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Article

The EOQ Model for Deteriorating Items with a Conditional Trade Credit Linked to Order Quantity in a Supply Chain System

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Abstract: For generality, we observed that some of the optimization methods lack the mathematical rigor and some of them are based on intuitive arguments which result in the solution procedures being questionable from logical viewpoints of a mathematical analysis such as those in the work by Ouyang et al. (2009). They consider an economic order quantity model for deteriorating items with partially permissible delays in payments linked to order quantity. Basically, their inventory models are interesting, however, they ignore explorations of interrelations of functional behaviors (continuity, monotonicity properties, differentiability, et cetera) of the total cost function to locate the optimal solution, so those shortcomings will naturally influence the implementation of their considered inventory model. Consequently, the main purpose of this paper is to provide accurate and reliable mathematical analytic solution procedures for different scenarios that overcome the shortcomings of Ouyang et al.

Keywords: inventory modeling; mathematical analytic solution procedures; economic order quantity (EOQ) model; deteriorating products; trade-credit financing; partially permissible delay in payments; object function (that is, total annual cost function); supply chain management

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1. Introduction

Deterioration plays an essential role in many inventory systems and deterioration refers to factors such as product damage, spoilage, dryness and evaporation which decreases the original quality and quantity of the product, so it is important to control and maintain the inventories of deteriorating items. Inventory problems for deteriorating items have been studied extensively by numerous researchers. Research in this area began with the work of Whitin [1], who considered fashion goods deteriorating at the end of their prescribed storage period. An exponentially decaying inventory was first developed by Ghare and Schrader [2]. A considerable amount of work has been conducted on deteriorating inventory systems, the details of which can be found in review articles by Nahmias [3], the perishable inventory theory, Raafat [4], Goyal and Giri [5], Bakker et al. [6], Mahata [7] and Janssen et al. [8].

Due to huge competition among the business enterprises in the local and in the global market, the business enterprises adopt various tolls to sell their products efficiently. The trade credit policy is one of the most effective promotional tools to push a product. In practice, suppliers usually offer some credit periods to retailers to stimulate the demand for items they produce and reduce the selling price of the item indirectly. They do not charge any interest on the outstanding amount if retailers settle their account within the permissible delay period. This brings some economic advantages to retailers because they earn interest from the revenue realized during the stated period. Offing a trade credit to the small and micro-retailers is commonplace and acceptable as these retailers lack the financial means to pay in full upon the receipt of the items. Meanwhile, the supplier prefers to provide better terms of trade credit such as a payment extension date when the retailer orders a large enough quantity. Furthermore, the trade credit plays a major role in the inventory system for both the supplier and the retailer. Haley and Higgins [9] introduce the first model to consider the economic order quantity (EOQ) model under conditions of a permissible delay of payments. Goyal [10] developed an EOQ model for a retailer when the supplier offers a permissible delay in payments, which differs from the viewpoint of Haley and Higgins [9]. In general, Goyal [10] is more popular than Haley and Higgins [9]; thus, Chang et al. [11] adopted the viewpoints of Goyal [10] for their review article on inventory models under trade credit. Stokes [12] indicates that trade credit represents one of the most flexible sources of short-term financing available to firms principally because it arises spontaneously with the firm's purchases. Khouja and Mehrez [13] were the first to discuss suppliers only offering a permissible delay in payment when the order quantity is larger than a predetermined quantity. Furthermore, Chang et al. [14] established an EOQ model for deteriorating items, in which the supplier provides a permissible delay to the purchaser if the order quantity is greater than or equal to a predetermined quantity. Many related articles can be found, such as those by Liao [15], Chung [16], Chung [17], Chung et al. [18], Chang et al. [19] and Liao et al. [20].

Numerous published papers assume that the supplier offers the retailer a fully permissible delay in payments independent of the order quantity. Huang [21] considers the case of a conditionally permissible delay, assuming that the supplier only offers the retailer a fully permissible delay in payments if they order more than a predetermined quantity. With the novel invention of Ghare and Schrader [2], the researchers developed various models for deteriorating items under trade credit policy in different circumstances such as Yang [22] further adopts the concept of Huang [21], considering inventory models for deteriorating items in a discount cash flow analysis under alternatives to conditionally permissible delays in payments and cash discounts. Ouyang et al. [23] incorporate the concepts of Ghare and Schrader [2], Goyal [10], Khouja and Mehrez [13], and Huang [21] to consider an EOQ model for deteriorating items with partially permissible delay in payments linked to order quantity. Liao et al. [24] explore an EOQ model for non-instantaneous deteriorating items with imperfect quality and trade credit financing. The purpose of their article is to find an optimal ordering policy for minimising the total relevant inventory cost for the retailer. For literature review, other related articles in this field were given in the articles of Taleizadeh et al. [25], Lashgari et al. [26], Taleizadeh [27], Tiwari et al. [28], Chang et al. [29], Li et al. [30] and Tiwari et al. [31].

Essentially, in order to explore the functional behaviors (such as continuity, monotonicity (increasing and decreasing) properties, differentiability, et cetera) of the object functions (that is, the total cost functions), one can and should apply the mathematically accurate and reliable solution procedures. In fact, if the object function (that is, the total cost functions) is convex, it is easier to find the optimal solution by using the convexity property. In this direction, Chung et al. [32] notice shortcomings in the solution procedure in Leung's proof based on the complete squares method used by Leung [33]. They then correct and improve the investigation by Leung [33], reiterating the well-established fact that mathematical analytical techniques guarantee accuracy as well as dependability in inventory modelling problems. Chung et al. [34] overcome the shortcomings of Chang and Teng [35] and derive all optimal solutions for the annual total relevant cost $Z(T)$; their paper also presents in detail the mathematically correct methods for deriving $Z(T)$ and locating all optimal solutions. Srivastava et al. [36] modify the annual total relevant cost $TRC(T)$ in the study of Teng et al. [37] and present the correct derivations of $TRC(T)$. They also expose logical and mathematical problems in Teng et al.'s proof of Theorem 1. Teng et al. [38] discuss two payment methods for the EPQ model; however, Chung et al. [39] find that the total annual profit for the manufacturer under payment method 1 is incorrect and provided the correct solution procedure for the correct total annual profit. Chung et al. [39] also adopt an alternative but much easier to understand method of characterising the total annual profit, and provide the correct solution algorithm for the total annual profit.

Based upon above arguments, we have observed that some of the optimization methods lack the mathematical rigor and some of them are based on intuitive arguments which result in the solution procedures are questionable from logical viewpoints of mathematical analysis such as Ouyang et al. [23]. They ignored explorations of interrelations of functional behaviors of the total cost function to locate the optimal solution, so those shortcomings will naturally influence the implementation of their considered inventory model. For this reason, it is worth mentioning that, by the usage of the mathematical analytic solution procedures, to overcome the shortcomings of Ouyang et al. [23]. Although Ping [40] explores the optimal solutions from the mathematical points to overcome the shortcomings of Ouyang et al. [23], we will present more complete solution procedures for improvement.

As a summary, the present study will show that the total annual cost function is convex by exploring the functional properties of the total annual cost function such as (for example) the continuity, convexity, monotonicity (increasing and decreasing) and differentiability properties.

Ouyang et al. [23] at least have the following shortcomings about the theoretical results and solution procedures.

(1) Theorem B (Varberg et al. [41], page 164) can be stated as follows:

Theorem B (Second Derivative Test)

Let f' and f'' exist at every point in an open interval (a, b) containing c, and suppose that $f'(c) = 0$.

(i) If $f''(c) < 0$, then $f(c)$ is a local maximum value of f.
(ii) If $f''(c) > 0$, then $f(c)$ is a local maximum value of f.

So, Theorem B(ii) demonstrates that Equations (A4) and (B4) in Ouyang et al. [23] only assure that T_1 and T_3 are local minimum points of both $TRC_1(T)$ and $TRC_3(T)$, respectively. Many examples (Varberg et al. [41]) show that Theorem B cannot draw a conclusion about maxima or minima without more information in general.

Consequently, although the results of Lemmas 1–8 and Theorems 1–3 in Ouyang et al. [23] are correct, the processes of proofs of them have shortcomings. Equations (A17a–c), (A26a–c), (A33a–c), (A43a–c) and (A53a–c) reveal the correct ways of discussions about the minimum points T_1, T_2, T_3, T_4 and T_5 of $TRC_i(T)$ $(i = 1, 2, 3, 4, 5)$.

(2) Ouyang et al. [23] do not demonstrate why Equations (43) and (44) hold. Lemma 1(B) in this paper overcomes these shortcomings.
(3) Equations (36) and (42) in this paper reveal that Equation (25) in Ouyang et al. [23] is wrong. The correct formulation of Δ_6 should be Equation (45) in this paper. Furthermore, Ouyang et al. [23] do not demonstrate why Equations (46) and (47) hold. Lemma 1(A, B, H) in this paper overcome these shortcomings.
(4) Equations (36) and (52) in this paper reveal that Equation (30) in Ouyang et al. [23] is wrong. The correct formulation of Δ_9 should be equation (53) in this paper. Furthermore,

Ouyang et al. [23] do not demonstrate why Equations (56) and (57) hold. However, Lemma 1(A, B) and Equations (53)–(55) imply that Equations (56) and (57) hold.

(5) Since Equation (30) in Ouyang et al.'s [23] is wrong, it should not be true in general that Ouyang et al. [23] conclude $\Delta_9 \geq \Delta_7$ and $\Delta_7 \geq \Delta_5$. Therefore, it influences the validity of Theorem 3 in Ouyang et al.'s [23]. Theorem 4 in this paper gives the correct results about Case 3.

(6) Ouyang et al. [23] do not explore the convexity of $TRC_i(T)$. However, Theorem 1 in this paper provides the proof of the convexity of $TRC_i(T)$ $(i = 1, 2, 3, 4, 5)$.

(7) From the discussion about Case 3 in Ouyang et al.'s [23], $TRC(T_0) = TRC_4(T_0)$. If $TRC_5(T_0) \neq TRC_4(T_0)$, then $TRC(T_0) \neq TRC_5(T_0)$. Therefore, if $TRC_5(T_0) \neq TRC_4(T_0)$, Equation (15) in Ouyang et al.'s [23] is invalid. The correct formulations of $TRC(T)$ for Case 3 should be Equation (18a–d) in this paper.

2. Mathematical Formulation

The fallowing notation and assumptions are used in the whole paper.

Assumptions

1. Replenishments are instantaneous.
2. Demand rate, D, is known and constant.
3. Shortages are not allowed.
4. The inventory system involves only one type of inventory.
5. The time horizon is infinite.
6. If $Q < W$, the partially delayed payment is permitted. Otherwise, the fully delayed payment is permitted. Hence, if $Q \geq W$, pay cQ after M time periods from the time the order is filled. Otherwise, as the order is filled, the retailer must make a partial payment, $(1-\alpha)cDT$, to the supplier. Then, the retailer must pay off the remaining balances, αcDT, at the end of the trade credit period. This assumption constitutes the major difference of the proposed model from previous ones.
7. During the time period that the account is not settled, the generated sales revenue is deposited in an interest-bearing account.
8. $I_k \geq I_e$.

Indeed, based upon the above detailed notations and assumptions, we present a rather brief description of the model used Ouyang et al. [23].

The inventory level decreases owing to demand as well as deterioration. Thus, the change of inventory level can be represented by the following differential equation:

$$\frac{dI(t)}{dt} + \theta I(t) = -D, \qquad 0 < t < T, \tag{1}$$

with the boundary condition $I(T) = 0$. The solution of Equation (1) is:

$$I(t) = \frac{D}{\theta}[e^{\theta(T-t)} - 1], \qquad 0 < t < T. \tag{2}$$

Hence, the order quantity for each cycle is:

$$Q = I(0) = \frac{D}{\theta}\left(e^{\theta T} - 1\right). \tag{3}$$

From Equation (3), we can obtain the time interval that W units are depleted to zero due to both demand and deterioration as:

$$T_W = \frac{1}{\theta}\ln(\frac{D}{\theta}W + 1). \tag{4}$$

If $Q \geq W$ (i.e., $T \geq T_W$), then a fully delayed payment is permitted. Otherwise, the partially delayed payment is permitted. Hence, if $Q < W$ (i.e., $T < T_W$), then the retailer must take a loan (with the interest charged of I_k) to pay the supplier the partial payment of $(1-\alpha)cQ$ when the order is filled at time 0. From the constant sale revenue pD, the retailer will be able to pay off the loan $(1-\alpha)cQ$ at time $(1-\alpha)(c/p)(e^{\theta T}-1)/\theta$.

Note that (1) if $T \geq T_W$, and the payoff time of the partial payment at $(1-\alpha)(c/p)(e^{\theta T}-1)/\theta$ is shorter or equal to the permissible delay M, then $T \leq T_0 \equiv \frac{1}{\theta}\ln\left(\frac{\theta\, pM}{(1-\alpha)c}+1\right)$, and vice versa.

Ouyang et al. [23] assume:

$$T_0 > M. \tag{5}$$

After that, based on the values of M, T_W and T_0, we had three possible cases: (1) $T_0 > M \geq T_W$, (2) $T_0 \geq T_W > M$ and (3) $T_W \geq T_0 > M$.

Case 1. $T_0 > M \geq T_W$

Ouyang et al. [23] reveal that the annual total relevant cost for the retailer in Case 1 can be expressed as:

$$TRC(T) = \begin{cases} TRC_1(T) & if \quad M \leq T & (6a) \\ TRC_2(T) & if \quad T_W \leq T \leq M & (6b) \\ TRC_3(T) & if \quad 0 < T < T_W & (6c) \end{cases}$$

where

$$TRC_1(T) = \frac{A}{T} + \frac{(c\theta+h)D}{\theta^2 T}(e^{\theta T}-\theta T-1) + \frac{cI_kD}{\theta^2 T}\left[e^{\theta(T-M)}-\theta(T-M)-1\right] - \frac{pI_eDM^2}{2T} \tag{7}$$

$$TRC_2(T) = \frac{A}{T} + \frac{(c\theta+h)D}{\theta^2 T}(e^{\theta T}-\theta T-1) - pI_eD(M-\frac{T}{2}) \tag{8}$$

$$\begin{aligned} TRC_3(T) = \; & \tfrac{A}{T} + \tfrac{(c\theta+h)D}{\theta^2 T}(e^{\theta T}-\theta T-1) + \tfrac{cI_k(c/p)(1-\alpha)^2D}{2\theta^2 T}(e^{\theta T}-1)^2 \\ & -\tfrac{pI_eD}{2T}[T-(1-\alpha)(c/p)(e^{\theta T}-1)/\theta]^2 - \tfrac{pI_eD(M-T)}{T}[T-(1-\alpha)(c/p)(e^{\theta T}-1)/\theta] \end{aligned} \tag{9}$$

Equations (7)–(9) show:

$$TRC_1(M) = TRC_2(M), \tag{10}$$

and

$$\begin{aligned} TRC_3(T) - TRC_2(T) = \; & \tfrac{cD(I_k-I_e)(\frac{c}{p})(1-\alpha)^2}{2\theta^2 T}(e^{\theta T}-1)^2 + pI_eD(1-\alpha)(\tfrac{c}{p})\tfrac{(e^{\theta T}-1)}{\theta} \\ & +\tfrac{pI_eD(M-T)(1-\alpha)c}{p\theta T}(e^{\theta T}-1) \\ & > 0 \quad if \quad 0 < T \leq M. \end{aligned} \tag{11}$$

Therefore,

$$TRC_3(T_W) > TRC_2(T_W). \tag{12}$$

Therefore, $TRC(T)$ is continuous except at $T = T_W$.

Case 2. $T_0 \geq T_W > M$

Similar to the approach used in Case 1, the annual total relevant cost for the retailer in this case can be expressed as:

$$TRC(T) = \begin{cases} TRC_1(T) & if \quad T_W \leq T, & (13a) \\ TRC_4(T) & if \quad M \leq T < T_W, & (13b) \\ TRC_3(T) & if \quad T \leq M. & (13c) \end{cases}$$

where

$$\begin{aligned} TRC_4(T) = \quad & \tfrac{A}{T} + \tfrac{(c\theta+h)D}{\theta^2 T}(e^{\theta T} - \theta T - 1) + \tfrac{cI_k(c/p)(1-\alpha)^2 D}{2\theta^2 T}(e^{\theta T} - 1)^2 \\ & + \tfrac{cI_k D}{\theta^2 T}[e^{\theta(T-M)} - \theta(T-M) - 1] \\ & - \tfrac{pI_e D}{2T}[M - (1-\alpha)(c/p)(e^{\theta T} - 1)/\theta]^2 \end{aligned} \tag{14}$$

Equations (7) and (14) show:

$$\begin{aligned} TRC_4(T) - TRC_1(T) = \quad & \frac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D}{2\theta^2 T}(e^{\theta T} - 1)^2 + \frac{I_e DcM(1-\alpha)}{\theta T}(e^{\theta T} - 1) \\ & > 0 \qquad if \quad T > 0, \end{aligned} \tag{15}$$

and

$$TRC_4(M) = TRC_3(M). \tag{16}$$

Therefore,

$$TRC_4(T_W) > TRC_1(T_W). \tag{17}$$

Therefore, $TRC(T)$ is continuous except at $T = T_W$.

Case 3. $T_W \geq T_0 > M$

Similar to the approach in Case 1, the annual total relevant cost for the retailer in Case 3 is:

$$TRC(T) = \begin{cases} TRC_1(T) & \textit{if} \quad T_W < T, & (18a) \\ TRC_5(T) & \textit{if} \quad T_0 < T \leq T_W, & (18b) \\ TRC_4(T) & \textit{if} \quad M \leq T \leq T_0, & (18c) \\ TRC_3(T) & \textit{if} \quad T \leq M. & (18d) \end{cases}$$

where

$$\begin{aligned} TRC_5(T) = \quad & \tfrac{A}{T} + \tfrac{(c\theta+h)D}{\theta^2 T}(e^{\theta T} - \theta T - 1) + \tfrac{cI_k(c/p)(1-2\alpha+2\alpha^2)D}{2\theta^2 T}(e^{\theta T} - 1)^2 \\ & + \tfrac{cI_k\alpha D(e^{\theta T}-1)}{\theta T}[(1-\alpha)(c/p)(e^{\theta T} - 1)/\theta - M]. \end{aligned} \tag{19}$$

Since $TRC_3(M) = TRC_4(M)$, $TRC_4(T_0) \neq TRC_5(T_0)$ and $TRC_5(T_W) \neq TRC_1(T_W)$, $TRC(T)$ is continuous except at $T = T_0$ and T_W.

3. The Functional Behaviors of $TRC_i(T)(i = 1 \sim 5)$

Lemma 1.

(A) $\theta Te^{\theta T} - e^{\theta T} + 1$ *is increasing on* $T > 0$.
(B) $\theta Te^{\theta T} - e^{\theta T} + 1 > 0$ *if* $T > 0$.
(C) $\theta^2 T^2 e^{\theta T} - 2\theta Te^{\theta T} + 2e^{\theta T} - 2$ *is increasing on* $T > 0$.
(D) $\theta^2 T^2 e^{\theta T} - 2\theta Te^{\theta T} + 2e^{\theta T} - 2 > 0$ *if* $T > 0$.
(E) $4\theta^2 T^2 e^{2\theta T} - 2\theta^2 T^2 e^{\theta T} - 2(e^{\theta T} - 1)(2\theta Te^{\theta T} - e^{\theta T} + 1)$ *is increasing on* $T > 0$.
(F) $4\theta^2 T^2 e^{2\theta T} - 2\theta^2 T^2 e^{\theta T} - 2(e^{\theta T} - 1)(2\theta Te^{\theta T} - e^{\theta T} + 1) > 0$ *if* $T > 0$.
(G) $\theta Te^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1$ *is increasing on* $T > 0$.
(H) $\theta Te^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1 > 0$ *if* $T > M$.
(I) $\theta^2 T^2 e^{\theta(T-M)} - 2\theta Te^{\theta(T-M)} + 2e^{\theta(T-M)} + 2\theta M - 2$ *is increasing on* $T > 0$.
(J) $\theta^2 T^2 e^{\theta(T-M)} - 2\theta Te^{\theta(T-M)} + 2e^{\theta(T-M)} + 2\theta M - 2 > 0$ *if* $T > M$.

Proof. The detailed proof of Lemma 1 has been proved in Appendix A.1. □

Case 1: $T_0 > M \geq T_W$

Equations (7)–(9) yield the first-order derivatives of $TRC_i(T)(i = 1 \sim 5)$ with respect to T as follows:

$$TRC_1'(T) = \frac{-A}{T^2} + \frac{(c\theta+h)D(\theta T e^{\theta T} - \theta T + 1)}{\theta^2 T^2} + \frac{cI_k D[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1]}{\theta^2 T^2} + \frac{pI_e DM^2}{2T^2}, \tag{20}$$

$$TRC_1''(T) = \frac{2A}{T^3} + \frac{D(c\theta+h)}{\theta^2 T^3}(\theta^2 T^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2) + \frac{cI_k D}{\theta^2 T^3}(\theta^2 T^2 e^{\theta(T-M)} - 2\theta T e^{\theta(T-M)} + 2e^{\theta(T-M)} + 2\theta M - 2) - \frac{pI_e DM^2}{T^3}, \tag{21}$$

$$TRC_2'(T) = \frac{-A}{T^2} + \frac{D(c\theta + h)(\theta T e^{\theta T} - e^{\theta T} + 1)}{\theta^2 T^2} + \frac{pI_e D}{2}, \tag{22}$$

$$TRC_2''(T) = \frac{2A}{T^3} + \frac{D(c\theta + h)}{\theta^2 T^3}(\theta^2 T^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2) > 0, \tag{23}$$

$$TRC_3'(T) = \frac{-A}{T^2} + \frac{[D(c\theta+h) + \theta I_e Dc(1-\alpha)M]}{\theta^2 T^2}(\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D}{2\theta^2 T^2}(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{pI_e D}{2}, \tag{24}$$

$$TRC_3''(T) = \frac{2A}{T^3} + [\frac{D(c\theta+h) + \theta I_e Dc(1-\alpha)M}{\theta^2 T^3}](\theta^2 T^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2) + \frac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D\left[4\theta^2 T^2 e^{2\theta T} - 2\theta^2 T^2 e^{\theta T} - 2(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1)\right]}{2\theta^2 T^3} > 0, \tag{25}$$

Lemma 1(B) and Equations (20), (22) and (24) imply:

$$TRC_1'(M) = TRC_2'(M) = \frac{\Delta_1}{M^2}, \tag{26}$$

$$TRC_3'(T) > TRC_2'(T) > 0 \text{ if } T > 0, \tag{27}$$

and

$$TRC_3'(T_W) > TRC_2'(T_W), \tag{28}$$

where

$$TRC_2'(T_W) = \frac{\Delta_2}{{T_W}^2}, \tag{29}$$

$$TRC_3'(T_W) = \frac{\Delta_3}{{T_W}^2}, \tag{30}$$

$$\Delta_1 = -A + \frac{(c\theta + h)D}{\theta^2}(\theta M e^{\theta M} - e^{\theta M} + 1) + \frac{pI_e DM^2}{2}, \tag{31}$$

$$\Delta_2 = -A + \frac{(c\theta + h)D}{\theta^2}(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1) + \frac{pI_e D{T_W}^2}{2}, \tag{32}$$

and

$$\Delta_3 = -A + \frac{[D(c\theta+h) + \theta I_e Dc(1-\alpha)M]}{\theta^2}(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1) + \frac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D}{2\theta^2}(e^{\theta T_W} - 1)(2\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1) + \frac{pI_e D{T_W}^2}{2}, \tag{33}$$

Since $M \geq T_W$, Lemma1(A,B) implies:

$$\Delta_1 \geq \Delta_2 \tag{34}$$

and

$$\Delta_3 > \Delta_2 \tag{35}$$

Let T_i denote the minimum point of $TRC_i(T)$ for all $i = 1, 2, 3, 4, 5$. Then, we had the following results.

Lemma 2.

(A) *(i)* *If $\Delta_1 > 0$, then $TRC_1(T)$ is increasing on $[M, \infty)$.*
(ii) *If $\Delta_1 \leq 0$, then $T_1 \in [M, \infty)$. We also have that $TRC_1(T)$ is decreasing on $(0, T_1]$ and increasing on $[T_1, \infty)$.*

(B) *(i)* *If $\Delta_2 > 0$, then $TRC_2(T)$ is increasing on $[T_W, \infty)$.*
(ii) *If $\Delta_2 \leq 0 < \Delta_1$, then $T_2 \in [T_W, M)$. We also have that $TRC_2(T)$ is decreasing on $[T_W, T_2]$ and increasing on $[T_2, M]$.*
(iii) *If $\Delta_1 \leq 0$, then $TRC_2(T)$ is decreasing on $[T_W, M)$.*

(C) *(i)* *If $\Delta_3 > 0$, then $T_3 \in (0, T_W)$. We also have that $TRC_3(T)$ is decreasing on $(0, T_3]$ and increasing on $[T_3, T_W)$.*
(ii) *If $\Delta_3 \leq 0$, then $TRC_3(T)$ is decreasing on $(0, T_W)$.*

Proof. The detailed proof of Lemma 2 has been proved in Appendix A.2. □

Case 2: $T_0 \geq T_W > M$

Equation (14) yields:

$$\begin{aligned} TRC_4'(T) = &\ \tfrac{-A}{T^2} + \tfrac{[D(c\theta+h)+\theta I_e Dc(1-\alpha)M]}{\theta^2 T^2}(\theta T e^{\theta T} - e^{\theta T} + 1) \\ &+ \tfrac{cI_k D}{\theta^2 T^2}[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1] \\ &+ \tfrac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D}{2\theta^2 T^2}(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1) + \tfrac{pI_e DM^2}{2T^2}. \end{aligned} \tag{36}$$

and

$$\begin{aligned} TRC_4''(T) = &\ \tfrac{2A}{T^3} + \tfrac{[D(c\theta+h)+\theta I_e Dc(1-\alpha)M]}{\theta^2 T^3}(\theta^2 T^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2) \\ &+ \tfrac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D[4\theta^2 T^2 e^{2\theta T} - 2\theta^2 T^2 e^{\theta T} - 2(e^{\theta T}-1)(2\theta T e^{\theta T} - e^{\theta T} + 1)]}{2\theta^2 T^3} \\ &+ \tfrac{cI_k D}{\theta^2 T^3}[\theta^2 T^2 e^{\theta(T-M)} - 2\theta T e^{\theta(T-M)} + 2e^{\theta(T-M)} + 2\theta M - 2] - \tfrac{pI_e DM^2}{T^3} \end{aligned} \tag{37}$$

Equations (20), (24), (36) and Lemma 1(A) imply:

$$TRC_4'(M) = TRC_3'(M) = \frac{\Delta_5}{M^2}, \tag{38}$$

and

$$TRC_4'(T) > TRC_1'(T) \text{ if } T > 0 \tag{39}$$

Of course,

$$TRC_4'(T_W) > TRC_1'(T_W), \tag{40}$$

where

$$TRC_1'(T_W) = \frac{\Delta_4}{{T_W}^2}, \tag{41}$$

$$TRC_4'(T_W) = \frac{\Delta_6}{{T_W}^2}, \tag{42}$$

$$\begin{aligned}\Delta_5 = \; & \frac{D(c\theta+h)}{\theta^2}(\theta M e^{\theta M} - e^{\theta M} + 1) + \frac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D}{2\theta^2}(e^{\theta M} - 1)(2\theta M e^{\theta M} - e^{\theta M} + 1) \\ & + \frac{pI_e D M^2}{2} + \frac{I_e D(1-\alpha)cM}{\theta}(\theta M e^{\theta M} - e^{\theta M} + 1) - A,\end{aligned} \tag{43}$$

$$\begin{aligned}\Delta_4 = \; & \frac{[D(c\theta+h)]}{\theta^2}\left(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1\right) + \frac{cI_k D}{\theta^2}\left(\theta T_W e^{\theta(T_W - M)} - e^{\theta(T_W - M)} - \theta M + 1\right) \\ & + \frac{pI_e D M^2}{2} - A\end{aligned} \tag{44}$$

and

$$\begin{aligned}\Delta_6 = \; & \frac{D(c\theta+h)}{\theta^2}\left(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1\right) + \frac{cI_k D}{\theta^2}\left[\theta T_W e^{\theta(T_W - M)} - e^{\theta(T_W - M)} - \theta M + 1\right] \\ & + \frac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)\Big)^2 D}{2\theta^2}\left(e^{\theta T_W} - 1\right)\left(2\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1\right) + \frac{pI_e D M^2}{2} \\ & + \frac{I_e D(1-\alpha)cM}{\theta}\left(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1\right) - A.\end{aligned} \tag{45}$$

Since $T_W > M$, Equations (41)–(45) and Lemma 1(A,B,H) imply:

$$\Delta_6 > \Delta_4, \tag{46}$$

and

$$\Delta_6 > \Delta_5. \tag{47}$$

Lemma 3.

- *(A)* *(i)* *If* $\Delta_4 > 0$, *then* $TRC_1(T)$ *is increasing on* $[T_W, \infty)$.
  - *(ii)* *If* $\Delta_4 \leq 0$, *then* $T_1 \in [T_W, \infty)$. *We also have that* $TRC_1(T)$ *is decreasing on* $[T_W, T_1]$ *and increasing on* $[T_1, \infty)$.
- *(B)* *(i)* *If* $\Delta_6 \leq 0$, *then* $TRC_4(T)$ *is decreasing on* $[M, T_W)$.
  - *(ii)* *If* $\Delta_5 \leq 0 < \Delta_6$, *then* $T_4 \in [M, T_W)$. *We also have that* $TRC_4(T)$ *is decreasing on* $[M, T_4]$ *and increasing on* $[T_4, T_W)$.
  - *(ii)* *If* $\Delta_5 > 0$, *then* $TRC_4(T)$ *is increasing on* $[M, \infty)$.
- *(C)* *(i)* *If* $\Delta_5 > 0$, *then* $T_3 \in (0, M)$. *We also have that* $TRC_3(T)$ *is decreasing on* $(0, T_3]$ *and increasing on* $[T_3, M]$.
  - *(ii)* *If* $\Delta_5 \leq 0$, *then* $TRC_3(T)$ *is decreasing on* $(0, M]$.

Proof. The detailed proof of Lemma 3 has been proved in Appendix A.3. □

Case 3. $T_W > T_0 > M$

Equation (19) yields:

$$\begin{aligned}TRC_5'(T) = \; & \frac{-A}{T^2} + \frac{[Dc\theta(1 - I_k \alpha M) + Dh]}{\theta^2 T^2}(\theta T e^{\theta T} - e^{\theta T} + 1) \\ & + \frac{\frac{c^2}{p} I_k D}{2\theta^2 T^2}(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1),\end{aligned} \tag{48}$$

and

$$\begin{aligned}TRC_5''(T) = \; & \frac{2A}{T^3} + \frac{[Dc\theta(1 - I_k \alpha M) + Dh]}{\theta^2 T^3}(\theta^2 T^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2) \\ & + \frac{(\frac{c^2}{p}) I_k D}{2\theta^2 T^3}[4\theta^2 T^2 e^{2\theta T} - 2\theta^2 T^2 e^{\theta T} - 2(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1)] \\ & > 0.\end{aligned} \tag{49}$$

Equations (36) and (48) show:

$$TRC_5'(T_W) = \frac{\Delta_8}{T_W^2}, \tag{50}$$

$$TRC_5'(T_0) = \frac{\Delta_7}{T_W^2} \tag{51}$$

$$TRC_4'(T_0) = \frac{\Delta_9}{T_W^2} \tag{52}$$

where

$$\begin{aligned} \Delta_8 = {} & \tfrac{[Dc\theta(1-I_k\alpha M)+Dh]}{\theta^2}(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1) \\ & + \tfrac{(\frac{c^2}{p})I_k D}{2\theta^2}(e^{\theta T_W} - 1)(2\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1) - A \end{aligned} \tag{53}$$

$$\begin{aligned} \Delta_7 = {} & \tfrac{[Dc\theta(1-I_k\alpha M)+Dh]}{\theta^2}(\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1) \\ & + \tfrac{(\frac{c^2}{p})I_k D}{2\theta^2}(e^{\theta T_0} - 1)(2\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1) - A \end{aligned} \tag{54}$$

and

$$\begin{aligned} \Delta_9 = {} & \tfrac{[D(c\theta+h)+\theta I_e Dc(1-\alpha)M]}{\theta^2}(\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1) \\ & + \tfrac{cI_k D}{\theta^2}[\theta T_0 e^{\theta(T_0-M)} - e^{\theta(T_0-M)} - \theta M + 1] \\ & + \tfrac{(\frac{c^2}{p})(I_k-I_e)(1-\alpha)^2 D}{2\theta^2}(e^{\theta T_0} - 1)(2\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1) + \tfrac{pI_e DM^2}{2} - A \end{aligned} \tag{55}$$

Since $T_W > T_0 > M$, Equations (38), (A40), (53)–(55) and Lemma 1(A,B) reveal that:

$$\Delta_8 > \Delta_7, \tag{56}$$

and

$$\Delta_9 > \Delta_5. \tag{57}$$

Lemma 4.

(A) *(i)* *If* $\Delta_4 > 0$, *then* $TRC_1(T)$ *is increasing on* $[T_W, \infty)$.
(ii) *If* $\Delta_4 \le 0$, *then* $T_1 \in [T_W, \infty)$. *We also have that* $TRC_1(T)$ *is decreasing on* $[T_W, T_1]$ *and increasing on* $[T_1, \infty)$.

(B) *(i)* *If* $\Delta_7 \ge 0$, *then* $TRC_5(T)$ *is increasing on* (T_0, ∞).
(ii) *If* $\Delta_7 < 0 < \Delta_8$, *then* $T_5 \in [T_0, T_W)$. *We also have that* $TRC_5(T)$ *is decreasing on* $(T_0, T_5]$ *and increasing on* $[T_5, T_W)$.
(iii) *If* $\Delta_8 \le 0$, *then* $TRC_5(T)$ *is decreasing on* $[T_0, T_W)$.

(C) *(i)* *If* $\Delta_5 > 0$, *then* $TRC_4(T)$ *is increasing on* $[M, \infty)$.
(ii) *If* $\Delta_5 \le 0 < \Delta_9$, *then* $T_4 \in [M, T_0)$. *We also have that* $TRC_4(T)$ *is decreasing on* $[M, T_4]$ *and increasing on* $[T_4, T_0]$.
(iii) *If* $\Delta_9 \le 0$, *then* $TRC_4(T)$ *is decreasing on* $[M, T_0]$.

(D) *(i)* *If* $\Delta_5 > 0$, *then* $T_3 \in (0, M)$. *We also have that* $TRC_3(T)$ *is decreasing on* $(0, T_3]$ *and increasing on* $[T_3, M]$.
(ii) *If* $\Delta_5 \le 0$, *then* $TRC_3(T)$ *is decreasing on* $(0, M]$.

Proof. The detailed proof of Lemma 4 has been proved in Appendix A.4. □

Theorem 1.

(A) *If* $2A - pI_eDM^2 \geq 0$*, then* $TRC_1(T)$ *is convex on* $T \geq M$.
(B) $TRC_2(T)$ *is convex on* $T > 0$.
(C) $TRC_3(T)$ *is convex on* $T > 0$.
(D) *If* $2A - pI_eDM^2 \geq 0$*, then* $TRC_4(T)$ *is convex on* $T \geq M$.
(E) $TRC_5(T)$ *is convex on* $T > 0$.

Proof.

(A) Equation (21) and Lemma 1(D,J) imply that (A) holds.
(B) Equation (23) and Lemma 1(D) imply that (B) holds.
(C) Equation (25) and Lemma 1(D,F) imply that (C) holds.
(D) Equation (37) and Lemma 1(D,F,J) imply that (D) holds.
(E) Equation (49) and Lemma 1(D,F) imply that (E) holds.

Incorporating (A)–(E), we completed the proof of Theorem 1. □

4. Theorems for the Optimal Replenishment Cycle Time T^* of $TRC(T)$

Theorem 2. *For* $T_0 > M \geq T_W$*, the optimal replenishment cycle time* T^* *that minimizes* $TRC(T)$ *is given as follows:*

(A) *If* $\Delta_1 \leq 0$ *and* $\Delta_3 \leq 0$*, then* $TRC(T^*) = TRC_1(T_1)$ *and* $T^* = T_1$.
(B) *If* $\Delta_1 \leq 0$ *and* $\Delta_3 > 0$*, then* $TRC(T^*) = min\{TRC_1(T_1), TRC_3(T_3)\}$ *and* $T^* = T_1$ *or* T_3 *associated with the least cost.*
(C) *If* $\Delta_1 > 0$, $\Delta_2 \leq 0$ *and* $\Delta_3 > 0$*, then* $TRC(T^*) = TRC_2(T_2)$ *and* $T^* = T_2$.
(D) *If* $\Delta_2 > 0$*, then* $TRC(T^*) = min\{TRC_2(T_W), TRC_3(T_3)\}$ *and* $T^* = T_W$ *or* T_3 *associated with the least cost.*
(E) *If* $\Delta_1 > 0$ *and* $\Delta_3 \leq 0$*, then* $TRC(T^*) = TRC_2(T_2)$ *and* $T^* = T_2$.

Proof. The detailed proof of Theorem 2 has been proved in Appendix A.5. □

Actually, based upon the above detailed arguments, we have the following remark:

Remark 1. *If* $\Delta_1 > 0$, $\Delta_2 \leq 0$ *and* $\Delta_3 > 0$*, then Ouyang et al. [23] imply* $T^* = T_2$ *or* T_3*. However, Theorem 2(C) in this paper concludes* $T^* = T_2$*. So, Theorem 2(C) in this paper simplifies the corresponding result of Theorem 1 in Ouyang et al. [23].*

Theorem 3. *For* $T_0 \geq T_W > M$*, the optimal replenishment cycle time* T^* *that minimizes* $TRC(T)$ *is given as follows:*

(A) *If* $\Delta_6 \leq 0$*, then* $TRC(T^*) = TRC_1(T_1)$ *and* $T^* = T_1$.
(B) *If* $\Delta_4 \leq 0$, $\Delta_5 \leq 0$ *and* $\Delta_6 > 0$*, then* $TRC(T^*) = TRC_1(T_1)$ *and* $T^* = T_1$.
(C) *If* $\Delta_4 \leq 0$ *and* $\Delta_5 > 0$*, then* $TRC(T^*) = min\{TRC_1(T_1), TRC_3(T_3)\}$ *and* $T^* = T_1$ *or* T_3.
(D) *If* $\Delta_4 > 0$ *and* $\Delta_5 \leq 0$*, then* $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_W)\}$ *and* $T^* = T_4$ *or* T_W *associated with the least cost.*
(E) *If* $\Delta_4 > 0$ *and* $\Delta_5 > 0$*, then* $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_W)\}$ *and* $T^* = T_3$ *or* T_W *associated with the least cost.*

Proof. The proof of Theorem 3 is similar to the proof of Theorem 2. □

Likewise, based upon the above theorem, we had the following remarks:

Consider the following two conditions:

(C1) $\Delta_6 \leq 0$;
(C2) $\Delta_4 \leq 0$, $\Delta_5 \leq 0$ and $\Delta_6 > 0$.

Remark 2. *About (C1): If (C1) holds, then Ouyang et al. [23] imply $T^* = T_1$ or M. However, Theorem 3(A) in this paper concludes $T^* = T_1$.*

Remark 3. *About (C2): If (C2) holds, then Ouyang et al.* [23] *imply $T^* = T_1$ or T_4. However, Theorem 3(B) in this paper concludes $T^* = T_1$.*

Combining the above arguments, we reveal that Theorem 3(A,B) in this paper simplify the corresponding results of Theorem 2 in Ouyang et al. [23], this conclusion is the same as Ping [40].

Next, let:

$$G_1 = TRC_5(T_0) - TRC_4(T_0), \text{ and} \tag{58}$$

$$H_1 = TRC_1(T_W) - TRC_5(T_W). \tag{59}$$

Then, there are four situations to occur.

(S1) $G_1 > 0$ and $H_1 > 0$ if, and only if:

$$TRC_5(T_0) > TRC_4(T_0), \text{ and} \tag{60}$$

$$TRC_1(T_W) > TRC_5(T_W). \tag{61}$$

(S2) $G_1 > 0$ and $H_1 \leq 0$ if, and only if:

$$\begin{array}{c} TRC_5(T_0) > TRC_4(T_0), \text{ and} \\ TRC_1(T_W) \leq TRC_5(T_W) \end{array} \tag{62}$$

(S3) $G_1 \leq 0$ and $H_1 > 0$ if, and only if:

$$\begin{array}{c} TRC5(T0) \leq TRC4(T0), \text{ and} \\ TRC1(TW) > TRC5(TW). \end{array} \tag{63}$$

(S4) $G_1 \leq 0$ and $H_1 \leq 0$ if, and only if:

$$\begin{array}{c} TRC5(T0) \leq TRC4(T0), \text{ and} \\ TRC1(TW) > TRC5(TW). \end{array}$$

Theorem 4. *For $T_W > T_0 > M$, the optimal replenishment cycle time T^* that minimizes $TRC(T)$ is given as follows:*

(A) Suppose $\Delta_4 > 0$, $\Delta_5 > 0$ and $\Delta_7 \geq 0$. Hence,
- *(a1) if $G_1 > 0$ and $H_1 > 0$, then $TRC(T^*) = TRC_3(T_3)$.*
- *(a2) if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_W)\}$.*
- *(a3) if $G_1 \leq 0$ and $H_1 > 0$, hence,*
  - *(i) if $TRC_5(T_0) \geq TRC_3(T_3)$, then $TRC(T^*) = TRC_3(T_3)$.*
  - *(ii) if $TRC_5(T_0) < TRC_3(T_3)$, then T^* does not exist.*
- *(a4) if $G_1 \leq 0$ and $H_1 \leq 0$, hence, (i)(ii)(iii)*
  - *(i) if $TRC_5(T_0) \geq min\{TRC_3(T_3), TRC_1(T_W)\}$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_W)\}$.*
  - *(ii) if $TRC_5(T_0) = TRC_4(T_0)$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_W)\}$.*
  - *(iii) if $TRC_5(T_0) < TRC_4(T_0)$ and $TRC_5(T_0) \geq min\{TRC_3(T_3), TRC_1(T_W)\}$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_W)\}$.*

(B) Suppose $\Delta_4 > 0$, $\Delta_5 > 0$ and $\Delta_7 < 0 < \Delta_8$. Hence,
- *(b1) if $G_1 > 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_5(T_5)\}$.*
- *(b2) if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_5(T_5), TRC_1(T_W)\}$.*
- *(b3) if $G_1 \leq 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_5(T_5)\}$.*
- *(b4) if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_5(T_5), TRC_1(T_W)\}$.*

(C) *Suppose* $\Delta_4 > 0$, $\Delta_5 > 0$ *and* $\Delta_8 \leq 0$. *Hence,*

- (c1) *if* $G_1 > 0$ *and* $H_1 > 0$, *hence,*
  - (i) *if* $TRC_5(T_W) \geq TRC_3(T_3)$, *then* $TRC(T^*) > TRC_3(T_3)$.
  - (ii) *if* $TRC_5(T_W) < TRC_3(T_3)$, *then* T^* *does not exist.*
- (c2) *if* $G_1 > 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_W)\}$.
- (c3) *if* $G_1 \leq 0$ *and* $H_1 > 0$, *hence,*
  - (i) *if* $TRC_5(T_W) \geq TRC_3(T_3)$, *then* $TRC(T^*) = TRC_3(T_3)$.
  - (ii) *if* $TRC_5(T_W) < TRC_3(T_3)$, *then* T^* *does not exist.*
- (c4) *if* $G_1 \leq 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_W)\}$.

(D) *Suppose* $\Delta_4 > 0$, $\Delta_5 \leq 0$, $\Delta_7 \geq 0$ *and* $\Delta_9 > 0$. *Hence,*

- (d1) *if* $G_1 > 0$ *and* $H_1 > 0$, *then* $TRC(T^*) = TRC_4(T_4)$.
- (d2) *if* $G_1 > 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_W)\}$.
- (d3) *if* $G_1 \leq 0$ *and* $H_1 > 0$, *hence,*
  - (i) *if* $TRC_5(T_0) \geq TRC_4(T_4)$, *then* $TRC(T^*) = TRC_4(T_4)$.
  - (ii) *if* $TRC_5(T_0) < TRC_3(T_3)$, *then* T^* *does not exist.*
  - (iii) *if* $TRC_5(T_0) = TRC_4(T_0)$, *then* $TRC(T^*) = TRC_4(T_4)$.
- (d4) *if* $G_1 \leq 0$ *and* $H_1 \leq 0$, *hence,*
  - (i) *if* $TRC_5(T_0) \geq min\{TRC_4(T_4), TRC_1(T_W)\}$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_W)\}$.
  - (ii) *if* $TRC_5(T_0) < min\{TRC_4(T_4), TRC_1(T_1)\}$, *then* T^* *does not exist.*
  - (iii) *if* $TRC_5(T_0) = TRC_4(T_0)$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_W)\}$.

(E) *Suppose* $\Delta_4 > 0$, $\Delta_5 \leq 0$, $\Delta_7 < 0 < \Delta_8$ *and* $\Delta_9 > 0$. *Hence,*

- (e1) *if* $G_1 > 0$ *and* $H_1 > 0$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_5(T_5)\}$.
- (e2) *if* $G_1 > 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_5(T_5), TRC_1(T_W)\}$.
- (e3) *if* $G_1 \leq 0$ *and* $H_1 > 0$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_5(T_5)\}$.
- (e4) *if* $G_1 \leq 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_5(T_5), TRC_1(T_W)\}$.

(F) *Suppose* $\Delta_4 > 0$, $\Delta_5 \leq 0$, $\Delta_8 \leq 0$ *and* $\Delta_9 > 0$. *Hence,*

- (f1) *if* $G_1 > 0$ *and* $H_1 > 0$, *hence,*
  - (i) *if* $TRC_5(T_W) \geq TRC_4(T_4)$, *then* $TRC(T^*) = TRC_4(T_4)$.
  - (ii) *if* $TRC_5(T_W) < TRC_4(T_4)$, *then* T^* *does not exist.*
- (f2) *if* $G_1 > 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_W)\}$.
- (f3) *if* $G_1 \leq 0$ *and* $H_1 > 0$, *hence,*
  - (i) *if* $TRC_5(T_W) \geq TRC_4(T_4)$, *then* $TRC(T^*) = TRC_4(T_4)$.
  - (ii) *if* $TRC_5(T_W) < TRC_4(T_4)$, *then* T^* *does not exist.*
- (f4) *if* $G_1 \leq 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_W)\}$.

(G) *Suppose* $\Delta_4 > 0$, $\Delta_5 \leq 0$, $\Delta_7 \geq 0$ *and* $\Delta_9 \leq 0$. *Hence,*

- (g1) *if* $G_1 > 0$ *and* $H_1 > 0$, *then* $TRC(T^*) = TRC_4(T_0)$.
- (g2) *if* $G_1 > 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_4(T_0), TRC_1(T_W)\}$.
- (g3) *if* $G_1 \leq 0$ *and* $H_1 > 0$, *hence,*
  - (i) *if* $TRC_5(T_0) < TRC_4(T_0)$, *then* T^* *does not exist.*
  - (ii) *if* $TRC_5(T_0) = TRC_4(T_0)$, *then* $TRC(T^*) = TRC_4(T_0)$.
- (g4) *if* $G_1 \leq 0$ *and* $H_1 \leq 0$, *hence,*
  - (i) *if* $TRC_5(T_0) \geq TRC_1(T_W)$, *then* $TRC(T^*) = TRC_1(T_W)$.
  - (ii) *if* $TRC_5(T_0) < TRC_4(T_0)$ *and* $TRC_5(T_0) < TRC_1(T_W)$, *then* T^* *does not exist.*

(H) *Suppose* $\Delta_4 > 0$, $\Delta_5 \leq 0$, $\Delta_7 < 0 < \Delta_8$ *and* $\Delta_9 \leq 0$. *Hence,*

- (h1) *if* $G_1 > 0$ *and* $H_1 > 0$, *then* $TRC(T^*) = min\{TRC_4(T_0), TRC_5(T_5)\}$
- (h2) *if* $G_1 > 0$ *and* $H_1 \leq 0$, *then* $TRC(T^*) = min\{TRC_4(T_0), TRC_5(T_5), TRC_1(T_W)\}$.

(h3) if $G_1 \leq 0$ and $H_1 > 0$, then $TRC(T^*) = TRC_5(T_5)$.
(h4) if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_5(T_5), TRC_1(T_W)\}$.

(I) Suppose $\Delta_4 \leq 0$, $\Delta_5 \leq 0$, $\Delta_8 \leq 0$ and $\Delta_9 \leq 0$. Hence,

(i1) if $G_1 > 0$ and $H_1 > 0$, hence,
  (i) if $TRC_5(T_W) \geq TRC_4(T_0)$, then $TRC(T^*) = TRC_4(T_0)$.
  (ii) if $TRC_5(T_W) < TRC_4(T_0)$, then T^*does not exist.
(i2) if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_1(T_W)\}$.
(i3) if $G_1 \leq 0$ and $H_1 > 0$, then T^* does not exist.
(i4) if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = TRC_1(T_W)$.

(J) Suppose $\Delta_4 > 0$, $\Delta_5 > 0$ and $\Delta_7 \geq 0$. Hence,

(j1) if $G_1 > 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.
(j2) if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.
(j3) if $G_1 \leq 0$ and $H_1 > 0$, hence,
  (i) if $TRC_5(T_0) \geq min\{TRC_3(T_3), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.
  (ii) if $TRC_5(T_0) < min\{TRC_3(T_3), TRC_1(T_1)\}$, then T^* does not exist.
  (iii) if $TRC_5(T_0) = TRC_4(T_0)$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.
(j4) if $G_1 \leq 0$ and $H_1 \leq 0$, hence,
  (i) if $TRC_5(T_0) \geq min\{TRC_3(T_3), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.
  (ii) if $TRC_5(T_0) < TRC_4(T_0)$ and $TRC_5(T_0) < min\{TRC_3(T_3), TRC_1(T_1)\}$, then T^* does not exist.
  (iii) if $TRC_5(T_0) = TRC_4(T_0)$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.

(K) Suppose $\Delta_4 \leq 0$, $\Delta_5 > 0$ and $\Delta_7 < 0 < \Delta_8$. Hence,

(k1) if $G_1 > 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_1(T_1), TRC_3(T_3), TRC_5(T_5)\}$.
(k2) if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_1(T_1), TRC_3(T_3), TRC_5(T_5)\}$.
(k3) if $G_1 \leq 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_1(T_1), TRC_3(T_3), TRC_5(T_5)\}$.
(k4) if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_1(T_1), TRC_3(T_3), TRC_5(T_5)\}$.

(L) Suppose $\Delta_4 \leq 0$, $\Delta_5 > 0$ and $\Delta_8 \leq 0$. Hence,

(l1) if $G_1 > 0$ and $H_1 > 0$, hence,
  (i) if $TRC_5(T_W) \geq min\{TRC_3(T_3), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.
  (ii) if $TRC_5(T_W) < min\{TRC_3(T_3), TRC_1(T_1)\}$, then T^* does not exist.
(l2) if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.
(l3) if $G_1 \leq 0$ and $H_1 > 0$, hence,
  (i) if $TRC_5(T_W) \geq min\{TRC_3(T_3), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.
  (ii) if $TRC_5(T_W) < min\{TRC_3(T_3), TRC_1(T_1)\}$, then T^* does not exist.
(l4) if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_3(T_3), TRC_1(T_1)\}$.

(M) Suppose $\Delta_4 \leq 0$, $\Delta_5 \leq 0$, $\Delta_7 \geq 0$ and $\Delta_9 > 0$. Hence,

(m1) if $G_1 > 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$.
(m2) if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$.
(m3) if $G_1 \leq 0$ and $H_1 > 0$, hence,
  (i) if $TRC_5(T_0) \geq min\{TRC_4(T_4), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$.
  (ii) if $TRC_5(T_0) < TRC_4(T_0)$ and $TRC_5(T_0) < min\{TRC_4(T_4), TRC_1(T_1)\}$, then T^* does not exist.
  (iii) if $TRC_5(T_0) = TRC_4(T_0)$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$.
(m4) if $G_1 \leq 0$ and $H_1 \leq 0$, hence,

- *(i)* if $TRC_5(T_0) \geq min\{TRC_4(T_4), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$.
- *(ii)* if $TRC_5(T_0) < TRC_4(T_0)$ and $TRC_5(T_0) < min\{TRC_4(T_4), TRC_1(T_1)\}$, then T^* does not exist.
- *(iii)* if $TRC_5(T_0) = TRC_4(T_0)$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$.

(N) *Suppose* $\Delta_4 \leq 0$, $\Delta_5 \leq 0$, $\Delta_7 < 0 < \Delta_8$ *and* $\Delta_9 > 0$. *Hence,*

- *(n1)* if $G_1 > 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_5(T_5), TRC_1(T_1)\}$.
- *(n2)* if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_5(T_5), TRC_1(T_1)\}$.
- *(n3)* if $G_1 \leq 0$and $H_1 > 0$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_5(T_5), TRC_1(T_1)\}$.
- *(n4)* if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_5(T_5), TRC_1(T_1)\}$.

(O) *Suppose* $\Delta_4 \leq 0$, $\Delta_5 \leq 0$, $\Delta_8 \leq 0$ *and* $\Delta_9 > 0$. *Hence,*

- *(o1)* if $G_1 > 0$ and $H_1 > 0$, hence,
  - *(i)* if $TRC_5(T_W) \geq min\{TRC_4(T_4), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$
  - *(ii)* if $TRC_5(T_W) < min\{TRC_4(T_4), TRC_1(T_1)\}$, then T^* does not exist.
- *(o2)* if $G_1 > 0$ and $H_1 \leq 0$, then $TRC_5(T_W) = min\{TRC_4(T_4), TRC_1(T_1)\}$.
- *(o3)* if $G_1 \leq 0$ and $H_1 > 0$, hence,
  - *(i)* if $TRC_5(T_W) \geq min\{TRC_4(T_4), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$
  - *(ii)* if $TRC_5(T_W) < min\{TRC_4(T_4), TRC_1(T_1)\}$, then T^* does not exist.
- *(o4)* if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_4(T_4), TRC_1(T_1)\}$.

(P) *Suppose* $\Delta_4 \leq 0$, $\Delta_5 \leq 0$, $\Delta_7 \geq 0$ *and* $\Delta_9 \leq 0$. *Hence,*

- *(p1)* if $G_1 > 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_1(T_1)\}$.
- *(p2)* if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_1(T_1)\}$.
- *(p3)* if $G_1 \leq 0$ and $H_1 > 0$, hence,
  - *(i)* if $TRC_5(T_0) \geq TRC_1(T_1)$, then $TRC(T^*) = TRC_1(T_1)$.
  - *(ii)* if $TRC_5(T_0) < TRC_4(T_0)$ and $TRC_5(T_0) < TRC_1(T_1)$, then T^* does not exist.
  - *(iii)* if $TRC_5(T_0) = TRC_4(T_0)$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_1(T_1)\}$.
- *(p4)* if $G_1 \leq 0$ and $H_1 \leq 0$, hence,
  - *(i)* if $TRC_5(T_0) \geq TRC_1(T_1)$, then $TRC(T^*) = TRC_1(T_1)$.
  - *(ii)* if $TRC_5(T_0) < TRC_4(T_0)$ and $TRC_5(T_0) < TRC_1(T_1)$, then T^* does not exist.
  - *(iii)* if $TRC_5(T_0) = TRC_4(T_0)$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_1(T_1)\}$.

(Q) *Suppose* $\Delta_4 \leq 0$, $\Delta_5 \leq 0$, $\Delta_7 < 0 < \Delta_8$ *and* $\Delta_9 \leq 0$. *Hence,*

- *(q1)* if $G_1 > 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_5(T_5), TRC_1(T_1)\}$.
- *(q2)* if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_5(T_5), TRC_1(T_1)\}$.
- *(q3)* if $G_1 \leq 0$ and $H_1 > 0$, then $TRC(T^*) = min\{TRC_5(T_5), TRC_1(T_1)\}$.
- *(q4)* if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_5(T_5), TRC_1(T_1)\}$.

(R) *Suppose* $\Delta_4 \leq 0$, $\Delta_5 \leq 0$, $\Delta_8 \leq 0$ *and* $\Delta_9 \leq 0$. *Hence,*

- *(r1)* if $G_1 > 0$ and $H_1 > 0$, hence,
  - *(i)* if $TRC_5(T_W) \geq min\{TRC_4(T_0), TRC_1(T_1)\}$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_1(T_1)\}$
  - *(ii)* if $TRC_5(T_W) < min\{TRC_4(T_0), TRC_1(T_1)\}$, then T^* does not exist.
- *(r2)* if $G_1 > 0$ and $H_1 \leq 0$, then $TRC(T^*) = min\{TRC_4(T_0), TRC_1(T_1)\}$.
- *(r3)* if $G_1 \leq 0$ and $H_1 > 0$, hence,
  - *(i)* if $TRC_5(T_W) \geq TRC_1(T_1)$, then $TRC(T^*) = TRC_1(T_1)$.
  - *(ii)* if $TRC_5(T_W) < TRC_1(T_1)$, then T^* does not exist.
- *(r4)* if $G_1 \leq 0$ and $H_1 \leq 0$, then $TRC(T^*) = TRC_1(T_1)$.

Proof. The detailed proof of Theorem 4 has been proved in Appendix A.6. □

To our best knowledge, the object function (that is, the total cost function) of this paper was a piecewise continuous function; therefore, the standard approach is to use calculus to explore functional behaviors (such as continuous, increasing, decreasing, convex, concave, etc.) of that object function and reveal it is increasing or decreasing in its own domain. After that, we discussed the continuity of the objective function specially at its extreme point(s). Consequently, the main purpose of this paper was to provide accurate and reliable mathematical analytic solution procedures for different scenarios that overcome the shortcomings of Ouyang et al.

5. Numerical Examples

Example 1. *Given A = USD11.25/order, D = 1000 units/year, M = 0.12 years, h = USD5/unit/year, I_k = USD0.1/USD/year, I_e = USD0.07/USD/year, p = USD50/unit, c = USD10/unit, W = 50 units, $\theta = 0.05$ and $\alpha = 0.2$, we then have $T_W = 0.049937603$, $T_0 = 0.736279462$, $T_0 > M > T_W$, $\Delta_1 = 53.708884 > 0$, $\Delta_2 = -38.76674266 < 0$ and $\Delta_3 = 0.035573674 > 0$. Following Step 2(3) of the algorithm described in Ouyang et al. [23], we obtained:*

$$TRC(T^*) = min\{TRC_2(T_2), TRC_3(T_3)\}. \tag{64}$$

However, if we followed Theorem 2(C) in this paper, we had:

$$TRC(T^*) = TRC_2(T_2). \tag{65}$$

Applying the Intermediate Value Theorem (Varberg et al. [41]) to $F_2(T)$ and $F_3(T)$, we obtained $T_2 = 0.0499766$, $T_3 = 0.049870522$ and $TRC_2(T_2) = 30.11469284 < 98.35926525 = TRC_3(T_3)$. Therefore, $T^* = T_2$. Theorem 2(C) in this paper simplified Step 2(3) of the algorithm described in Ouyang et al. [1].

Example 2. *Given A = USD150/order, D = 1000 units/year, M = 0.12 years, h = USD5/unit/year, I_k = USD0.1/USD/year, I_e = USD0.07/USD/year, p = USD50/unit, c = USD20/unit, W = 150 units, $\theta = 0.05$ and $\alpha = 0.5$, we then have $T_W = 0.149440296$, $T_0 = 0.591176044$, $T_0 > M > T_W$ and $\Delta_6 = -48.81286509 < 0$. Following Step 3(1) of the algorithm described in Ouyang et al. [23], we obtained:*

$$TRC(T^*) = min\{TRC_1(T_1), TRC_3(M)\}. \tag{66}$$

However, if we followed Theorem 3(A) in this paper, we had:

$$TRC(T^*) = TRC_1(T_1). \tag{67}$$

Applying the Intermediate Value Theorem (Varberg et al. [41]) to $F_1(T)$, we obtained $T_1 = 0.18609161$ and $TRC_1(T_1) = 1254.146557 < 1488.595181 = TRC_3(M)$. Therefore, $T^* = T_1$. Theorem 3(A) in this paper simplified Step 3(1) of the algorithm described in Ouyang et al. [23].

Example 3. *Given A = USD50/order, D = 1000 units/year, M = 0.12 years, h = USD5/unit/year, I_k = USD0.1/USD/year, I_e = USD0.07/USD/year, p = USD50/unit, c = USD30/unit, W = 250 units, $\theta = 0.05$ and $\alpha = 0.1$, we then have $T_0 = 0.220996723$, $T_W = 0.2484504$, $T_W > T_0 > M$, $TRC_4(T_0) = 1029.265035$, $TRC_5(T_0) = 1112.049231$, $TRC_1(T_W) = 1010.464592$, $TRC_5(T_W) = 1197.715922$,*

$$G_1 = TRC_5(T_0) - TRC_4(T_0) = 82.784196 > 0 \tag{68}$$

$$H_1 = TRC_1(T_W) - TRC_5(T_W) = -187.25133 < 0 \tag{69}$$

Δ_4 = 248.729873, Δ_5 = 25.4570077, Δ_7 = 154.7950047, Δ_8 = 209.1780964 and Δ_9 = 198.1043919. Following Steps 4–2(6) of the algorithm described in Ouyang et al. [23], then:

$$TRC(T^*) = min\{TRC_1(T_W), TRC_3(T_3), TRC_5(T_0)\}. \tag{70}$$

However, Equation (18b) revealed $TRC(T_0) \neq TRC_5(T_0)$ and $TRC(T_0) = TRC_4(T_0)$. Therefore, Equation (70) can be modified as:

$$TRC(T^*) = min\{TRC_1(T_W), TRC_3(T_3), TRC_4(T_0)\}. \tag{71}$$

On the other hand, by Theorem 4(A-a2), we obtained:

$$TRC(T^*) = min\{TRC_1(T_W), TRC_3(T_3)\}. \tag{72}$$

Applying the Intermediate Value Theorem (Varberg et al. [41]) to $F_3(T)$, we obtained:

T_3 = 0.097774218 and $TRC_3(T_3)$ = 829.3667074 < $TRC_1(T_W) < TRC_4(T_0) < TRC_5(T_0)$. Therefore, $T^* = T_3$. Theorem 4(A-a2) in this paper simplified Step 4.2(6) of the algorithm described in Ouyang et al. [23].

6. Conclusions

Ouyang et al. [23] developed two solution approaches to solve the problem. The first approach was to use any standard nonlinear programming software to solve ten subcases described in Ouyang et al. [23]. However, the second approach was to develop algorithms by using the characteristics of Theorems 1–3 in Ouyang et al.'s [23].

(I) The nonlinear programming software approach: Referring to Equations (6c), (13b) and (18b) in this paper, we found that the valid domains of $TRC_3(T)$ for Case 1, $TRC_4(T)$ for Case 2 and $TRC_5(T)$ for Case 3 should have been $(0, T_W)$, $[M, T_W)$ and $(0, T_W)$, respectively. Therefore, problems S–3 (Case 1), S–5 (Case 2) and S–8 (Case 3) should be modified as follows:

$\overline{S}$-3 (Case 1): minimize $TRC_3(T)$
subject to $0 < T < T_W$.
$\overline{S}$-5 (Case 2): minimize $TRC_4(T)$
subject to $M \leq T < T_W$.
$\overline{S}$-8 (Case 3): minimize $TRC_5(T)$
subject to $T_0 < T < T_W$.

(A) About problem $\overline{S}$-3: Lemma 2(C(ii)) implies that if $\Delta_3 \leq 0$, then $TRC_3(T)$ is decreasing on (T_0, T_W). Therefore, $TRC_3(T)$ will have no minimum point on $(0, T_W)$. Therefore, if $\Delta_3 \leq 0$, then, the minimum point of problem $\overline{S}$-3 does not exist.

(B) About problem $\overline{S}$-5: Lemma 3(B(i)) implies that if $\Delta_6 \leq 0$, then $TRC_4(T)$ is decreasing on $[M, T_W)$. Therefore, $TRC_4(T)$ will have no minimum point on $[M, T_W)$. Therefore, if $\Delta_6 \leq 0$, then, the minimum point of problem $\overline{S}$-5 does not exist.

(C) About problem $\overline{S}$-8: Lemma 4(B(iii)) implies that if $\Delta_8 \leq 0$, then $TRC_5(T)$ is decreasing on $(0, T_W)$. Therefore, $TRC_5(T)$ will have no minimum point on (T_0, T_W). Therefore, if $\Delta_8 \leq 0$, then, the minimum point of problem $\overline{S}$-8 does not exist.

Incorporating the above arguments, we concluded that the nonlinear programming software approach may not be necessarily valid.

(II) The algorithm approach:

(A) About Step 2(3) in Ouyang et al.'s [23]: Following Theorem 2(C) in this paper, Step 2(3) in Ouyang et al.'s [23] can be modified as follows:

(3)′ If $\Delta_1 > 0$, $\Delta_2 \leq 0$ and $\Delta_3 > 0$, then $TRC(T^*) = TRC_2(T_2)$ and $T^* = T_2$. Go to Step 5.

(B) About Step 3(1) in Ouyang et al.'s [23]: Following Theorem 3(A) in this paper, Step 3(1) in Ouyang et al.'s [23] can be modified as follows:

(1)′ If $\Delta_6 \leq 0$, then $TRC(T^*) = TRC_1(T_1)$ and $T^* = T_1$. Go to Step 5.

(C) About Step 3(2) in Ouyang et al.'s [23]: Following Theorem 3(B) in this paper, Step 3(2) in Ouyang et al.'s [23] can be modified as follows:

(2)′ If $\Delta_4 \leq 0$, $\Delta_5 \leq 0$ and $\Delta_6 > 0$, then $TRC(T^*) = TRC_1(T_1)$ and $T^* = T_1$. Go to Step 5.

(D) About Step 4 in Ouyang et al.'s [1]: Example 3 reveals $TRC(T_0) \neq TRC_5(T_0)$ and $TRC_1(T_W) \neq TRC_5(T_W)$. Equation (18a–d) implies that $TRC(T)$ is not continuous at $T = T_0$ and T_W, in general. Furthermore, Ouyang et al. [1] do not demonstrate whether both $\Delta_9 \geq \Delta_7$ and $\Delta_7 \geq \Delta_5$ hold. Therefore, this paper divided the discussion into four parts:

(1) If $G_1 > 0$ and $H_1 > 0$,
(2) If $G_1 > 0$ and $H_1 \leq 0$,
(3) If $G_1 \leq 0$ and $H_1 > 0$,
(4) If $G_1 \leq 0$ and $H_1 \leq 0$.

In order to obtain the thorough solution procedures to obtain the optimal solution of $TRC(T)$ for Case 3, Ouyang et al. [23] ignore the discontinuity of $TRC(T)$ for Case 3. It may be such that Theorem 3 in Ouyang et al.'s [23] may not be complete. For $T_W > T_0 > M$, if $\Delta_4 \leq 0$, $\Delta_5 > 0$, $\Delta_7 < 0 < \Delta_8$ and $\Delta_9 > 0$, then Theorem 4(K) in this paper implies:

$$TRC(T^*) = min\{TRC_1(T_1), TRC_3(T_3), TRC_5(T_5)\} \tag{73}$$

However, under this case, Theorem 3 in Ouyang et al.'s [23] cannot provide the optimal solution of $TRC(T)$ for Case 3 since Ouyang et al. [23] always treated $\Delta_7 \geq \Delta_5$. Therefore, Step 4 of the algorithm described in Ouyang et al. [23] may not necessarily be valid as well.

In general, facing an optimal problem of an objective function, the standard approach is to use calculus to explore functional behaviors (such as continuous, increasing, decreasing, convex, concave, etc.) of that objective function. Ouyang et al. [23] adopted the first-order necessary condition and the second derivative test (such as equations (A4) and (B4) in Ouyang et al.'s [23] to conclude that T_i is the minimum point of $TRC_i(T)(i = 1-5)$. However, many examples revealed that Theorem B (Second Derivative Test) (Varberg et al. [41], page 164) cannot draw a conclusion about the maxima or minima without more information in general. Therefore, the processes of proofs of Lemmas 1–8 and Theorems 1–3 in Ouyang et al.'s [23] have shortcomings. Lemmas 2–4 in this paper adopted calculus to explore functional behaviors of $TRC_i(T)$ $(i = 1,2,3,4,5)$ to present the correct proofs for Theorems 2–4 in this paper to overcome shortcomings occurring in Lemmas 1–8 and Theorems 1–3 of Ouyang et al.'s [23]. Incorporating the above arguments, we conclude that this paper improves Ouyang et al.'s [23].

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Notation

D	the annual demand
A	the ordering cost per order
W	the quantity at which the fully delay payment permitted per order
c	the purchasing cost per unit
h	the unit holding cost per year excluding interest charge
p	the selling price per unit
I_e	the interest earned per dollar per year
I_k	the interest charged per dollar in stocks per year
M	the period of permissible delay in settling accounts
α	the fraction of the delay payments permitted by the supplier per order, $0 \le \alpha \le 1,\ 1 - I_k\alpha M > 0$
θ	the deterioration rate, $0 \le \theta < 1$
T	the replenishment cycle time in years
Q	the order quantity
$TRC(T)$	the annual total relevant cos t, which is a function of T
T^*	the optimal replenishment cycle time of $TRC(T)$

Appendix A.

Appendix A.1. Proof of Lemma 1

Proof.

(A) and (B): Let:

$$f_1(T) = \theta T e^{\theta T} - e^{\theta T} + 1 \tag{A1}$$

Equation (A1) yields:

$$f_1'(T) = \theta^2 e^{\theta T} > 0 \text{ if } T > 0$$

Therefore, we have:

(i) $f_1(T)$ is increasing on $T > 0$.
(ii) $f_1(T) > f_1(0) = 0$ if $T > 0$.

Both (i) and (ii) conclude that (A) and (B) hold.

(C) and (D): Let:

$$f_2(T) = \theta^2 T^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2 \tag{A2}$$

Equation (A2) yields:

$$f_2'(T) = \theta^3 T^2 e^{\theta T} > 0 \text{ if } T > 0 \tag{A3}$$

Therefore, we have:

(iii) $f_2(T)$ is increasing on $T > 0$.
(iv) $f_2(T) > f_2(0) = 0$ if $T > 0$.

Both (iii) and (iv) conclude that (C) and (D) hold.

(E) and (F): Let:

$$f_3(T) = 4\theta^2 T^2 e^{2\theta T} - 2\theta^2 T^2 e^{\theta T} - 2(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1). \tag{A4}$$

Therefore, we have:

$$f_3'(T) = 2\theta^3 T^2 e^{\theta T}(4e^{\theta T} - 1) > 0 \text{ if } T > 0. \tag{A5}$$

Furthermore, Equation (A5) implies:

(v) $f_3(T)$ is increasing on $T > 0$.

(vi) $f_3(T) > f_3(0) = 0$.

Both (v) and (vi) conclude that (E) and (F) hold.

(G) and (H): Let:

$$f_4(T) = \theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1 \text{ if } T > 0 \tag{A6}$$

Equation (A6) yields:

$$f_4'(T) = \theta^2 T e^{\theta(T-M)} > 0 \text{ if } T > 0 \tag{A7}$$

Furthermore, Equation (A7) implies:

(vii) $f_4(T)$ is increasing on $T > 0$.

(viii) $f_4(T) > f_4(M) = 0$ if $T > M$.

Both (vii) and (viii) conclude that (G) and (H) hold.

(I) and (J): Let:

$$f_5(T) = \theta^2 T^2 e^{\theta(T-M)} - 2\theta T e^{\theta(T-M)} + 2e^{\theta(T-M)} + 2\theta M - 2 \tag{A8}$$

Then, Equation (A8) yields:

$$f_5'(T) = \theta^3 T^2 e^{\theta(T-M)} > 0 \text{ if } T > 0. \tag{A9}$$

Furthermore, Equation (A9) implies:

(ix) $f_5(T)$ is increasing on $T > 0$.

(x) $f_5(M) = \theta^2 M^2 > 0$

Both (ix) and (x) conclude that (I) and (J) hold.

Incorporating the above arguments, we completed the proof of Lemma 1. □

Appendix A.2. Proof of Lemma 2

Proof.

(A) Let:

$$\begin{aligned} F_1(T) = \; & -A + \tfrac{D(c\theta+h)}{\theta^2}(\theta T e^{\theta T} - e^{\theta T} + 1) + \tfrac{cI_k D}{\theta^2}(\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1) \\ & + \tfrac{pI_e D M^2}{2}. \end{aligned} \tag{A10}$$

Then:

$$TRC_1'(T) = \frac{F_1(T)}{T^2}. \tag{A11}$$

Equation (A10) yields:

$$F_1'(T) = DT[(c\theta + h)e^{\theta T} + cI_k e^{\theta(T-M)}] > 0. \tag{A12}$$

Therefore, $F_1(T)$ is increasing on $T > 0$. Equations (31) and (A10) imply:

$$F_1(M) = \Delta_1, \tag{A13}$$

and

$$\lim_{T\to\infty} F_1(T) = \infty. \tag{A14}$$

(i) If $\Delta_1 > 0$, then:

$$F_1(T) > F_1(M) > 0 \text{ if } T > M. \tag{A15}$$

Equation (A11) reveals $TRC_1'(T) > 0$ if $T \geq M$. Therefore, $TRC_1(T)$ is increasing on $[M, \infty)$.

(ii) If $\Delta_1 \leq 0$, the Intermediate Value Theorem (Varberg et al. [41]) implies that there exists a unique point $\overline{T}_1 \in [M, \infty)$ such that $F_1(\overline{T}_1) = 0$. Therefore, we have:

$$F_1(T)\begin{cases} <0 & if \quad 0 < T < \overline{T}_1, & \text{(A16a)} \\ =0 & if \quad T = \overline{T}_1, & \text{(A16b)} \\ >0 & if \quad T > \overline{T}_1. & \text{(A16c)} \end{cases}$$

Equation (A11) shows:

$$TRC_1'(T)\begin{cases} <0 & if \quad 0 < T < \overline{T}_1, & \text{(A17a)} \\ =0 & if \quad T = \overline{T}_1, & \text{(A17b)} \\ >0 & if \quad T > \overline{T}_1. & \text{(A17c)} \end{cases}$$

Equations (A17a–c) demonstrates that $TRC_1(T)$ is decreasing on $(0, \overline{T}_1]$ and increasing on $[\overline{T}_1, \infty)$. Therefore, $T = \overline{T}_1$. Combining (i) and (ii), we completed the proof of (A).

(B) Let:

$$F_2(T) = -A + \frac{D(c\theta + h)(\theta T e^{\theta T} - e^{\theta T} + 1)}{\theta^2} + \frac{pI_e DT^2}{2}. \tag{A18}$$

Then:

$$TRC_2'(T) = \frac{F_2(T)}{T^2}. \tag{A19}$$

Equation (A18) yields:

$$F_1'(T) = D(c\theta + h)Te^{\theta T} + pI_e DT > 0. \tag{A20}$$

Therefore, $F_2(T)$ is increasing on $T \geq 0$. Equations (31), (32) and (A18) imply:

$$F_2(M) = \Delta_1, \tag{A21}$$

$$F_2(T_W) = \Delta_2, \tag{A22}$$

and

$$\lim_{T \to \infty} F_2(T) = \infty. \tag{A23}$$

(i) If $\Delta_2 > 0$, then:

$$F_2(T) > F_2(T_W) \text{ if } T > T_W. \tag{A24}$$

Equation (A19) reveals $TRC_2'(T) > 0$ if $T \geq T_W$. Therefore, $TRC_2(T)$ is increasing on $[T_W, \infty)$.

(ii) If $\Delta_2 \leq 0 < \Delta_1$, the Intermediate Value Theorem (Varberg et al. [41]) implies that there exists a unique point $\overline{T}_2 \in [T_W, M)$ such that $F_2(\overline{T}_2) = 0$. Therefore, we have:

$$F_2(T)\begin{cases} <0 & if \quad 0 < T < \overline{T}_2, & \text{(A25a)} \\ =0 & if \quad T = \overline{T}_2, & \text{(A25b)} \\ >0 & if \quad T > \overline{T}_2. & \text{(A25c)} \end{cases}$$

Equation (A19) shows:

$$TRC_2'(T)\begin{cases} <0 & if \quad 0 < T < \overline{T}_2, & \text{(A26a)} \\ =0 & if \quad T = \overline{T}_2, & \text{(A26b)} \\ >0 & if \quad T > \overline{T}_2. & \text{(A26c)} \end{cases}$$

Equation (A26a–c) demonstrates that $TRC_2(T)$ is decreasing on $(0, \overline{T}_2]$ and increasing on $[\overline{T}_2, \infty)$. Therefore, $T_2 = \overline{T}_2$. Of course, $TRC_2(T)$ is decreasing on $[T_W, \overline{T}_2]$ and increasing on $[\overline{T}_2, M]$.

(iii) If $\Delta_1 \leq 0$, then:

$$0 \geq \Delta_1 = F_2(M) > F_2(T) \text{ if } 0 < T < M.$$

Equation (A19) reveals $TRC_2'(T) < 0$ if $T_W < T < M$. Of course, $TRC_2(T)$ is decreasing on $[T_W, M]$.

Combining (i)-(iii), we completed the proof of (B).

(C) Let:

$$\begin{aligned} F_3(T) = \quad & -A + \frac{D[(c\theta+h)+\theta I_e c(1-\alpha)M]}{\theta^2}(\theta T e^{\theta T} - e^{\theta T} + 1) \\ & + \frac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D}{2\theta^2}(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{pI_e DT^2}{2} \end{aligned} \tag{A27}$$

Then:

$$TRC_3'(T) = \frac{F_3(T)}{T^2}. \tag{A28}$$

Equation (A27) and Lemma 1(B) yield:

$$\begin{aligned} F_3'(T) = \quad & D[(c\theta+h)+\theta I_e c(1-\alpha)M](Te^{\theta T}) + \frac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D}{2\theta^2} \\ & \times[\theta e^{\theta T}(2\theta T e^{\theta T} - e^{\theta T} + 1) + (e^{\theta T} - 1)(2\theta^2 T e^{\theta T} + \theta e^{\theta T})] + pI_e DT \\ & > 0 \quad \text{ if } \quad T > 0. \end{aligned} \tag{A29}$$

Therefore, $F_3(T)$ is increasing on $T \geq 0$. Equations (33) and (A27) imply:

$$F_3(T_W) = \Delta_3, \tag{A30}$$

and

$$F_3(0) = -A < 0 \tag{A31}$$

(i) If $\Delta_3 > 0$, the Intermediate Value Theorem (Varberg et al. [41]) concludes that there exists a unique point $\overline{T}_3 \in (0, T_W)$ such that $F_3(\overline{T}_3) = 0$. Therefore, we have:

$$F_3(T)\begin{cases} < 0 & if \quad 0 < T < \overline{T}_3, & \text{(A32a)} \\ = 0 & if \quad T = \overline{T}_3, & \text{(A32b)} \\ > 0 & if \quad T > \overline{T}_3. & \text{(A32c)} \end{cases}$$

Equation (A28) shows:

$$TRC_3'(T)\begin{cases} < 0 & if \quad 0 < T < \overline{T}_3, & \text{(A33a)} \\ = 0 & if \quad T = \overline{T}_3, & \text{(A33b)} \\ > 0 & if \quad T > \overline{T}_3. & \text{(A33c)} \end{cases}$$

Equation (A33a–c) demonstrates that $TRC(T_3)$ is decreasing on $(0, \overline{T}_3]$ and increasing on $[\overline{T}_3, \infty)$. Therefore, $T_3 = \overline{T}_3$. Of course, $TRC(T_3)$ is decreasing on $(0, \overline{T}_3]$ and increasing on $[\overline{T}_3, T_W)$.

(i) If $\Delta_3 \leq 0$, then:

$$0 \geq F_3(T_W) > F_3(T) \quad \text{ if } \quad 0 < T < T_W. \tag{A34}$$

Equation (A28) reveals $TRC_3'(T) < 0$ if $0 < T < T_W$. Therefore, $TRC(T_3)$ is decreasing on $(0, T_W)$.

Incorporating the above arguments, we completed the proof of Lemma 2. □

Appendix A.3. Proof of Lemma 3

Proof.

(A) Since $F_1(T_W) = \Delta_4$, following the same arguments as those of Lemma 1 (A(i),(ii)), Lemma 2 (A(i),(ii)) holds.

(B) Let:

$$\begin{aligned} F_4(T) = \quad & -A + \tfrac{D(c\theta+h)}{\theta^2}(\theta T e^{\theta T} - e^{\theta T} + 1) + \tfrac{cI_k D}{\theta^2}(\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1) \\ & + \tfrac{(\frac{c^2}{p})(I_k - I_e)(1-\alpha)^2 D}{2\theta^2}(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1) + \tfrac{pI_e DM^2}{2} \\ & + \tfrac{I_e D(1-\alpha)cM}{\theta}(\theta T e^{\theta T} - e^{\theta T} + 1). \end{aligned} \tag{A35}$$

Then:

$$TRC_4'(T) = \frac{F_4(T)}{T^2} \tag{A36}$$

Equation (A35) and Lemma 1(B) yield:

$$\begin{aligned} F_4'(T) = \quad & D(c\theta + h)(Te^{\theta T}) + cI_k D(Te^{\theta(T-M)}) + (\tfrac{c^2}{p})(I_k - I_e)(1-\alpha)^2 DTe^{\theta T}(2e^{\theta T} - 1) \\ & + I_e Dc(1-\alpha)M(\theta T e^{\theta T}) \\ & > 0 \end{aligned} \tag{A37}$$

Therefore, $F_4(T)$ is increasing on $T \geq 0$. Equations (45) and (A35) imply:

$$F_4(M) = \Delta_5, \tag{A38}$$

$$F_4(T_W) = \Delta_6, \tag{A39}$$

and

$$\lim_{T \to \infty} F_4(T) = \infty. \tag{A40}$$

(i) If $\Delta_6 \leq 0$, then:

$$0 \geq F_4(T_W) > F_4(T) \text{ if } M \leq T < T_W. \tag{A41}$$

Equation (A37) reveals $TRC_4'(T) < 0$ if $M < T < T_W$. Therefore, $TRC_4(T)$ is decreasing on $[M, T_W)$.

(ii) If $\Delta_5 \leq 0 < \Delta_6$, the Intermediate Value Theorem (Varberg et al. [41]) implies that there exists a unique point $\overline{T}_4 \in [M, T_W)$ such that $F_4(\overline{T}_4) = 0$. Therefore, we have:

$$F_4(T)\begin{cases} < 0 & if \quad 0 < T < \overline{T_4}, & \text{(A42a)} \\ = 0 & if \quad T = \overline{T_4}, & \text{(A42b)} \\ > 0 & if \quad T > \overline{T_4}. & \text{(A42c)} \end{cases}$$

Equation (A31) shows:

$$TRC_4'(T)\begin{cases} < 0 & if \quad 0 < T < \overline{T_4}, & \text{(A43a)} \\ = 0 & if \quad T = \overline{T_4}, & \text{(A43b)} \\ > 0 & if \quad T > \overline{T_4}. & \text{(A43c)} \end{cases}$$

Equation (A43a–c) demonstrates that $TRC_4(T)$ is decreasing on $(0, \overline{T}_4]$ and increasing on $[\overline{T}_4, \infty)$. Therefore, $T = \overline{T_4}$. Of course, Lemma 2 B(ii) holds.

(iii) If $\Delta_5 > 0$, then:

$$F_4(T) > F_4(M) > 0 \text{ if } T > M. \tag{A44}$$

Equation (40) reveals $TRC_4'(T) > 0$ if $T > M$. Therefore, $TRC_4(T)$ is increasing on $[M, \infty)$.

(C) The proof is the same as that of Lemma 2(C).

Incorporating the above arguments, we completed the proof of Lemma 3. □

Appendix A.4. Proof of Lemma 4

Proof.

(A) The proofs of (A)(i,ii) are the same as those of Lemma 3A(i,ii).

(B) Let:

$$\begin{aligned} F_5(T) = \quad & -A + \frac{[Dc\theta(1-I_k\alpha M)+Dh]}{\theta^2}(\theta Te^{\theta T} - e^{\theta T} - 1) \\ & + \frac{(\frac{c^2}{p})I_kD}{2\theta^2}(e^{\theta T} - 1)(2\theta Te^{\theta T} - e^{\theta T} + 1). \end{aligned} \tag{A45}$$

Then:

$$TRC_5'(T) = \frac{F_5(T)}{T^2} \tag{A46}$$

Since $1 - I_k\alpha M > 0$ in general, Equation (A38) yields:

$$F'_5(T) = [Dc\theta(1 - I_k\alpha M) + Dh]Te^{\theta T} + (\frac{c^2}{p})I_kDTe^{\theta T}(2e^{\theta T} - 1) > 0. \tag{A47}$$

According to Equation (A47), $F_5(T)$ is increasing on $T > 0$. Equations (53) and (54) imply:

$$F_5(T_W) = \Delta_8, \tag{A48}$$

$$F_5(T_0) = \Delta_7, \tag{A49}$$

and

$$\lim_{T\to\infty} F_5(T) = \infty. \tag{A50}$$

(i) If $\Delta_7 \geq 0$, then:

$$F_5(T) > F_5(T_0) = \Delta_7 \geq 0 \text{ if } T > T_0. \tag{A51}$$

Equation (A46) reveals $TRC_5'(T) > 0$ if $T > T_0$. Therefore, $TRC_5(T)$ is increasing on $[T_0, \infty)$.

(ii) If $\Delta_7 < 0 < \Delta_8$, the Intermediate Value Theorem (Varberg et al. [41]) implies that there exists a unique point $\overline{T}_5 \in (T_0, T_W)$ such that $F_5(\overline{T}_5) = 0$. Therefore, we have:

$$F_5(T)\begin{cases} < 0 & if \quad 0 < T < \overline{T}_5, & \text{(A52a)} \\ = 0 & if \quad T = \overline{T}_5, & \text{(A52b)} \\ > 0 & if \quad T > \overline{T}_5. & \text{(A52c)} \end{cases}$$

Equation (A46) shows:

$$TRC_5'(T)\begin{cases} < 0 & if \quad 0 < T < \overline{T}_5, & \text{(A53a)} \\ = 0 & if \quad T = \overline{T}_5, & \text{(A53b)} \\ > 0 & if \quad T > \overline{T}_5. & \text{(A53c)} \end{cases}$$

Equation (A53a–c) demonstrates that $TRC_5(T)$ is decreasing on $(0, \overline{T}_5]$ and increasing on $[\overline{T}_5, \infty)$. Therefore, $T_5 = \overline{T}_5$. Of course, Lemma 4 B(ii) holds.

(iii) If $\Delta_8 \leq 0$, then:

$$0 \geq \Delta_8 = F_5(T_W) > F_5(T) \text{ if } 0 < T < T_W. \tag{A54}$$

Equation (A46) reveals $TRC_5(T) < 0$ if $0 < T < T_W$. Of course, $TRC_5(T)$ is decreasing on (T_0, T_W).

(C) (i) The proof is the same as that of Lemma 3 B(iii).

(ii) Equations (43), (A35) and (55) imply:

$$F_4(M) = \Delta_5, \tag{A55}$$

$$F_4(T_0) = \Delta_9, \tag{A56}$$

and

$$\lim_{T\to\infty} F_4(T) = \infty. \tag{A57}$$

If $\Delta_5 \leq 0 < \Delta_9$, the Intermediate Value Theorem (Varberg et al. [41]) implies that there exists a unique point $\widetilde{T}_4 \in [M, T_0]$ such that $F_4(\widetilde{T}_4) = 0$. Therefore, we have

$$F_4(T)\begin{cases} < 0 & if \quad 0 < T < \widetilde{T}_4, & \text{(A58a)} \\ = 0 & if \quad T = \widetilde{T}_4, & \text{(A58b)} \\ > 0 & if \quad T > \widetilde{T}_4. & \text{(A58c)} \end{cases}$$

Equation (A36) shows:

$$TRC_4'(T)\begin{cases} < 0 & if \quad 0 < T < \widetilde{T}_4, & \text{(A59a)} \\ = 0 & if \quad T = \widetilde{T}_4, & \text{(A59b)} \\ > 0 & if \quad T > \widetilde{T}_4. & \text{(A59c)} \end{cases}$$

According to Equation (A59a–c), we have that $TRC_4(T)$ is decreasing on $(0, \widetilde{T}_4]$ and increasing on $[\widetilde{T}_4, \infty)$. Therefore, $T_4 = \widetilde{T}_4$. Of course, $TRC_4(T)$ is decreasing on $[M, T_4]$ and increasing on $[T_4, T_0]$.

(i) If $\Delta_9 \leq 0$, then:

$$0 = \Delta_9 = F_4(T_0) > F_4(T) \text{ if } 0 < T < T_0. \tag{A60}$$

Equation (A36) reveals $TRC_4'(T) < 0$ if $0 < T < T_0$. Of course, $TRC_4(T)$ is decreasing on $[M, T_0]$.

(D) The proof is the same as that of Lemma 2(C).
Incorporating the above arguments, we completed the proof of Lemma 4. □

Appendix A.5. Proof of Theorem 2

Proof.

(A) If $\Delta_1 \leq 0$ and $\Delta_3 \leq 0$, then $\Delta_1 \leq 0$, $\Delta_2 \leq 0$ and $\Delta_3 \leq 0$. With Lemma 2 (A(ii), B(iii), C(ii)), we have:

(a1) $TRC_3(T)$ is decreasing on $(0, T_W)$.
(a2) $TRC_2(T)$ is decreasing on $[T_W, M]$.
(a3) $TRC_1(T)$ is decreasing on $[M, T_1]$ and increasing on $[T_1, \infty)$.

Then, (a1)–(a3) and Equation (13a–c) imply $T^* = T_1$ and $TRC(T^*) = TRC_1(T_1)$.

(B) If $\Delta_1 \leq 0$ and $\Delta_3 > 0$, then $\Delta_1 \leq 0$, $\Delta_2 < 0$ and $\Delta_3 > 0$. With Lemma 2(A(ii), B(iii), C(i)), we have:

(b1) $TRC_3(T)$ is decreasing on $(0, T_3]$ and increasing on $[T_3, T_W)$.
(b2) $TRC_2(T)$ is decreasing on $[T_W, M]$.
(b3) $TRC_1(T)$ is decreasing on $[M, T_1]$ and increasing on $[T_1, \infty)$.

Then, (b1)–(b3) and Equation (13a–c) imply $TRC(T^*) = \min\{TRC_1(T_1), TRC_3(T_3)\}$ and $T^* = T_1$ or T_3 associated with the least cost.

(C) If $\Delta_1 > 0$, $\Delta_2 \leq 0$ and $\Delta_3 > 0$, with Lemma 2(A(i), B(ii), C(i)), we have:

(c1) $TRC_3(T)$ is decreasing on $(0, T_3]$ and increasing on $[T_3, T_W)$.
(c2) $TRC_2(T)$ is decreasing on $[T_W, T_2]$ and increasing on $[T_2, M]$.
(c3) $TRC_1(T)$ is increasing on $[M, \infty)$.

Then, (c1)–(c3) and Equation (13a–c) imply:

$$TRC(T^*) = \min\{TRC_3(T_3), TRC_2(T_2)\} \tag{A61}$$

Since $0 < T_3 < T_2 < M$, Equation (11) implies:

$$TRC_3(T_3) > TRC_2(T_3) > TRC_2(T_2). \tag{A62}$$

Equations (A61) and (A62) yield $TRC(T^*) = TRC_2(T_2)$ and $T^* = T_2$.

(D) If $\Delta_2 > 0$, then $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 > 0$. With Lemma 2(A(i), B(i), C(i)), we have:

(d1) $TRC_3(T)$ is decreasing on $(0, T_3]$ and increasing on $[T_3, T_W)$.
(d2) $TRC_2(T)$ is increasing on $[T_W, M]$.
(d3) $TRC_1(T)$ is increasing on $[M, \infty)$.

Then, (d1)–(d3) and Equation (13a–c) imply $TRC(T^*) = \min\{TRC_2(T_W), TRC_3(T_3)\}$ and $T^* = T_W$ or T_3 associated with the least cost.

(E) If $\Delta_1 > 0$ and $\Delta_3 \leq 0$, then $\Delta_1 > 0$, $\Delta_2 \leq 0$ and $\Delta_3 \leq 0$. With Lemma 2(A(i), B(ii), C(ii)), we have:

(e1) $TRC_3(T)$ is decreasing on $(0, T_W]$.
(e2) $TRC_2(T)$ is decreasing on $[T_W, T_2]$ and increasing on $[T_2, M]$.
(e3) $TRC_1(T)$ is increasing on $[M, \infty)$.

Then, (e1)–(e3) and Equation (13a–c) imply $TRC(T^*) = TRC_2(T_2)$ and $T^* = T_2$.

Incorporating (A)–(E), we completed the proof of Theorem 2. □

Appendix A.6. Proof of Theorem 4

Proof.

(A(a1)) If $\Delta_4 > 0$, $\Delta_5 > 0$ and $\Delta_7 \geq 0$, then Equations (56) and (57) imply $\Delta_4 > 0$, $\Delta_5 > 0$, $\Delta_7 \geq 0$, $\Delta_8 > 0$ and $\Delta_9 > 0$ With Lemma 49A(i), B(i), C(i), D(i)), we have:

(a11) $TRC_3(T)$ is decreasing on $(0, T_3]$ and increasing on $[T_3, M]$.
(a12) $TRC_4(T)$ is increasing on $[M, T_0]$.
(a13) $TRC_5(T)$ is increasing on (T_0, T_W).
(a14) $TRC_1(T)$ is increasing on $[T_W, \infty)$.

(a11–a14), Equations (16), (60) and (61) reveal $TRC(T^*) = TRC_3(T_3)$.

(A(a2)–A(a3)) Similar to the approach used in (A(a1)), it can be shown that (A(a2)-A(a3)) hold.

(B(b1)) If $\Delta_4 > 0$, $\Delta_5 > 0$ and $\Delta_7 < 0 < \Delta_8$, then $\Delta_4 > 0$, $\Delta_5 > 0$, $\Delta_7 < 0$, $\Delta_8 > 0$ and $\Delta_9 > 0$. With Lemma 4(A(i), B(ii), C(i), D(i)), we have:

(b11) $TRC_3(T)$ is decreasing on $(0, T_3]$ and increasing on $[T_3, M]$.
(b12) $TRC_4(T)$ is increasing on $[M, T_0]$.
(b13) $TRC_5(T)$ is decreasing on $(T_0, T_5]$ and increasing on (T_5, T_W).
(b14) $TRC_1(T)$ is increasing on $[T_W, \infty)$.

(b11)–(b14), Equations (16), (60) and (61) reveal $TRC(T^*) = \min\{TRC_3(T_3), TRC_5(T_5)\}$.

(B(b2)–(b4)) Similar to the approach used in (B(b1)), it can be shown that (B(b2)–(b4)) hold.

(C(c1)) If $\Delta_4 > 0$, $\Delta_5 > 0$ and $\Delta_8 \leq 0$, then $\Delta_4 > 0$, $\Delta_5 > 0$, $\Delta_7 < 0$, $\Delta_8 \leq 0$ and $\Delta_9 > 0$. With Lemma 4(A(i), B(iii), C(i), D(i)0, we have:

(c11) $TRC_3(T)$ is decreasing on $(0, T_3]$ and increasing on $[T_3, M]$.
(c12) $TRC_4(T)$ is increasing on $[M, T_0]$.
(c13) $TRC_5(T)$ is decreasing on (T_0, T_W).
(c14) $TRC_1(T)$ is increasing on $[T_W, \infty)$.

(c11)–(c14), Equations (16), (60) and (61) reveal that there are two cases to occur:

(i) If $TRC_5(T_W) \geq TRC_3(T_3)$, then $TRC(T^*) = TRC_3(T_3)$.
(ii) If $TRC_5(T_W) < TRC_3(T_3)$, then T^* does not exist.

(C(c2)–(c4)) Similar to the approach used in (C(c1)), it can be shown that (C(c2)–(c4)) hold.

(D(d1)–(d4)) Similar to the approach used in (A(a1)–(a4))), it can be shown that (D(d1)–(d4)) hold.

(E(e1)–(e4)) Similar to the approach used in (B(b1)–(b4)), it can be shown that (E(e1)–(e4)) hold.

(F(f1)–(f4)) Similar to the approach used in (C(c1)–(c4)), it can be shown that (F(f1)–(f4)) hold.

(G(g1)–(g4)) Similar to the approach used in (A(a1)–(a4)), it can be shown that (G(g1)–(g4)) hold.

(H(h1)–(h4)) Similar to the approach used in (B(b1)–(b4)), it can be shown that (H(h1)–(h4)) hold.

(I(i1)–(i4)) Similar to the approach used in (C(c1)–(c4)), it can be shown that (I(i1)–(i4)) hold.

(J(j1)–(j4)) Similar to the approach used in (B(b1)–(b4)), it can be shown that (J(j1)–(j4)) hold.

(K(k1)) If $\Delta_4 \le 0$, $\Delta_5 > 0$ and $\Delta_7 < 0 < \Delta_8$, then $\Delta_4 \le 0$, $\Delta_5 > 0$, $\Delta_7 < 0 < \Delta_8$ and $\Delta_9 > 0$. With Lemma 4(A(ii), B(ii), C(i), D(i)), we have:

(k11) $TRC_3(T)$ is decreasing on $(0, T_3]$ and increasing on $[T_3, M]$.

(k12) $TRC_4(T)$ is increasing on $[M, T_0]$.

(k13) $TRC_5(T)$ is decreasing on $(T_0, T_5]$ and increasing on $[T_5, T_W)$.

(k14) $TRC_1(T)$ is decreasing on $[T_W, T_1]$ and increasing on $[T_1, \infty)$.

(k11)–(k14), Equations (16), (60) and (61) reveal $TRC(T^*) = \min\{TRC_3(T_3), TRC_5(T_5),\}$ $\{TRC_1(T_1)\}$.

(K(k2)–(k4)) Similar to the approach used in (K(k1)), it can be shown that (K(k2)–(k4)) hold.

(L(l1)–(l4)) Similar to the approach used in (C(c1)–(c4)), it can be shown that (L(l1)–(l4)) hold.

(M(m1)–(m4)) Similar to the approach used in (B(b1)–(b4)), it can be shown that (M(m1)–(m4)) hold.

(N(n1)–(n4)) Similar to the approach used in (K(k1)–(k4)), it can be shown that (N(n1)–(n4)) hold.

(O(o1)–(o4)) Similar to the approach used in (C(c1)–(c4)), it can be shown that (O(o1)–(o4)) hold.

(P(p1)–(p4)) Similar to the approach used in (B(b1)–(b4)), it can be shown that (P(p1)–(p4)) hold.

(Q(q1)–(q4)) Similar to the approach used in (K(k1)–(k4)), it can be shown that (Q(q1)–(q4)) hold.

(R(r1)–(r4)) Similar to the approach used in (C(c1)–(c4)), it can be shown that (R(r1)–(r4)) hold.

Incorporating (A)–(R), we completed the proof of Theorem 4. □

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