mathematics

MDPI

Article

Strong Differential Superordination Results Involving Extended Sălăgean and Ruscheweyh Operators

Alina Alb Lupaş *,†, Georgia Irina Oros †

Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania; georgia_oros_ro@yahoo.co.uk or goros@uoradea.ro
* Correspondence: dalb@uoradea.ro
† These authors contributed equally to this work.

Abstract: The notion of strong differential subordination was introduced in 1994 and the theory related to it was developed in 2009. The dual notion of strong differential superordination was also introduced in 2009. In a paper published in 2012, the notion of strong differential subordination was given a new approach by defining new classes of analytic functions on $U \times \overline{U}$ having as coefficients holomorphic functions in $\overline{U}$. Using those new classes, extended Sălăgean and Ruscheweyh operators were introduced and a new extended operator was defined as $L_\alpha^m : \mathcal{A}_{n\zeta}^* \to \mathcal{A}_{n\zeta}^*$, $L_\alpha^m f(z,\zeta) = (1-\alpha)R^m f(z,\zeta) + \alpha S^m f(z,\zeta), z \in U, \zeta \in \overline{U}$, where $R^m f(z,\zeta)$ is the extended Ruscheweyh derivative, $S^m f(z,\zeta)$ is the extended Sălăgean operator and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$. This operator was previously studied using the new approach on strong differential subordinations. In the present paper, the operator is studied by applying means of strong differential superordination theory using the same new classes of analytic functions on $U \times \overline{U}$. Several strong differential superordinations concerning the operator L_α^m are established and the best subordinant is given for each strong differential superordination.

Keywords: strong differential superordination; convex function; best subordinant; extended Sălăgean differential operator; extended Ruscheweyh derivative

Citation: Alb Lupaş, A.; Oros, G.I. Strong Differential Superordination Results Involving Extended Sălăgean and Ruscheweyh Operators. *Mathematics* **2021**, *9*, 2487. https://doi.org/10.3390/math9192487

Academic Editors: Nak Eun Cho, Om P. Ahuja and Adam Lecko

Received: 14 September 2021
Accepted: 2 October 2021
Published: 4 October 2021

1. Introduction

Strong differential subordination is a concept introduced by J.A. Antonino and S. Romaguera in 1994 [1] based on the classical notion of subordination defined by S.S. Miller and P.T. Mocanu [2,3]. When Antonino and Romaguera introduced the notion, only the special case of Briot–Bouquet strong differential subordination was considered. The subject was further developed by J.A. Antonino in 2006 [4], but it was only in 2009 that the classical theory of differential subordination was followed by G.I. Oros and Gh. Oros [5] in order to study the general case of strong differential subordination.

In the paper [6] published in 2012, the notion of strong differential subordination was given a new approach by defining new classes of analytic functions on $U \times \overline{U}$ having as coefficients holomorphic functions in $\overline{U}$. These classes are given below as they appear in [6]:

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge 2$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

In 2009, G.I. Oros [7] proposed the concept of strong differential superordination as a dual concept, building on the general theory of differential superordination established by S.S. Miller and P.T. Mocanu [8].

Definition 1 ([7])**.** *Let $f(z,\zeta)$, $H(z,\zeta)$ be analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. In such a case, we write $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1 ([7])**.** *(i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1 is equivalent to $H(0,\zeta) = f(0,\zeta)$, for all $\zeta \in \overline{U}$, and $H(U \times \overline{U}) \subset f(U \times \overline{U})$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 2 ([7])**.** *Let $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z,\zeta)$ are univalent in U for all $\zeta \in \overline{U}$ and satisfy the (second-order) strong differential superordination*

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z,\zeta), \quad z \in U, \zeta \in \overline{U} \tag{1}$$

then p is called a solution of the strong differential superordination (1). An analytic function q is called a subordinant of the solutions of the strong differential superordination or more simply a subordinant, if $q \prec\prec p$ for all p satisfying (1). A subordinant $\tilde{q}$ that satisfies $q \prec\prec \tilde{q}$ for all subordinants q of (1) is said to be the best subordinant of (1).

Definition 3 ([7])**.** *We denote by Q^* the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, where $E(f,\zeta) = \{y \in \partial U : \lim_{z \to y} f(z,\zeta) = \infty\}$, and are such that $f'_z(y,\zeta) \neq 0$ for $y \in \partial U \times \overline{U} \backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta) = a$ is denoted by $Q^*(a)$.*

Results involving strong differential superordination investigated with operators began to be published shortly after the concept was introduced [9], continued to demonstrate the topic's interest in the following years ([10,11]) and are still in development, as evidenced by the numerous papers published in recent years ([12–17]). The differential operator studied in [18] was extended in the paper published in 2012 [19] to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$ using the definitions given below. It will be further studied in this paper and several strong differential superordinations will be established.

Definition 4 ([19])**.** *For $f \in \mathcal{A}^*_{n\zeta}$, $n, m \in \mathbb{N}$, the Sălăgean operator S^m is defined by $S^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned} S^0 f(z,\zeta) &= f(z,\zeta), \\ S^1 f(z,\zeta) &= z f'_z(z,\zeta), \ldots, \\ S^{m+1} f(z,\zeta) &= z(S^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 2 ([19])**.** *If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then the Sălăgean operator has the following form*
$S^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} j^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Definition 5 ([19])**.** *For $f \in \mathcal{A}^*_{n\zeta}$, $n, m \in \mathbb{N}$, the Ruscheweyh operator R^m is defined by $R^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta), \\ R^1 f(z,\zeta) &= z f'_z(z,\zeta), \ldots, \\ (m+1)R^{m+1} f(z,\zeta) &= z(R^m f(z,\zeta))'_z + m R^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 3 ([19])**.** *If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then the Ruscheweyh operator has the following form*

$$R^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C^m_{m+j-1} a_j(\zeta) z^j, z \in U, \zeta \in \overline{U}.$$

The extended operator introduced as a linear combination of Sălăgean and Ruscheweyh operators and studied using the notions related to strong differential subordination in [18] is shown in the next definition:

Definition 6 ([19])**.** *Let $\alpha \geq 0$, $m \in \mathbb{N}$. Denote by L^m_α the operator defined as a linear combination of Sălăgean and Ruscheweyh operators, given by $L^m_\alpha : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$L^m_\alpha f(z,\zeta) = (1-\alpha) R^m f(z,\zeta) + \alpha S^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Remark 4 ([19])**.** *If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then*

$$L^m_\alpha f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C^m_{m+j-1} \right) a_j(\zeta) z^j, z \in U, \zeta \in \overline{U}.$$

In order to prove the strong differential superordination results, the following lemmas are required:

Lemma 1 ([19])**.** *Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\,\gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta)$ is univalent in $U \times \overline{U}$ and*

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and is the best subordinant.

Lemma 2 ([19])**.** *Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + \frac{1}{\gamma} z q'_z(z,\zeta)$, $z \in U, \zeta \in \overline{U}$, where $Re\,\gamma \geq 0$.*

If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta)$ is univalent in $U \times \overline{U}$ and

$$q(z,\zeta) + \frac{1}{\gamma} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U, \zeta \in \overline{U}$. The function q is the best subordinant.

2. Main Results

The original results contained in this section are presented in theorems and corollaries that involve the operator $L_\alpha^m f(z,\zeta)$, its derivative with respect to z, and the operator of order $m+1$ $L_\alpha^{m+1} f(z,\zeta)$ alongside its derivative with respect to z. Results related to the operator $L_\alpha^m f(z,\zeta)$ are obtained in Theorem 1 and concerning its derivative with respect to z, $(L_\alpha^m f(z,\zeta))'_z$, in Theorems 2–4. Different orders of the operator are considered in Theorems 5 and 6 and strong differential superordinations involving the derivative with respect to z of the form $\left(\frac{zL_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)'_z$ are investigated providing the best subordinant for each strong differential superordination. Special strong differential superordinations are considered in Theorems 7 and 8 where the operator $L_\alpha^{m+1} f(z,\zeta)$ and its derivative with respect to z, $[L_\alpha^{m+1} f(z,\zeta)]'_z$, are used. The best subordinants of those strong differential superordinations are also provided. Interesting corollaries are obtained for special functions used as auxiliary function $h(z,\zeta)$ in the strong differential superordinations investigated in the theorems.

Theorem 1. *Let $h(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $h(0,\zeta) = 1$. Let $\alpha \geq 0$, $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta)dt$, $z \in U$, $\zeta \in \overline{U}$, $Re\, c > -2$, and suppose that $(L_\alpha^m f(z,\zeta))'_z$ is univalent in $U \times \overline{U}$, $(L_\alpha^m F(z,\zeta))'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and*

$$h(z,\zeta) \prec\prec (L_\alpha^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{2}$$

then

$$q(z,\zeta) \prec\prec (L_\alpha^m F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof. We have

$$z^{c+1} F(z,\zeta) = (c+2) \int_0^z t^c f(t,\zeta)dt$$

and differentiating it, with respect to z, we obtain $(c+1)F(z,\zeta) + zF'_z(z,\zeta) = (c+2)f(z,\zeta)$ and

$$(c+1)L_\alpha^m F(z,\zeta) + z(L_\alpha^m F(z,\zeta))'_z = (c+2)L_\alpha^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Differentiating the last relation with respect to z, we have

$$(L_\alpha^m F(z,\zeta))'_z + \frac{1}{c+2} z (L_\alpha^m F(z,\zeta))''_{z^2} = (L_\alpha^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{3}$$

Using (3), the strong differential superordination (2) becomes

$$h(z,\zeta) \prec\prec (L_\alpha^m F(z,\zeta))'_z + \frac{1}{c+2} z (L_\alpha^m F(z,\zeta))''_{z^2}. \tag{4}$$

Denote

$$p(z,\zeta) = (L_\alpha^m F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{5}$$

Replacing (5) in (4), we obtain

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1 for $\gamma = c+2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.,} \quad q(z,\zeta) \prec\prec (L_\alpha^m F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is convex and it is the best subordinant. $\square$

Corollary 1. *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$, where $\beta\in[0,1)$. Let $\alpha\geq 0$, $m\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)dt$, $z\in U$, $\zeta\in\overline{U}$, Re $c>-2$, and suppose that $(L^m_\alpha f(z,\zeta))'_z$ is univalent in $U\times\overline{U}$, $(L^m_\alpha F(z,\zeta))'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ and*

$$h(z,\zeta)\prec\prec(L^m_\alpha f(z,\zeta))'_z,\ z\in U,\ \zeta\in\overline{U}, \tag{6}$$

then

$$q(z,\zeta)\prec\prec(L^m_\alpha F(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z)=2\beta-\zeta+\frac{2(c+2)(\zeta-\beta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 1 and considering $p(z,\zeta)=(L^m_\alpha F(z,\zeta))'_z$, the strong differential superordination (6) becomes

$$h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1 for $\gamma=c+2$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.,

$$q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}t^{\frac{c+2}{n}-1}dt$$

$$=2\beta-\zeta+\frac{2(c+2)(\zeta-\beta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{t+1}dt\prec\prec(L^m_\alpha F(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant. □

Theorem 2. *Let $q(z,\zeta)$ be a convex function in $U\times\overline{U}$ and let $h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq'_z(z,\zeta)$, where $z\in U$, $\zeta\in\overline{U}$, Re $c>-2$. Let $\alpha\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)dt$, $z\in U$, $\zeta\in\overline{U}$, and suppose that $(L^m_\alpha f(z,\zeta))'_z$ is univalent in $U\times\overline{U}$, $(L^m_\alpha F(z,\zeta))'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ and*

$$h(z,\zeta)\prec\prec(L^m_\alpha f(z,\zeta))'_z,\ z\in U,\ \zeta\in\overline{U}, \tag{7}$$

then

$$q(z,\zeta)\prec\prec(L^m_\alpha F(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is the best subordinant.

Proof. We obtain that

$$z^{c+1}F(z,\zeta)=(c+2)\int_0^z t^c f(t,\zeta)dt. \tag{8}$$

Differentiating (8), with respect to z, we have $(c+1)F(z,\zeta)+zF'_z(z,\zeta)=(c+2)f(z,\zeta)$ and

$$(c+1)L^m_\alpha F(z,\zeta)+z(L^m_\alpha F(z,\zeta))'_z=(c+2)L^m_\alpha f(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}. \tag{9}$$

Differentiating (9) with respect to z, we have

$$(L^m_\alpha F(z,\zeta))'_z+\frac{1}{c+2}z(L^m_\alpha F(z,\zeta))''_{z^2}=(L^m_\alpha f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U}. \tag{10}$$

Using (10), the strong differential superordination (7) becomes

$$h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq'_z(z,\zeta)\prec\prec(L^m_\alpha F(z,\zeta))'_z+\frac{1}{c+2}z(L^m_\alpha F(z,\zeta))''_{z^2}. \tag{11}$$

Denote

$$p(z,\zeta) = (L_\alpha^m F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{12}$$

Replacing (12) in (11), we obtain

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 2 for $\gamma = c+2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.,} \quad q(z,\zeta) \prec\prec (L_\alpha^m F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is the best subordinant. □

Theorem 3. *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and suppose that $(L_\alpha^m f(z,\zeta))'_z$ is univalent and $\frac{L_\alpha^m f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec (L_\alpha^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{13}$$

then

$$q(z,\zeta) \prec\prec \frac{L_\alpha^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof. Using the properties of operator L_α^m, we have

$$L_\alpha^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j(\zeta) z^j, \quad z \in U,\ \zeta \in \overline{U}.$$

Consider $p(z,\zeta) = \frac{L_\alpha^m f(z,\zeta)}{z} = \frac{z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j(\zeta) z^j}{z} = 1 + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots,$ $z \in U, \zeta \in \overline{U}$.

We deduce that $p \in \mathcal{H}^*[1,n,\zeta]$.

Let $L_\alpha^m f(z,\zeta) = zp(z,\zeta)$, $z \in U$ $\zeta \in \overline{U}$. Differentiating with respect to z, we obtain $(L_\alpha^m f(z,\zeta))'_z = p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U, \zeta \in \overline{U}$.

Then, (13) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.,} \quad q(z,\zeta) \prec\prec \frac{L_\alpha^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant. □

Corollary 2. *Let $h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \leq \beta < 1$. Let $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and suppose that $(L_\alpha^m f(z,\zeta))'_z$ is univalent and $\frac{L_\alpha^m f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec (L_\alpha^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{14}$$

then

$$q(z,\zeta) \prec\prec \frac{L_\alpha^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{t+1} dt$, $z \in U$, $\zeta \in \overline{U}$. *The function q is convex and it is the best subordinant.*

Proof. Using the same procedure as in the proof of Theorem 3, and taking into account $p(z,\zeta) = \frac{L_\alpha^m f(z,\zeta)}{z}$, the strong differential superordination (14) becomes

$$h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z} \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.,

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{\zeta + (2\beta-\zeta)t}{1+t} t^{\frac{1}{n}-1} dt$$

$$= 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{t+1} dt \prec\prec \frac{L_\alpha^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant. □

Theorem 4. *Let* $q(z,\zeta)$ *be convex in* $U \times \overline{U}$ *and let h be defined by* $h(z,\zeta) = q(z,\zeta) + zq_z'(z,\zeta)$. *If* $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}_{n\zeta}^*$, *suppose that* $(L_\alpha^m f(z,\zeta))_z'$ *is univalent and* $\frac{L_\alpha^m f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ *and satisfies the strong differential superordination*

$$h(z,\zeta) = q(z,\zeta) + zq_z'(z,\zeta) \prec\prec (L_\alpha^m f(z,\zeta))_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{15}$$

then

$$q(z,\zeta) \prec\prec \frac{L_\alpha^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. *The function q is the best subordinant.*

Proof. Let

$$p(z,\zeta) = \frac{L_\alpha^m f(z,\zeta)}{z} = \frac{z + \sum_{j=n+1}^\infty \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j(\zeta) z^j}{z}$$

$$= 1 + \sum_{j=n+1}^\infty \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j(\zeta) z^{j-1} = 1 + \sum_{j=n}^\infty p_j(\zeta) z^j, \quad z \in U, \zeta \in \overline{U}.$$

Differentiating with respect to z we obtain $(L_\alpha^m f(z,\zeta))_z' = p(z,\zeta) + zp_z'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and (15) becomes

$$q(z,\zeta) + zq_z'(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 2 for $\gamma = 1$, we have

$$\begin{aligned} q(z,\zeta) &\prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.,} \\ q(z,\zeta) &= \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt \prec\prec \frac{L_\alpha^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U}, \end{aligned}$$

and q is the best subordinant. □

Theorem 5. *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zL^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}\right)'_z$ is univalent and $\frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(\frac{zL^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{16}$$

then

$$q(z,\zeta) \prec\prec \frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof. For $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, we have
$L^m_\alpha f(z,\zeta) = z + \sum_{j=n+1}^{\infty}\left(\alpha j^m + (1-\alpha) C^m_{m+j-1}\right) a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider

$$p(z,\zeta) = \frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)} = \frac{z + \sum_{j=n+1}^{\infty}\left(\alpha j^{m+1} + (1-\alpha) C^{m+1}_{m+j}\right) a_j(\zeta) z^j}{z + \sum_{j=n+1}^{\infty}\left(\alpha j^{m} + (1-\alpha) C^{m}_{m+j-1}\right) a_j(\zeta) z^j}.$$

We have $p'_z(z,\zeta) = \frac{\left(L^{m+1}_\alpha f(z,\zeta)\right)'_z}{L^m_\alpha f(z,\zeta)} - p(z,\zeta) \cdot \frac{\left(L^m_\alpha f(z,\zeta)\right)'_z}{L^m_\alpha f(z,\zeta)}$ and we obtain $p(z,\zeta) + z \cdot p'_z(z,\zeta) = \left(\frac{zL^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}\right)'_z$.

Relation (16) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.,} \quad q(z,\zeta) \prec \frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant. □

Corollary 3. *Let $h(z,\zeta) = \frac{\zeta + (2\beta-\zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \leq \beta < 1$. Let $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zL^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}\right)'_z$ is univalent and $\frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(\frac{zL^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{17}$$

then

$$q(z,\zeta) \prec\prec \frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{t+1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 5 and considering $p(z,\zeta) = \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}$, the strong differential superordination (17) becomes

$$h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z} \prec\prec p(z,\zeta) + z p_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.,

$$q(z,\zeta) = \frac{1}{n z^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} = \frac{1}{n z^{\frac{1}{n}}} \int_0^z \frac{\zeta + (2\beta - \zeta)t}{1+t} t^{\frac{1}{n}-1} dt$$

$$= 2\beta - \zeta + \frac{2(\zeta - \beta)}{n z^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{t+1} dt \prec\prec \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant. □

Theorem 6. *Let $q(z,\zeta)$ be a convex function and h be defined by $h(z,\zeta) = q(z,\zeta) + z q_z'(z,\zeta)$. Let $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}_{n\zeta}^*$ and suppose that $\left(\frac{z L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z'$ is univalent and $\frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) = q(z,\zeta) + z q_z'(z,\zeta) \prec\prec \left(\frac{z L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{18}$$

then

$$q(z,\zeta) \prec\prec \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{n z^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is the best subordinant.

Proof. For $f \in \mathcal{A}_{n\zeta}^*$, $f(z,\zeta) = z + \sum_{j=n+1}^\infty a_j(\zeta) z^j$ we have
$L_\alpha^m f(z,\zeta) = z + \sum_{j=n+1}^\infty \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider

$$p(z,\zeta) = \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)} = \frac{z + \sum_{j=n+1}^\infty \left(\alpha j^{m+1} + (1-\alpha) C_{m+j}^{m+1}\right) a_j(\zeta) z^j}{z + \sum_{j=n+1}^\infty \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j(\zeta) z^j}.$$

We have $p_z'(z,\zeta) = \frac{\left(L_\alpha^{m+1} f(z,\zeta)\right)_z'}{L_\alpha^m f(z,\zeta)} - p(z,\zeta) \cdot \frac{\left(L_\alpha^m f(z,\zeta)\right)_z'}{L_\alpha^m f(z,\zeta)}$ and we obtain $p(z,\zeta) + z \cdot p_z'(z,\zeta) = \left(\frac{z L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z'$.

Relation (18) becomes

$$h(z,\zeta) = q(z,\zeta) + z q_z'(z,\zeta) \prec\prec p(z,\zeta) + z p_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 2 for $\gamma = 1$, we have

$$\begin{aligned} q(z,\zeta) &\prec \prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.,} \\ q(z,\zeta) &= \frac{1}{n z^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt \prec\prec \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U}, \end{aligned}$$

and q is the best subordinant. □

Theorem 7. *Let $h(z,\zeta)$ be a convex function in $U \times \overline{U}$, with $h(0,\zeta) = 1$, and let $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$, $\left(L^{m+1}_\alpha f(z,\zeta)\right)'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}$ is univalent and $[L^m_\alpha f(z,\zeta)]'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(L^{m+1}_\alpha f(z,\zeta)\right)'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}, \quad z \in U,\ \zeta \in \overline{U}, \tag{19}$$

holds, then

$$q(z,\ \zeta) \prec\prec [L^m_\alpha f(z,\zeta)]'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1}$. The function q is convex and it is the best subordinant.

Proof. Using the properties of operator L^m_α, we obtain

$$L^{m+1}_\alpha f(z,\zeta) = (1-\alpha)R^{m+1} f(z,\zeta) + \alpha S^{m+1} f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \tag{20}$$

Then, (19) becomes

$$h(z,\zeta) \prec\prec \left((1-\alpha)R^{m+1} f(z,\zeta) + \alpha S^{m+1} f(z,\zeta)\right)'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1},$$

with $z \in U$, $\zeta \in \overline{U}$.

After a short calculation, we obtain

$$h(z,\zeta) \prec\prec (1-\alpha)(R^m f(z,\zeta))'_z + \alpha(S^m f(z,\zeta))'_z + z\left((1-\alpha)(R^m f(z,\zeta))''_{z^2} + \alpha(S^m f(z,\zeta))''_{z^2}\right), z \in U, \zeta \in \overline{U}.$$

Let

$$\begin{aligned} p(z,\zeta) &= (1-\alpha)(R^m f(z,\zeta))'_z + \alpha(S^m f(z,\zeta))'_z = (L^m_\alpha f(z,\zeta))'_z \\ &= 1 + \sum_{j=n+1}^{\infty} \left(\alpha j^{m+1} + (1-\alpha) j C^m_{m+j-1}\right) a_j(\zeta) z^{j-1} \\ &= 1 + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \dots . \end{aligned} \tag{21}$$

Using the notation in (21), the strong differential superordination becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta).$$

Using Lemma 1 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.,} \quad q(z,\zeta) \prec\prec (L^m_\alpha f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1}$. The function q is convex and it is the best subordinant. □

Corollary 4. *Let $h(z) = \frac{\zeta + (2\beta - \zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \leq \beta < 1$. Let $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$. Suppose that $\left(L^{m+1}_\alpha f(z,\zeta)\right)'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}$ is univalent in $U \times \overline{U}$ and $[L^m_\alpha f(z,\zeta)]'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec [L^{m+1}_\alpha f(z,\zeta)]'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}, \quad z \in U,\ \zeta \in \overline{U}, \tag{22}$$

then

$$q(z,\zeta) \prec\prec (L^m_\alpha f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{t+1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 7 and considering $p(z,\zeta) = (L_\alpha^m f(z,\zeta))'_z$, the strong differential superordination (22) becomes

$$h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.,

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1} = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \frac{\zeta + (2\beta-\zeta)t}{1+t}t^{\frac{1}{n}-1}dt$$

$$= 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{t+1}dt \prec\prec (L_\alpha^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant. □

Theorem 8. *Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and $h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta)$. Let $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$, suppose that $\left(L_\alpha^{m+1} f(z,\zeta)\right)'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}$ is univalent in $U \times \overline{U}$ and $[L_\alpha^m f(z,\zeta)]'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec [L_\alpha^{m+1} f(z,\zeta)]'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}, \tag{23}$$

$z \in U, \zeta \in \overline{U}$, then

$$q(z,\zeta) \prec\prec (L_\alpha^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}$. The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 7 and considering $p(z,\zeta) = (L_\alpha^m f(z,\zeta))'_z$, the strong differential superordination (23) becomes

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 2 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.,

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1} \prec\prec (L_\alpha^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}.$$

The function q is the best subordinant. □

3. Conclusions

The results presented in this paper continue the line of research which combines strong differential superordinations and operators. A previously introduced and studied operator L_α^m given in Definition 6 is further studied in this paper in view of obtaining strong differential superordinations for which best subordinants are found. Interesting corollaries follow the proved theorems containing the original results. In future research, it may be possible to look for univalence requirements for the studied operator utilizing certain functions as best subordinants. It is also possible to explore the potential of adding new classes of analytic functions using this operator.

Author Contributions: Conceptualization, A.A.L. and G.I.O.; methodology, G.I.O.; software, A.A.L.; validation, A.A.L. and G.I.O.; formal analysis, A.A.L. and G.I.O.; investigation, A.A.L.; resources, G.I.O.; data curation, G.I.O.; writing—original draft preparation, A.A.L.; writing—review and editing, A.A.L. and G.I.O.; visualization, A.A.L.; supervision, G.I.O.; project administration, A.A.L.; funding acquisition, G.I.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Antonino, J.A.; Romaguera, S. Strong differential subordination to Briot-Bouquet differential equations. *J. Differ. Equ.* **1994**, *114*, 101–105. [CrossRef]
2. Miller, S.S.; Mocanu, P.T. Second order-differential inequalities in the complex plane. *J. Math. Anal. Appl.* **1978**, *65*, 298–305. [CrossRef]
3. Miller, S.S.; Mocanu, P.T. Differential subordinations and univalent functions. *Michig. Math. J.* **1981**, *28*, 157–171. [CrossRef]
4. Antonino, J.A. Strong differential subordination and applications to univalency conditions. *J. Korean Math. Soc.* **2006**, *43*, 311–322. [CrossRef]
5. Oros, G.I.; Oros, G. Strong differential subordination. *Turk. J. Math.* **2009**, *33*, 249–257.
6. Oros, G.I. On a new strong differential subordination. *Acta Univ. Apulensis* **2012**, *32*, 243–250.
7. Oros, G.I. Strong differential superordination. *Acta Univ. Apulensis* **2009**, *19*, 101–106.
8. Miller, S.S.; Mocanu, P.T. Subordinations of differential superordinations. *Complex Var.* **2003**, *48*, 815–826.
9. Cho, N.E.; Kwon, O.S.; Srivastava, H. M. Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu–Srivastava operator. *Integral Transform. Spec. Funct.* **2010**, *21*, 589–601. [CrossRef]
10. Oshah, A.; Darus, M. Strong differential subordination and superordination of new generalized derivative operator. *Korean J. Math.* **2015**, *23*, 503–519. [CrossRef]
11. Aghalary, R.; Arjomandinia, P.; Ebadian, A. Application of strong differential superordination to a general equation. *Rocky Mt. J. Math.* **2017**, *47*, 383–390. [CrossRef]
12. Alb Lupaş, A. Inequalities for special strong differential superordinations using a generalized Sălăgean operator and Ruscheweyh derivative. In *Frontiers in Functional Equations and Analytic Inequalities*; Anastassiou, G., Rassias, J., Eds.; Springer: Cham, Switzerland, 2019. [CrossRef]
13. Oros, G.I. Strong differential superordination and sandwich theorem obtained with some new integral operators. *J. Comput. Anal. Appl.* **2019**, *26*, 256–261.
14. Srivastava, H. M.; Wanas, A.K. Strong differential sandwich results of λ-pseudo-starlike functions with respect to symmetrical points. *Math. Morav.* **2019**, *23*, 45–58. [CrossRef]
15. Wanas, A.K.; Frasin, B.A. Strong differential sandwich results for Frasin operator. *Earthline J. Math. Sci.* **2020**, *3*, 95–104. [CrossRef]
16. Wanas, A.K.; Majeed, A. H. New strong differential subordination and superordination of meromorphic multivalent quasi-convex functions. *Kragujev. J. Math.* **2020**, *44*, 27–39. [CrossRef]
17. Abd, E. H., Atshan, W.G. Strong subordination for p-valent functions involving a linear operator. *J. Phys. Conf. Ser.* **2021**, *1818*, 012113. [CrossRef]
18. Alb Lupaş, A. On special differential subordinations using Sălăgean and Ruscheweyh operators. *Math. Inequal. Appl.* **2009**, *12*, 781–790. [CrossRef]
19. Alb Lupaş, A.; Oros, G.I.; Oros, G. On special strong differential subordinations using Sălăgean and Ruscheweyh operators. *J. Comput. Anal. Appl.* **2012**, *14*, 266–270.