$ is an even natural number;*

Hypothesis 2 (H2). *a and q are continuous real functions on $\mathbf{I}_0:=[t_0,\infty)$, $a(t)>0$, $a'(t)\geq 0$, $q(t)\geq 0$, and:*

$$\int_{t_0}^{\infty}a^{-1}(\eta)\mathrm{d}\eta<\infty;$$

Hypothesis 3 (H3). *g is a continuous nondecreasing real function on $\mathbf{I}_0$, $g(t)\leq t$, and $\lim_{t\to\infty}g(t)=\infty$.*

By a proper solution of (1), we mean a function $\psi\in C^{n-1}(\mathbf{I}_0)$ with:

$$a\psi^{(n-1)}\in C^1(\mathbf{I}_0)\text{ and }\sup\{|\psi(t)|:t\geq t_*\}>0,\text{ for }t_*\in\mathbf{I}_0,$$

and ψ satisfies (1) on $\mathbf{I}_0$.

DDEs are a type of functional differential equation that takes into account the effect of past time. Therefore, DDEs are a better way to describe natural phenomena and time-related problems. For example, the oscillation of contacts of electromagnetic switches could be described by the oscillation of solutions of the second-order DDE (see [1]), and in mathematical ecology, by DDE, Israelsson and Johnsson [2] introduced a model for geotropic circumnutations of Helianthus annuus.

Recently, a research movement has been active that deals with the qualitative properties of solutions such as these equations, especially their oscillatory behavior. Baculíková [3,4], Džrina and Jadlovská [5], and Chatzarakis et al. [6] developed approaches and techniques for studying oscillatory behavior in order to improve the oscillation criteria of second-order delay/advanced differential equations. Bohner et al. [7], Grace et al. [8], and Moaaz et al. [9,10] also extended this evolution to DDEs of the neutral type. On the other hand, Džurina et al. [11,12] and Moaaz et al. [13] dealt by different methods with the asymptotic properties of the solutions of DDEs of the odd-order.

For even-order DDEs, Moaaz et al. [14] and Park et al. [15] were interested in studying the oscillation of the even-order DDE:

$$(a(t)(\psi^{(n-1)}(t))^{\gamma})' + q(t)\psi^{\beta}(g(t)) = 0, \tag{2}$$

(or some of its special cases) where γ, β are the ratio of odd natural numbers. They only focused on studying the oscillation of (2) in the canonical case, that is,

$$\int_{t_0}^{\infty} a^{-1/\beta}(\eta)\mathrm{d}\eta = \infty.$$

For the canonical DDE of the neutral type, see [16]. In the noncanonical case, that is,

$$\int_{t_0}^{\infty} a^{-1/\beta}(\eta)\mathrm{d}\eta < \infty,$$

Zhang et al. [17] studied the qualitative properties of (2). They obtained conditions that ensured that all nonoscillatory solutions of Equation (2) tend to *zero*. In [18], Zhang et al. established criteria for the oscillation of all solutions of (2) by using Riccati substitution. By establishing comparison theorems that compare the nth-order equation with one or a couple of first-order delay differential equations, Baculíková et al. [19] studied the oscillatory properties of the DDE:

$$(a(t)(\psi^{(n-1)}(t))^{\gamma})' + q(t)f(\psi(g(t))) = 0, \tag{3}$$

where f is nondecreasing and $-f(-x\psi) \geq f(x\psi) \geq f(x)f(\psi)$, for $x\psi > 0$. Moreover, by introducing a generalized Riccati substitution, Moaaz and Muhib [20] extended the technique used in [21] to study the oscillation of (2).

In this study, we first obtain new asymptotic properties of the positive solutions of DDE (1). Then, we improve these asymptotic properties by using an iterative technique. Finally, we use these new properties to study the oscillatory behavior of the solutions of (1). Our results in this paper extend and complement the results in [17–19].

Remark 1. *Note that, in (1), the delay appears only in the solution ψ, but not in its derivatives, which makes it quite a special high-order DDE.*

2. Main Results

For brevity, we denote the set of all eventually positive solutions of (1) by S^+. Moreover, we define the operators φ_m by:

$$\varphi_0(u) := \int_u^{\infty} a^{-1}(\eta)\mathrm{d}\eta, \ \varphi_k(u) := \int_u^{\infty} \varphi_{m-1}(\eta)\mathrm{d}\eta, \text{ for } k = 1, 2, ..., n-2.$$

Theorem 1. *Assume that $\psi \in S^+$ and ψ satisfies:*

$$\psi^{(s)}(t)\psi^{(s+1)}(t) < 0 \ \textit{for } s = 0, 1, ..., n-2, \tag{4}$$

for $t \geq t_1 \in \mathbf{I}_0$. If:

$$\int_{t_0}^{\infty} \frac{1}{a(u)} \left(\int_{t_0}^{u} q(\eta)\mathrm{d}\eta \right) \mathrm{d}u = \infty \tag{5}$$

and there exists a $\delta_0 \in (0,1)$ *such that:*

$$q(t)\varphi_{n-2}^{2}(t)\varphi_{n-3}^{-1}(t) \geq \delta_0 \tag{6}$$

then:

$(\mathbf{B}_1)$ $(-1)^{\kappa+1}\psi^{(n-\kappa-2)}(t) \leq a(t)\psi^{(n-1)}(t)\varphi_\kappa(t)$ *for* $\kappa = 0,1,...,n-2$;
$(\mathbf{B}_2)$ $\lim_{t\to\infty}\psi(t) = 0$;
$(\mathbf{B}_3)$ ψ/φ_{n-2} *is increasing;*
$(\mathbf{B}_4)$ $\psi/\varphi_{n-2}^{\delta_0}$ *is decreasing;*
$(\mathbf{B}_5)$ $\lim_{t\to\infty}\psi(t)/\varphi_{n-2}^{\delta_0}(t) = 0$.

Proof. Assume that $\psi \in S^+$ and satisfies (4) for $t \geq t_1$ for some $t_1 \in \mathbf{I}_0$. Then, there is a $t_2 \geq t_1$ with $\psi(g(t)) > 0$ for all t_2, and hence, from (1),

$$\left(a(t)\psi^{(n-1)}(t)\right)' = -q(t)\psi(g(t)) \leq 0.$$

$(\mathbf{B}_1)$: Using (4), we have that:

$$a(t)\psi^{(n-1)}(t)\varphi_0(t) \geq \int_t^\infty \frac{a(\eta)\psi^{(n-1)}(\eta)}{a(\eta)}\mathrm{d}\eta \geq -\psi^{(n-2)}(t),$$

or equivalently:

$$\psi^{(n-2)}(t) \geq -a(t)\psi^{(n-1)}(t)\varphi_0(t).$$

Integrating this relationship $n-2$-times over $[t,\infty)$ and taking into account the behavior of the derivatives in (4), we arrive at $(\mathbf{B}_1)$.
$(\mathbf{B}_2)$: Since ψ is positive decreasing, we obtain that $\lim_{t\to\infty}\psi(t) = k \geq 0$. Assume the contrary, that $k > 0$. Then, there is a $t_2 \geq t_1$ with $\psi(t) \geq k$ for $t \geq t_2$. Then, (1) becomes $\left(a(t)\psi^{(n-1)}(t)\right)' \leq -kq(t)$. Integrating this inequality twice over $[t_2,t)$, we obtain:

$$a(t)\psi^{(n-1)}(t) - a(t_2)\psi^{(n-1)}(t_2) \leq -k\int_{t_2}^{t} q(\eta)\mathrm{d}\eta.$$

From (4), we have $\psi^{(n-1)}(t) < 0$ for $t \geq t_1$. Then, $a(t_2)\psi^{(n-1)}(t_2) < 0$, and so:

$$\psi^{(n-1)}(t) \leq -\frac{k}{a(t)}\int_{t_2}^{t} q(\eta)\mathrm{d}\eta,$$

and then:

$$\psi^{(n-2)}(t) \leq \psi^{(n-2)}(t_2) - k\int_{t_2}^{t}\frac{1}{a(u)}\left(\int_{t_2}^{u} q(\eta)\mathrm{d}\eta\right)\mathrm{d}u,$$

which with (5) gives $\lim_{t\to\infty}\psi^{(n-2)}(t) = -\infty$, a contradiction with the positivity of $\psi^{(n-2)}(t)$. Therefore, $\psi(t)$ converges to *zero*.
$(\mathbf{B}_3)$: Using $(\mathbf{B}_1)$ at $\kappa = 0$, we obtain that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\psi^{(n-2)}(t)}{\varphi_0(t)} = \frac{1}{\varphi_0^2(t)}\left(\varphi_0(t)\psi^{(n-1)}(t) + a^{-1}(t)\psi^{(n-2)}(t)\right) \geq 0,$$

which leads to:

$$-\psi^{(n-3)}(t) \geq \int_t^\infty \varphi_0(\eta)\frac{\psi^{(n-2)}(\eta)}{\varphi_0(\eta)}\mathrm{d}\eta \geq \frac{\psi^{(n-2)}(t)}{\varphi_0(t)}\varphi_1(t).$$

This implies:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\psi^{(n-3)}(t)}{\varphi_1(t)} = \frac{1}{\varphi_1^2(t)}\Big(\varphi_1(t)\psi^{(n-2)}(t) + \varphi_0(t)\psi^{(n-3)}(t)\Big) \leq 0.$$

By repeating a similar approach, we obtain $(\mathbf{B}_3)$.
$(\mathbf{B}_4)$: Integrating (1) over $[t_2, t)$ and using (6), we find:

$$\begin{aligned} a(t)\psi^{(n-1)}(t) &= a(t_2)\psi^{(n-1)}(t_2) - \int_{t_2}^{t} q(\eta)\psi(g(\eta))\mathrm{d}\eta \\ &\leq a(t_2)\psi^{(n-1)}(t_2) - \psi(t)\int_{t_2}^{t} q(\eta)\mathrm{d}\eta \\ &\leq a(t_2)\psi^{(n-1)}(t_2) - \delta_0\psi(t)\int_{t_2}^{t}\frac{\varphi_{n-3}(\eta)}{\varphi_{n-2}^2(\eta)}\mathrm{d}\eta \\ &\leq a(t_2)\psi^{(n-1)}(t_2) + \delta_0\frac{\psi(t)}{\varphi_{n-2}(t_2)} - \delta_0\frac{\psi(t)}{\varphi_{n-2}(t)}, \end{aligned}$$

which, with $(\mathbf{B}_2)$, gives:

$$a(t)\psi^{(n-1)}(t) \leq -\delta_0\frac{\psi(t)}{\varphi_{n-2}(t)}. \tag{7}$$

Thus, from $(\mathbf{B}_1)$ at $\kappa = n-3$, we obtain:

$$\frac{\psi'(t)}{\varphi_{n-3}(t)} \leq -\delta_0\frac{\psi(t)}{\varphi_{n-2}(t)}.$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\psi(t)}{\varphi_{n-2}^{\delta_0}(t)} = \frac{1}{\varphi_{n-2}^{\delta_0+1}(t)}\big(\varphi_{n-2}(t)\psi'(t) + \delta_0\varphi_{n-3}(t)\psi(t)\big) \leq 0.$$

$(\mathbf{B}_5)$: Now, since $\psi/\varphi_{n-2}^{\delta_0}$ is a positive decreasing function, we see that $\lim_{t\to\infty}\psi(t)/\varphi_{n-2}^{\delta_0}(t) = k_1 \geq 0$. Assume the contrary, that $k_1 > 0$. Then, there is a $t_2 \geq t_1$ with $\psi(t)/\varphi_{n-2}^{\delta_0}(t) \geq k_1$ for $t \geq t_2$. Next, we define:

$$F(t) := \frac{\psi(t) + a(t)\psi^{(n-1)}(t)\varphi_{n-2}(t)}{\varphi_{n-2}^{\delta_0}(t)}.$$

Then, from $(\mathbf{B}_1)$, $F(t) > 0$ for $t \geq t_2$. Differentiating $F(t)$ and using (6) and $(\mathbf{B}_1)$, we obtain:

$$\begin{aligned} F'(t) &= \frac{1}{\varphi_{n-2}^{2\delta_0}(t)}\Big[\varphi_{n-2}^{\delta_0}(t)\Big(\psi'(t) - a(t)\psi^{(n-1)}(t)\varphi_{n-3}(t) + \Big(a(t)\psi^{(n-1)}(t)\Big)'\varphi_{n-2}(t)\Big) \\ &\quad +\delta_0\varphi_{n-2}^{\delta_0-1}(t)\varphi_{n-3}(t)\Big(\psi(t) + a(t)\psi^{(n-1)}(t)\varphi_{n-2}(t)\Big)\Big] \\ &\leq \frac{1}{\varphi_{n-2}^{\delta_0+1}(t)}\Big[-\varphi_{n-2}^2(t)q(t)\psi(g(t)) + \delta_0\varphi_{n-3}(t)\Big(\psi(t) + a(t)\psi^{(n-1)}(t)\varphi_{n-2}(t)\Big)\Big] \\ &\leq \frac{1}{\varphi_{n-2}^{\delta_0+1}(t)}\Big[-\delta_0\varphi_{n-3}(t)\psi(t) + \delta_0\varphi_{n-3}(t)\psi(t) + \delta_0\varphi_{n-3}(t)a(t)\psi^{(n-1)}(t)\varphi_{n-2}(t)\Big] \\ &\leq \frac{\delta_0}{\varphi_{n-2}^{\delta_0}(t)}\varphi_{n-3}(t)a(t)\psi^{(n-1)}(t). \end{aligned} \tag{8}$$

Using the fact that $\psi(t)/\varphi_{n-2}^{\delta_0}(t) \geq k_1$ with (7), we obtain:

$$a(t)\psi^{(n-1)}(t) \leq -\delta_0\frac{\psi(t)}{\varphi_{n-2}(t)} \leq -\delta_0 k_1\varphi_{n-2}^{\delta_0-1}(t). \tag{9}$$

Combining (8) and (9), we obtain:

$$F'(t) \leq -\delta_0^2 k_1 \frac{\varphi_{n-3}(t)}{\varphi_{n-2}(t)} < 0.$$

Integrating this inequality over $[t_2, t)$, we find:

$$-F(t_2) \leq -\delta_0^2 k_1 \log \frac{\varphi_{n-2}(t_2)}{\varphi_{n-2}(t)} \to \infty \text{ as } t \to \infty.$$

Then, we arrive at a contradiction, and so, $k_1 = 0$.
Therefore, the proof is complete. □

Theorem 2. *Assume that $\psi \in S^+$, ψ satisfies (4), and that (5) and (6) hold for some $\delta_0 \in (0,1)$. If $\delta_{i-1} \leq \delta_i < 1$ for all $i = 1, 2, ..., m-1$, then:*

$$(\mathbf{B}_{1,m})\ \psi/\varphi_{n-2}^{\delta_m} \text{ is decreasing;}$$
$$(\mathbf{B}_{2,m})\ \lim_{t\to\infty} \psi(t)/\varphi_{n-2}^{\delta_m}(t) = 0,$$

where:

$$\delta_j = \delta_0 \frac{\lambda^{\delta_{j-1}}}{1-\delta_{j-1}},\quad j = 1, 2, ..., m,$$

and:

$$\frac{\varphi_{n-2}(g(t))}{\varphi_{n-2}(t)} \geq \lambda,\ \text{for all } t \geq t_0, \tag{10}$$

for some $\lambda \geq 1$.

Proof. Assume that $\psi \in S^+$ and satisfies (4) for $t \geq t_1$ for some $t_1 \in \mathbf{I}_0$. Then, from Theorem 1, we have that $(\mathbf{B}_1) - (\mathbf{B}_5)$ hold. Using induction, we have from Theorem 1 that $(\mathbf{B}_{1,0})$ and $(\mathbf{B}_{2,0})$ hold. Now, we assume that $(\mathbf{B}_{1,m-1})$ and $(\mathbf{B}_{2,m-1})$ hold. Integrating (1) over $[t_2, t)$, we find:

$$a(t)\psi^{(n-1)}(t) = a(t_2)\psi^{(n-1)}(t_2) - \int_{t_2}^{t} q(\eta)\psi(g(\eta))\mathrm{d}\eta. \tag{11}$$

Using $(\mathbf{B}_{1,m-1})$, we have that:

$$\psi(g(t)) \geq \varphi_{n-2}^{\delta_{m-1}}(g(t)) \frac{\psi(t)}{\varphi_{n-2}^{\delta_{m-1}}(t)}.$$

Then, (11) becomes:

$$a(t)\psi^{(n-1)}(t) \leq a(t_2)\psi^{(n-1)}(t_2) - \int_{t_2}^{t} q(\eta)\varphi_{n-2}^{\delta_{m-1}}(g(\eta)) \frac{\psi(\eta)}{\varphi_{n-2}^{\delta_{m-1}}(\eta)}\mathrm{d}\eta.$$

which, with the fact that $\psi/\varphi_{n-2}^{\delta_{m-1}}$ is a decreasing function, gives:

$$a(t)\psi^{(n-1)}(t) \leq a(t_2)\psi^{(n-1)}(t_2) - \frac{\psi(t)}{\varphi_{n-2}^{\delta_{m-1}}(t)} \int_{t_2}^{t} q(\eta)\varphi_{n-2}^{\delta_{m-1}}(\eta) \frac{\varphi_{n-2}^{\delta_{m-1}}(g(\eta))}{\varphi_{n-2}^{\delta_{m-1}}(\eta)}\mathrm{d}\eta.$$

Hence, from (6) and (10), we obtain:

$$\begin{aligned} a(t)\psi^{(n-1)}(t) &\leq a(t_2)\psi^{(n-1)}(t_2) - \delta_0\lambda^{\delta_{m-1}}\frac{\psi(t)}{\varphi_{n-2}^{\delta_{m-1}}(t)}\int_{t_2}^{t}\frac{\varphi_{n-3}(\eta)}{\varphi_{n-2}^{2-\delta_{m-1}}(\eta)}\mathrm{d}\eta \\ &= a(t_2)\psi^{(n-1)}(t_2) - \delta_0\frac{\lambda^{\delta_{m-1}}}{1-\delta_{m-1}}\frac{\psi(t)}{\varphi_{n-2}^{\delta_{m-1}}(t)}\left(\frac{1}{\varphi_{n-2}^{1-\delta_{m-1}}(t)} - \frac{1}{\varphi_{n-2}^{1-\delta_{m-1}}(t_2)}\right) \\ &= a(t_2)\psi^{(n-1)}(t_2) + \delta_m\frac{\psi(t)}{\varphi_{n-2}^{\delta_{m-1}}(t)}\frac{1}{\varphi_{n-2}^{1-\delta_{m-1}}(t_2)} - \delta_m\frac{\psi(t)}{\varphi_{n-2}(t)}, \end{aligned}$$

which, with the fact that $\lim_{t\to\infty}\psi(t)/\varphi_{n-2}^{\delta_{m-1}}(t) = 0$, gives:

$$a(t)\psi^{(n-1)}(t) \leq -\delta_m\frac{\psi(t)}{\varphi_{n-2}(t)} \tag{12}$$

Thus, from $(\mathbf{B}_1)$ at $\kappa = n-3$, we obtain:

$$\frac{\psi'(t)}{\varphi_{n-3}(t)} \leq -\delta_m\frac{\psi(t)}{\varphi_{n-2}(t)}.$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\psi(t)}{\varphi_{n-2}^{\delta_m}(t)} = \frac{1}{\varphi_{n-2}^{\delta_m+1}(t)}\left(\varphi_{n-2}(t)\psi'(t) + \delta_m\varphi_{n-3}(t)\psi(t)\right) \leq 0.$$

Proceeding as in the proof of $(\mathbf{B}_5)$ in Theorem 1, we can prove that $\lim_{t\to\infty}\psi(t)/\varphi_{n-2}^{\delta_m}(t) = 0$.

Therefore, the proof is complete. □

Theorem 3. *Assume that $\psi \in S^+$, ψ satisfies (4), and that (5) and (6) hold for some $\delta_0 \in (0,1)$. If $\delta_{i-1} \leq \delta_i < 1$ for all $i = 1,2,...,m-1$, then the DDE:*

$$H'(t) + \frac{1}{1-\delta_m}q(t)\varphi_{n-2}(t)H(g(t)) = 0, \tag{13}$$

has a positive solution, where δ_j and λ are defined as in Theorem 2.

Proof. Assume that $\psi \in S^+$ and satisfies (4) for $t \geq t_1$ for some $t_1 \in \mathbf{I}_0$. Then, from Theorem 2, we have that $(\mathbf{B}_{1,m})$ and $(\mathbf{B}_{2,m})$ hold.

Now, we define:

$$H(t) := a(t)\psi^{(n-1)}(t)\varphi_{n-2}(t) + \psi(t). \tag{14}$$

Then, from $(\mathbf{B}_1)$ at $\kappa = n-2$, $H(t) > 0$ for $t \geq t_2$, and:

$$H'(t) = \left(a(t)\psi^{(n-1)}(t)\right)'\varphi_{n-2}(t) - a(t)\psi^{(n-1)}(t)\varphi_{n-3}(t) + \psi'(t),$$

which, with $(\mathbf{B}_1)$ at $\kappa = n-3$, leads to:

$$H'(t) \leq \left(a(t)\psi^{(n-1)}(t)\right)'\varphi_{n-2}(t) \leq -q(t)\varphi_{n-2}(t)\psi(g(t)). \tag{15}$$

As in the proof of Theorem 2, we arrive at (12). From (14) and (12), we obtain:

$$H(t) \leq (1-\delta_m)\psi(t).$$

Thus, (15) becomes:

$$H'(t) + \frac{1}{1-\delta_m} q(t)\varphi_{n-2}(t)H(g(t)) \leq 0. \tag{16}$$

Hence, H is a positive solution of the differential inequality (16). Using [22] (Theorem 1), Equation (13) has also a positive solution, and this completes the proof. □

3. Applications in Oscillation Theory

In the following, we use our results in the previous section to obtain the criteria of the oscillation for the solutions of (1). A solution u of (1) is called *nonoscillatory* if it is eventually positive or eventually negative; otherwise, it is called *oscillatory*.

Theorem 4. *Assume that (5) and (6) hold for some $\delta_0 \in (0,1)$ and that δ_j, λ are defined as in Theorem 2. If $\delta_{i-1} \leq \delta_i < 1$ for all $i = 1, 2, ..., m-1$, and all solutions of the DDEs (13),*

$$w'(t) + q(t)\frac{\epsilon_1 g^{n-1}(t)}{(n-1)!(a(g(t)))} w(g(t)) = 0 \tag{17}$$

and:

$$\omega'(t) + \frac{\epsilon_2}{(n-2)!a(t)}\left(\int_{t_0}^{t} q(\eta)g^{n-2}(\eta)\mathrm{d}\eta\right)\omega(g(t)) = 0, \tag{18}$$

are oscillatory, for some $\epsilon_1, \epsilon_2, \delta_m \in (0,1)$, then every solution of (1) is oscillatory.

Proof. Assume the contrary, that $\psi \in S^+$. Then, from [23], we have the following three cases, eventually:
(**i**) $\psi^{(j)}(t) > 0$ for $j = 0, 1, n-1$ and $\psi^{(n)}(t) < 0$;
(**ii**) $\psi^{(j)}(t) > 0$ for $j = 0, 1, n-2$ and $\psi^{(n-1)}(t) < 0$;
(**iii**) $(-1)^j \psi^{(j)}(t) > 0$ for $j = 0, 1, ..., n-1$.

In view of [19] (Theorem 3) , the fact that the solutions of Equations (17) and (18) oscillate rules out the cases (**i**) and (**ii**), respectively. Then, we have that (**iii**) hold. Using Theorem 3, we obtain that Equation (13) has a positive solution, a contradiction. Therefore, the proof is complete. □

Corollary 1. *Assume that (5) and (6) hold for some $\delta_0 \in (0,1)$ and that δ_j, λ are defined as in Theorem 2. If $\delta_{i-1} \leq \delta_i < 1$ for all $i = 1, 2, ..., m-1$,*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} q(\eta)\varphi_{n-2}(\eta)\mathrm{d}\eta > \frac{1-\delta_m}{\mathrm{e}}, \tag{19}$$

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{1}{a(g(\eta))} q(\eta)g^{n-1}(\eta)\mathrm{d}\eta > \frac{(n-1)!}{\mathrm{e}}, \tag{20}$$

and:

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{1}{a(u)}\left(\int_{t_0}^{u} q(\eta)g^{n-2}(\eta)\mathrm{d}\eta\right)\mathrm{d}u > \frac{(n-2)!}{\mathrm{e}}, \tag{21}$$

for some $\epsilon, \delta_m \in (0,1)$, then every solution of (1) is oscillatory.

Proof. In view of [24] (Corollary 2.1) , Conditions (19)–(21) imply the oscillation of the solutions of (13), (17), and (18), respectively. Therefore, from Theorem 4, every solution of (1) is oscillatory. □

Example 1. *Consider the DDE:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\psi\right)\right) + q_0\mathrm{e}^t\psi(g_0 t) = 0, \tag{22}$$

where $t \geq 1$, $q_0 \in (0,1)$ *and* $g_0 \in (0,1)$. *It is easy to verify that* $\varphi_i(u) = \mathrm{e}^{-t}$, $i = 0,1,2$, *and then:*

$$\int_{t_0}^{\infty} \frac{1}{a(u)} \left(\int_{t_0}^{u} q(\eta) \mathrm{d}\eta \right) \mathrm{d}u = q_0 \int_{1}^{\infty} \frac{1}{\mathrm{e}^u} (\mathrm{e}^u - \mathrm{e}) \mathrm{d}u = \infty.$$

By choosing $\delta_0 = q_0$, *we find that (6) holds. Moreover, by a simple computation, we have that (20) and (21) are satisfied. Now, using Corollary 1, every solution of (22) is oscillatory if (19) holds, that is,*

$$q_0 \log \frac{1}{g_0} > \frac{1 - \delta_m}{\mathrm{e}}. \tag{23}$$

Remark 2. *By reviewing the results in [18,20], we have that Equation (22) is oscillatory if* $q_0 > 1/4$. *It is easy to note that this condition essentially neglects the influence of delay argument* $g(t)$. *However, our criterion (23) takes into account the influence of* g_0. *Furthermore, using (23), every solution of the DDE:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^t \left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \psi \right) \right) + \frac{\mathrm{e}^t}{5} \psi \left(\frac{t}{5} \right) = 0$$

is oscillatory, despite the failure of the results [18,20].

Remark 3. *Consider the DDE:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^t \left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \psi \right) \right) + 0.185 \mathrm{e}^t \psi \left(\frac{t}{5} \right) = 0. \tag{24}$$

Note that, the condition (23), with $m = 0$, *reduces to* $0.185 \log 5 > \frac{0.815}{\mathrm{e}}$, *which is not satisfied, and thus, the oscillatory behavior of (24) cannot be verified. However, using the iterative nature of (23), we find that:*

$$\mathrm{e}^{(1-g_0)t} > \mathrm{e}(1 - g_0)t \geq \mathrm{e}(1 - g_0) := \lambda, \text{ for all } t \geq 1,$$

and $\delta_1 = 0.26208$. *Now, the condition (23) with* $m = 1$ *reduces to* $0.185 \log 5 > \frac{0.73792}{\mathrm{e}}$, *which is satisfied. Hence, every solution of (24) is oscillatory.*

Remark 4. *By using comparison principles, Baculíková et al. [19] studied the oscillatory properties of DDE (3). In order to rule out the existence of positive solutions in class (4), they assumed that there is a* $\xi \in C^1([t_0, \infty))$ *with* $\xi(t) > t$, $\xi'(t) \geq 0$ *and* $\xi_{n-2}(\sigma(t)) < t$ *such that the DDE:*

$$z'(t) + \frac{1}{a(t)} \left(\int_{t_0}^{t} q(\eta) \mathrm{d}\eta \right) J_{n-2}(g(t))(z(\xi_{n-2}(g(t)))) = 0$$

is oscillatory, where:

$$\xi_1 = \xi,\ \xi_{i+1} = \xi_i \circ \xi,\ J_1 = \xi - t \text{ and } J_{i+1}(t) = \int_t^{\xi} J_i(\eta) \mathrm{d}\eta,$$

for $i = 1, 2, ..., n - 3$. *Since there is no general rule as to how to choose* ξ *and* ξ_i *satisfying the imposed conditions, our results in this paper improve the results in [19], as our results do not require unknown functions.*

4. Conclusions

In this work, new results of studying the oscillatory behavior of a class of even-order DDEs were presented. In the noncanonical case, using the principles of comparison, we obtained new criteria that guarantee the oscillation of all solutions of the studied equation. By comparing with previous results in the literature that used the same approach, we found that our results are easy to apply and do not require unknown functions. Moreover, the new criteria have an iterative nature. An interesting problem is to extend our results to even-order DDEs of the neutral type.

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