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Fuzzy Stability Results of Generalized Quartic Functional Equations

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Abstract: In the present paper, we introduce a new type of quartic functional equation and examine the Hyers–Ulam stability in fuzzy normed spaces by employing the direct method and fixed point techniques. We provide some applications in which the stability of this quartic functional equation can be controlled by sums and products of powers of norms. In particular, we show that if the control function is the fuzzy norm of the product of powers of norms, the quartic functional equation is hyperstable.

Keywords: quartic functional equation; Hyers–Ulam stability; fixed point; fuzzy normed space

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1. Introduction

In modeling applied issues, only fractional data might be known, or there might be a level of vulnerability in the boundaries of the model, or a few estimations might be loose. Due to such features, we would like to investigate functional equations in fuzzy settings. In the last 40 years, the fuzzy hypothesis has become an important examination tool and a lot of progress has been made in the theory of fuzzy sets to find the fuzzy analogues of the old style set theory.This branch finds many uses in the sciences. Katsaras [1] and Felbin [2] presented the notion of fuzzy norms on linear spaces. Recently, many authors have investigated the functional equations in fuzzy normed linear spaces (See e.g., [3–7]).

The stability problem of functional equations began with a question of Ulam [8] regarding the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? That is, under what condition does there exist a homomorphism near an approximate homomorphism? Hyers [9] provided a first solution to the question of Ulam for additive mappings between Banach spaces. After Hyers, various functional equations have been studied by many authors. We refer the readers to [3,7,10–17] for recent results and history on the stability.

Consider the following functional equation:

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1}$$

Since it is quite easy to prove that the function $f(x) = x^4$ fulfills (1), it is called a quartic functional equation. Each solution of the quartic functional equation is called a quartic mapping.

Many mathematicians have investigated the quartic functional equations. Lee et al. [18] derived the general solution of (1) and examined its stability results in Banach spaces.

Eshaghi Gordji et al. [19] investigated the stability of mixed type quartic–cubic–quadratic functional equations in non-Archimedean normed spaces. Ravi et al. [20] studied the stability of mixed type cubic–quartic equations in Banach spaces. Lee et al. [21] investigated the quartic functional equations in the space of generalized functions. Yang et al. [7] investigated the stability in the fuzzy β-normed spaces. Wang et al. [17] showed the stability of a mixed type cubic–quartic functional equation in 2-Banach spaces.

In this work, we introduce the generalized quartic functional equation of the form

$$\begin{aligned} &\phi\left(\sum_{i=1}^{m} it_i\right) \\ &= \sum_{1\le i<j<k<l\le m} \phi(it_i + jt_j + kt_k + lt_l) - \psi_1 \sum_{1\le i<j<k\le m} \phi(it_i + jt_j + kt_k) \\ &\quad + \psi_2 \sum_{1\le i<j\le m} \phi(it_i + jt_j) + \psi_3 \sum_{i=0}^{m-1} (i+1)^4 \left[\frac{\phi(t_{i+1}) + \phi(-t_{i+1})}{2}\right], \end{aligned} \tag{2}$$

where $m \geq 5$ and

$$\begin{aligned} \psi_1 &= (m-4), \\ \psi_2 &= \left(\frac{m^2 - 7m + 12}{2}\right), \\ \psi_3 &= \left(\frac{-m^3 + 9m^2 - 26m + 24}{6}\right). \end{aligned}$$

The main purpose of this paper is to investigate the Hyers–Ulam stability of (2) in fuzzy normed spaces with the help of direct and fixed point methods. We also provide some corollaries in which the stability of this equation can be controlled by sums and products of powers of norms. In one of the corollaries, we obtain the hyperstability of (2).

This work is coordinated as follows. In Section 2, we derive the general solution of (2) between real vector spaces. In Section 3, we investigate the fuzzy stability results of (2) by using the direct method. In Section 4, we examine the fuzzy stability results of (2) by using a fixed point method.

We will use some preliminary definitions and notions of [22–24] to study the Hyers–Ulam stability of (2) in fuzzy normed spaces.

Definition 1 ([22–24])**.** *Let E be a real vector space. A function $F : E \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on E if for every $p, q \in E$ and $u, v \in \mathbb{R}$,*

(F_1) $F(p, v) = 0$ *for* $v \leq 0$;
(F_2) $p = 0 \Leftrightarrow F(p, v) = 1$ *for all* $v > 0$;
(F_3) $F(cp, v) = F(p, \frac{v}{|c|})$ *if* $c \neq 0$;
(F_4) $F(p + q, u + v) \geq min\{F(p, u), F(q, v)\}$;
(F_5) $F(p, \cdot)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{v\to\infty} F(p, v) = 1$;
(F_6) *for* $p \neq 0$, $F(p, \cdot)$ *is continuous on* $\mathbb{R}$.

The pair (E, F) is called a fuzzy normed vector space.

The following fixed point theorem plays a crucial role in the investigation of the stability of (2).

Theorem 1 (Alternative fixed point theorem [25])**.** *Let (E, d) be a generalized complete metric space and $\Gamma : E \to E$ be a strictly contractive function with Lipschitz constant $L < 1$. Suppose that for a given element $a \in E$ there exists a positive integer m such that $d(\Gamma^{m+1}a, \Gamma^m a) < \infty$. Then,*

(i) *the sequence* $\{\Gamma^m a\}_{m=1}^{\infty}$ *converges to a fixed point* $b \in E$ *of* Γ;
(ii) b *is the unique fixed point of* Γ *in the set* $F = \{q \in E : d(\Gamma^m a, q) < \infty\}$;
(iii) $d(q, b) \leq \frac{1}{1-L} d(q, \Gamma q)$, $q \in F$.

2. General Solution

Theorem 2. *Let E, W be real vector spaces. If $\phi : E \to W$ is a mapping which fulfills (2) for all $t_1, t_2, \cdots, t_m \in E$, then, the mapping ϕ is quartic.*

Proof. Taking $t_1 = t_2 = \cdots = t_m = 0$ in (2), we have $\phi(0) = 0$. Now, substituting $(t, 0, \cdots, 0)$ for $(t_1, t_2, \cdots, t_m)$ in (2), we get

$$\phi(-t) = \phi(t), \quad t \in E.$$

Hence, the function ϕ is even. Replacing $(t_1, t_2, \cdots, t_m)$ with $(0, t, 0, \cdots, 0)$ in (2), we get

$$\phi(2t) = 2^4 \phi(t), \quad t \in E. \tag{3}$$

Then, by induction on $m \in \mathbb{N}$,

$$\phi(2^m t) = 2^{4m} \phi(t), \quad t \in E. \tag{4}$$

Replacing t with $\frac{t}{2^m}$ in (4), we get

$$\phi\left(\frac{t}{2^m}\right) = \frac{1}{2^{4m}} \phi(t), \quad t \in E. \tag{5}$$

Substituting $\left(x, -\frac{x}{2}, \frac{x}{3}, -\frac{x}{4}, \frac{y}{5}, 0, \cdots, 0\right)$ for $(t_1, t_2, t_3, t_4, t_5, \ldots, t_m)$ in (2), utilizing the evenness of ϕ and using (4) and (5), we obtain that (1) holds for every $x, y \in E$. Therefore, the mapping ϕ is quartic. □

In what follows, we assume that E is a linear space, (Z, F) is a fuzzy normed space, and (W, G) is a fuzzy Banach space. For notational convenience, we define the mapping $D\phi : E^m \to W$ by

$$\begin{aligned} &D\phi(t_1, t_2, \cdots, t_m) \\ &= \phi\left(\sum_{i=1}^{m} i t_i\right) - \sum_{1 \le i < j < k < l \le m} \phi(i t_i + j t_j + k t_k + l t_l) \\ &\quad + \psi_1 \sum_{1 \le i < j < k \le m} \phi(i t_i + j t_j + k t_k) - \psi_2 \sum_{1 \le i < j \le m} \phi(i t_i + j t_j) \\ &\quad - \psi_3 \sum_{0 \le i \le m-1} (i+1)^4 \left[\frac{\phi(t_{i+1}) + \phi(-t_{i+1})}{2}\right] \end{aligned}$$

for all $t_1, t_2, \cdots, t_m \in E$. Here, ψ_1, ψ_2, and ψ_3 are those in (2).

We denote $\psi = \psi_3 < 0$.

3. Results: Direct Technique

Theorem 3. *Let a mapping $\chi : E^m \to Z$ satisfy*

$$F(\chi(0, 2t, 0, \cdots, 0), \delta) \ge F(\rho \chi(0, t, 0, \cdots, 0), \delta) \tag{6}$$

for all $t \in E$ and all $\delta > 0$, and

$$\lim_{l \to \infty} F\left(\chi\left(2^l t_1, 2^l t_2, \cdots, 2^l t_m\right), 2^{4l} \delta\right) = 1 \tag{7}$$

for all $t_1, t_2, \cdots, t_m \in E$ and all $\delta > 0$, where $0 < \rho < 2^4$. Suppose that an even mapping $\phi : E \to W$ with $\phi(0) = 0$ satisfies

$$G(D\phi(t_1, t_2, \cdots, t_m), \delta) \ge F(\chi(t_1, t_2, \cdots, t_m), \delta) \tag{8}$$

for all $t_1, t_2, \cdots, t_m \in E$ *and all* $\delta > 0$. *Then, the limit*

$$Q_4(t) = G - \lim_{l\to\infty} \frac{\phi\left(2^l t\right)}{2^{4l}}, \quad t \in E \tag{9}$$

exists and there exists a unique quartic mapping $Q_4 : E \to W$ *such that*

$$G(\phi(t) - Q_4(t), \delta) \geq F\left(\chi(0, t, 0, \cdots, 0), |\psi|\delta(2^4 - \rho)\right), \tag{10}$$

for all $t \in E$ *and* $\delta > 0$.

Proof. Replacing $(t_1, t_2, \cdots, t_m)$ with $(0, t, 0, \cdots, 0)$ in (8), we have

$$G(\psi\phi(2t) - 16\psi\phi(t), \delta) \geq F(\chi(0, t, 0, \cdots, 0), \delta), \ t \in E, \ \delta > 0. \tag{11}$$

From (11), we get

$$G\left(\frac{\phi(2t)}{2^4} - \phi(t), \frac{\delta}{2^4|\psi|}\right) \geq F(\chi(0, t, 0, \cdots, 0), \delta), \ \ t \in E, \ \delta > 0. \tag{12}$$

Substituting $2^l t$ for t in (12), we obtain

$$G\left(\frac{\phi(2^{l+1}t)}{2^{4(l+1)}} - \frac{\phi(2^l t)}{2^{4l}}, \frac{\delta}{2^{4(l+1)}|\psi|}\right) \geq F\left(\chi(0, 2^l t, 0, \cdots, 0), \delta\right). \tag{13}$$

Using (6) and F_3 in (13), we reach

$$G\left(\frac{\phi(2^{l+1}t)}{2^{4(l+1)}} - \frac{\phi(2^l t)}{2^{4l}}, \frac{\delta}{2^{4(l+1)}|\psi|}\right) \geq F\left(\chi(0, t, 0, \cdots, 0), \frac{\delta}{\rho^l}\right). \tag{14}$$

Replacing δ with $\rho^l \delta$ in (14), we attain

$$G\left(\frac{\phi(2^{l+1}t)}{2^{4(l+1)}} - \frac{\phi(2^l t)}{2^{4l}}, \frac{\rho^l \delta}{2^{4(l+1)}|\psi|}\right) \geq F(\chi(0, t, 0, \cdots, 0), \delta), \ \ t \in E, \ \delta > 0. \tag{15}$$

Note that

$$\frac{\phi(2^l t)}{2^{4l}} - \phi(t) = \sum_{i=0}^{l-1} \frac{\phi(2^{i+1}t)}{2^{4(i+1)}} - \frac{\phi(2^i t)}{2^{4i}}. \tag{16}$$

From (15) and (16), we get

$$\begin{aligned} &G\left(\frac{\phi(2^l t)}{2^{4l}} - \phi(t), \sum_{i=0}^{l-1} \frac{\rho^i \delta}{2^{4(i+1)}|\psi|}\right) \\ &\geq \min \bigcup_{i=0}^{l-1} \left\{ G\left(\frac{\phi(2^{(i+1)}t)}{2^{4(i+1)}} - \frac{\phi(2^i t)}{2^{4i}}, \frac{\rho^i \delta}{|\psi| 2^{4(i+1)}}\right)\right\} \\ &\geq F(\chi(0, t, 0, \cdots, 0), \delta), \ t \in E, \ \delta > 0. \end{aligned} \tag{17}$$

Replacing t with $2^n t$ in (17) and with the help of (6) and F_3, we arrive at

$$G\left(\frac{\phi(2^{(l+n)}t)}{2^{4l}} - \phi(2^n t), \sum_{i=0}^{l-1} \frac{\rho^i \delta}{2^{4(i+1)}|\psi|}\right) \geq F(\chi(0, 2^n t, 0, \cdots, 0), \delta).$$

Hence, we have

$$\begin{aligned} &G\left(\frac{\phi(2^{(l+n)}t)}{2^{4(l+n)}}-\frac{\phi(2^n t)}{2^{4n}},\frac{1}{2^{4n}}\sum_{i=0}^{l-1}\frac{\rho^i\delta}{2^{4(i+1)}|\psi|}\right) \\ &\geq F(\rho^n\chi(0,t,0,\cdots,0),\delta) \\ &\geq F\left(\chi(0,t,0,\cdots,0),\frac{\delta}{\rho^n}\right). \end{aligned} \tag{18}$$

Replacing δ with $\rho^n\delta$ in (18), we obtain for all $l,n\geq 0$

$$G\left(\frac{\phi(2^{(l+n)}t)}{2^{4(l+n)}}-\frac{\phi(2^n t)}{2^{4n}},\sum_{i=n}^{l+n-1}\frac{\rho^i\delta}{2^{4(i+1)}|\psi|}\right)\geq F(\chi(0,t,0,\cdots,0),\delta).$$

Replacing δ with $\frac{\delta}{\sum_{i=n}^{l+n-1}\frac{\rho^i}{2^{4(i+1)}|\psi|}}$ in the above inequality, we get

$$G\left(\frac{\phi(2^{(l+n)}t)}{2^{4(l+n)}}-\frac{\phi(2^n t)}{2^{4n}},\delta\right)\geq F\left(\chi(0,t,0,\cdots,0),\frac{\delta}{\sum_{i=n}^{l+n-1}\frac{\rho^i}{2^{4(i+1)}|\psi|}}\right) \tag{19}$$

for all $l,n\geq 0$. As $0<\rho<2^4$ and $\sum_{i=0}^{\infty}\left(\frac{\rho}{2^4}\right)^i<\infty$, F_5 implies that the right-hand side of (19) goes to 1 as $n\to\infty$. Hence, $\left\{\frac{\phi(2^m t)}{2^{4m}}\right\}$ is a Cauchy sequence in (W,G). As (W,G) is a fuzzy Banach space, this sequence converges to some point $Q_4(t)\in W$. Now, we can define a mapping $Q_4:E\to W$ by

$$Q_4(t)=G-\lim_{l\to\infty}\frac{\phi\left(2^l t\right)}{2^{4l}},\quad t\in E.$$

Putting $n=0$ and taking the limit l tends to ∞ in (19), with the help of F_6, we get

$$G(\phi(t)-Q_4(t),\delta)\geq F\left(\chi(0,t,0,\cdots,0),|\psi|\delta(2^4-\rho)\right),\quad t\in E,\delta>0.$$

Now, we show that Q_4 is quartic. Note that ϕ and Q_4 are even mappings. Replacing $(t_1,t_2,\cdots,t_m)$ with $(2^l t_1,2^l t_2,\cdots,2^l t_m)$ in (8), we have

$$\begin{aligned} &G\left(\frac{1}{2^{4l}}D\phi\left(2^l t_1,2^l t_2,\cdots,2^l t_m\right),\delta\right) \\ &\geq F\left(\frac{1}{2^{4l}}\chi\left(2^l t_1,2^l t_2,\cdots,2^l t_m\right),\delta\right) \\ &\geq F\left(\chi\left(2^l t_1,2^l t_2,\cdots,2^l t_m\right),2^{4l}\delta\right) \end{aligned}$$

for all $t_1,t_2,\cdots,t_m\in E$ and all $\delta>0$. Note that

$$\lim_{l\to\infty}F\left(\chi(2^l t_1,2^l t_2,\cdots,2^l t_m),2^{4l}\delta\right)=1.$$

Hence, Q_4 satisfies the functional Equation (2). Therefore, $Q_4:E\to W$ is quartic.

Next, we show the uniqueness of Q_4. Let $R_4:E\to W$ be another quartic mapping satisfying (10). Then,

$$
\begin{aligned}
&G(Q_4(t) - R_4(t), \delta) \\
&= G\left(\frac{Q_4(2^l t)}{2^{4l}} - \frac{R_4(2^l t)}{2^{4l}}, \delta\right) \\
&= G\left(\frac{Q_4(2^l t)}{2^{4l}} - \frac{\phi(2^l t)}{2^{4l}} + \frac{\phi(2^l t)}{2^{4l}} - \frac{R_4(2^l t)}{2^{4l}}, \delta\right) \\
&\geq \min\left\{G\left(\frac{Q_4(2^l t)}{2^{4l}} - \frac{\phi(2^l t)}{2^{4l}}, \frac{\delta}{2}\right), G\left(\frac{\phi(2^l t)}{2^{4l}} - \frac{R_4(2^l t)}{2^{4l}}, \frac{\delta}{2}\right)\right\} \\
&\geq F\left(\chi(0, 2^l t, 0, \cdots, 0), \frac{2^{4l}|\psi|\delta(2^4 - \rho)}{2}\right) \\
&\geq F\left(\rho^l \chi(0, t, 0, \cdots, 0), \frac{2^{4l}|\psi|\delta(2^4 - \rho)}{2}\right) \\
&\geq F\left(\chi(0, t, 0, \cdots, 0), \frac{2^{4l}|\psi|\delta(2^4 - \rho)}{2\rho^l}\right), \ t \in E, \ \delta > 0.
\end{aligned}
$$

Since $\lim_{l\to\infty} \frac{2^{4l}|\psi|\delta(2^4-\rho)}{2\rho^l} = \infty$, we get

$$
\lim_{l\to\infty} F\left(\chi(0, t, 0, \cdots, 0), \frac{2^{4l}|\psi|\delta(2^4 - \rho)}{2\rho^l}\right) = 1.
$$

Thus, $G(Q_4(t) - R_4(t), \delta) = 1$. Hence, $Q_4(t) = R_4(t)$. Therefore, the proof is now completed. □

We have the following result similar to Theorem 3, which corresponds to the case $\rho > 2^4$.

Theorem 4. *Let a mapping $\chi : E^m \to Z$ satisfy*

$$
F\left(\chi\left(0, \frac{t}{2}, 0, \cdots, 0\right), \delta\right) \geq F\left(\frac{1}{\rho}\chi(0, t, 0, \cdots, 0), \delta\right) \tag{20}
$$

for all $t \in E$ and all $\delta > 0$, and

$$
\lim_{l\to\infty} F\left(\chi\left(\frac{t_1}{2^l}, \frac{t_2}{2^l}, \cdots, \frac{t_m}{2^l}\right), \frac{\delta}{2^{4l}}\right) = 1, \quad t_1, t_2, \cdots, t_m \in E, \delta > 0, \tag{21}
$$

where $\rho > 2^4$. Suppose that an even mapping $\phi : E \to W$ with $\phi(0) = 0$ fulfills

$$
G(D\phi(t_1, t_2, \cdots, t_m), \delta) \geq F(\chi(t_1, t_2, \cdots, t_m), \delta) \tag{22}
$$

for all $t_1, t_2, \cdots, t_m \in E$ and all $\delta > 0$. Then, the limit

$$
Q_4(t) = G - \lim_{l\to\infty} 2^{4l}\phi\left(\frac{t}{2^l}\right) \tag{23}
$$

exists for each $t \in E$ and defines a unique quartic mapping $Q_4 : E \to W$ such that

$$
G(\phi(t) - Q_4(t), \delta) \geq F\left(\chi(0, t, 0, \cdots, 0), |\psi|\delta(\rho - 2^4)\right), \quad t \in E, \ \delta > 0. \tag{24}
$$

Proof. Following the same method as in Theorem 3, we obtain the result. □

In the remaining parts of this section, we apply the theorems to get some corollaries.

Corollary 1. *Let $\tau > 0$ be a real constant. If an even mapping $\phi : E \to W$ with $\phi(0) = 0$ fulfills*

$$G(D\phi(t_1, t_2, \cdots, t_m), \delta) \geq F\Big(\tau, \delta\Big),$$

for all $t_1, t_2, \cdots, t_m \in E$, $\delta > 0$, then there exists a unique quartic mapping $Q_4 : E \to W$ such that

$$G(\phi(t) - Q_4(t), \delta) \geq F(\tau, 15|\psi|\delta), \quad t \in E, \ \delta > 0.$$

Proof. Let us define $\chi(t_1, t_2, \cdots, t_m) = \tau$ and $\rho = 2^0$. Then, by Theorem 3, we have

$$G(\phi(t) - Q_4(t), \delta) \geq F(\tau, 15|\psi|\delta), \ t \in E, \ \delta > 0.$$

□

Corollary 2. *Let ε and p be real constants with $p \in (0, 4) \cup (4, +\infty)$. If an even mapping $\phi : E \to W$ with $\phi(0) = 0$ fulfills*

$$G(D\phi(t_1, t_2, \cdots, t_m), \delta) \geq F\Big(\varepsilon \sum_{i=1}^{m} \|t_i\|^p, \delta\Big),$$

for all $t_1, t_2, \cdots, t_m \in E$, $\delta > 0$, then there exists a unique quartic mapping $Q_4 : E \to W$ such that

$$G(\phi(t) - Q_4(t), \delta) \geq F(\varepsilon\|t\|^p, |2^4 - 2^p||\psi|\delta), \quad t \in E, \ \delta > 0.$$

Proof. Let us define $\chi(t_1, t_2, \cdots, t_m) = \varepsilon \sum_{i=1}^{m} \|t_i\|^p$, $\rho = 2^p$ and apply Theorems 3 and 4. Then, we get the result. □

Corollary 3. *Let ε, θ, p, and q be real constants with $mp, mq \in (0, 4) \cup (4, +\infty)$. If an even mapping $\phi : E \to W$ with $\phi(0) = 0$ fulfills*

$$G(D\phi(t_1, t_2, \cdots, t_m), \delta) \geq F\Big(\varepsilon \sum_{i=1}^{m} \|t_i\|^{mp} + \theta \prod_{i=1}^{m} \|t_i\|^q, \delta\Big),$$

for all $t_1, t_2, \cdots, t_m \in E$, $\delta > 0$, then there exists a unique quartic mapping $Q_4 : E \to W$ such that

$$G(\phi(t) - Q_4(t), \delta) \geq F(\varepsilon\|t\|^{mp}, |2^4 - 2^{mp}||\psi|\delta), \quad t \in E, \ \delta > 0.$$

Proof. Defining $\chi(t_1, t_2, \cdots, t_m) = \varepsilon \sum_{i=1}^{m} \|t_i\|^{mp} + \theta \prod_{i=1}^{m} \|t_i\|^q$, $\rho = 2^{mp}$ and applying Theorems 3 and 4, we get the result. □

We obtain the hyperstability of ϕ if $\varepsilon = 0$ in Corollary 3.

Corollary 4. *Let θ and q be real constants with $0 < mq \neq 4$. If an even mapping $\phi : E \to W$ with $\phi(0) = 0$ fulfills*

$$G(D\phi(t_1, t_2, \cdots, t_m), \delta) \geq F\Big(\theta \prod_{i=1}^{m} \|t_i\|^q, \delta\Big),$$

for all $t_1, t_2, \cdots, t_m \in E$, $\delta > 0$, then ϕ is quartic.

Proof. We consider Corollary 3 with $\varepsilon = 0$. Then, we have

$$\begin{aligned} & G(\phi(t) - Q_4(t), \delta) \\ & \geq F(\varepsilon\|t\|^{mp}, |2^4 - 2^{mp}||\psi|\delta) \\ & = F(0, |2^4 - 2^{mp}||\psi|\delta) = 1, \end{aligned}$$

for all $t \in E$, $\delta > 0$, and hence, $\phi = Q_4$ is quartic. $\square$

4. Results: Fixed Point Technique

In this section, we consider the stability of the functional equation (2) using Theorem 1. For notational convenience, we define ξ_a as follows:

$$\xi_a = \begin{cases} 2 & \text{if} \quad a = 0, \\ \frac{1}{2} & \text{if} \quad a = 1 \end{cases}$$

and set $\Lambda = \{g : E \to W : \ g(0) = 0\}$.

Now, we prove the main outcome of this section.

Theorem 5. *Let $\phi : E \to W$ be an even mapping such that $\phi(0) = 0$ and there exists a mapping $\chi : E^m \to Z$ satisfying*

$$\lim_{l \to \infty} F\Big(\chi(\xi_a^l t_1, \xi_a^l t_2, \cdots, \xi_a^l t_m), \xi_a^{4l}\delta\Big) = 1 \tag{25}$$

and

$$G(D\phi(t_1, t_2, \cdots, t_m), \delta) \geq F(\chi(t_1, t_2, \cdots, t_m), \delta) \tag{26}$$

for all $t_1, t_2, \cdots, t_m \in E$ and $\delta > 0$. Let $\varphi(t) = \frac{1}{\psi}\chi(0, \frac{t}{2}, 0, \cdots, 0)$. Assume there exists $L \in (0,1)$ such that

$$F\left(\frac{1}{\xi_a^4}\varphi(\xi_a t), \delta\right) \geq F(L\varphi(t), \delta), \quad t \in E,\ \delta > 0. \tag{27}$$

Then, there exist a unique quartic mapping $Q_4 : E \to W$ satisfying

$$G(\phi(t) - Q_4(t), \delta) \geq F\left(\frac{L^{1-a}}{1-L}\varphi(t), \delta\right), \quad t \in E,\ \delta > 0. \tag{28}$$

Proof. Let $\gamma : \Lambda \to [0, \infty]$ be given by

$$\gamma(f, g) = \inf\{w \in (0, \infty) : \ G(f(t) - g(t), \delta) \geq F(w\varphi(t), \delta),\ t \in X, \delta > 0\},$$

and as standard, $\inf \emptyset = +\infty$.

The same method used in ([26], Lemma 2.1) gives a complete generalized metric space (Λ, γ).

Let us define $\Psi_a : \Lambda \longrightarrow \Lambda$ by

$$\Psi_a f(t) = \frac{1}{\xi_a^4} f(\xi_a t),\ t \in E.$$

Let f, g be elements of Λ such that

$$\gamma(f, g) \leq \varepsilon.$$

Then,

$$G(f(t) - g(t), \delta) \geq F(\varepsilon\varphi(t), \delta), \quad t \in E,\ \delta > 0,$$

whence

$$G(\Psi_a f(t) - \Psi_a g(t), \delta) \geq F\left(\frac{\varepsilon}{\xi_a^4}\varphi(\xi_a t), \delta\right), \quad t \in E,\ \delta > 0.$$

It follows from (27) that

$$G(\Psi_a f(t) - \Psi_a g(t), \delta) \geq F(\varepsilon L\varphi(t), \delta), \quad t \in E,\ \delta > 0.$$

Hence, we have $\gamma(\Psi_a f, \Psi_a g) \leq \varepsilon L$. This shows

$$\gamma(\Psi_a f, \Psi_a g) \leq L\gamma(f,g),$$

that is, Ψ_a is a strictly contractive mapping on Λ with Lipschitz constant L. Substituting $(0,t,0,\cdots,0)$ for $(t_1,t_2,\cdots,t_m)$ in (26), we get

$$G\Big(\psi\phi(2t) - 2^4\psi\phi(t), \delta\Big) \geq F(\chi(0,t,0,\cdots,0),\delta), \quad t \in E,\ \delta > 0. \tag{29}$$

Using (27) and (F_3) when $a = 0$, it follows from (29) that

$$G\left(\frac{\phi(2t)}{2^4} - \phi(t), \frac{\delta}{2^4|\psi|}\right) \geq F(\chi(0,t,0,\cdots,0),\delta).$$

Hence,

$$\begin{aligned} G\left(\frac{\phi(2t)}{2^4} - \phi(t), \delta\right) &\geq F\Big(\chi(0,t,0,\cdots,0), 2^4|\psi|\delta\Big) \\ &\geq F\left(\frac{\chi(0,t,0,\cdots,0)}{2^4\psi}, \delta\right) \\ &\geq F(L\varphi(t),\delta), \quad t \in E,\ \delta > 0. \end{aligned}$$

Therefore,

$$\gamma(\Psi_0\phi, \phi) \leq L = L^{1-a}. \tag{30}$$

Replacing t with $\frac{t}{2}$ in (29) (i.e., when $a = 1$) and using (F_3), we obtain

$$G\left(\phi(t) - 2^4\phi\left(\frac{t}{2}\right), \frac{\delta}{|\psi|}\right) \geq F\left(\chi\left(0, \frac{t}{2}, 0, \cdots, 0\right), \delta\right).$$

Hence,

$$\begin{aligned} G\left(\phi(t) - 2^4\phi\left(\frac{t}{2}\right), \delta\right) &\geq F\left(\chi\left(0, \frac{t}{2}, 0, \cdots, 0\right), |\psi|\delta\right) \\ &\geq F\left(\frac{1}{\psi}\chi\left(0, \frac{t}{2}, 0, \cdots, 0\right), \delta\right) \\ &\geq F(\varphi(t),\delta), \quad t \in E,\ \delta > 0. \end{aligned}$$

Therefore,

$$\gamma(\Psi_1\phi, \phi) \leq 1 = L^{1-a}. \tag{31}$$

Then, from (30) and (31), we conclude

$$\gamma(\Psi_a\phi, \phi) \leq L^{1-a} < \infty.$$

Now, from Theorem 1, it follows that there exists a fixed point Q_4 of Ψ_a in Λ such that

(i) $\Psi_a Q_4 = Q_4$ and $\lim_{l\to\infty} \gamma(\Psi_a^l \phi, Q_4) = 0$;
(ii) Q_4 is the unique fixed point of Ψ_a in the set $E = \{g \in \Lambda : \ d(\phi, g) < \infty\}$;
(iii) $\gamma(\phi, Q_4) \leq \frac{1}{1-L}\gamma(\phi, \Psi_a\phi)$.

Letting $\gamma(\Psi_a^l\phi, Q_4) = \varepsilon_l$, we get $G(\Psi_a^l\phi(t) - Q_4(t), \delta) \geq F(\varepsilon_l\varphi(t), \delta)$ for all $t \in E$ and all $\delta > 0$. Since $\lim_{l\to\infty} \varepsilon_l = 0$, we infer

$$Q_4(t) = G - \lim_{l\to\infty} \frac{\phi(\xi_a^l t)}{\xi_a^{4l}}, \quad t \in E.$$

Replacing $(t_1, t_2, \cdots, t_m)$ with $(\xi_a^l t_1, \xi_a^l t_2, \cdots, \xi_a^l t_m)$ in (26), we obtain

$$G\left(\frac{1}{\xi_a^{4l}} D\phi(\xi_a^l t_1, \xi_a^l t_2, \cdots, \xi_a^l t_m), \delta\right) \geq F\left(\frac{1}{\xi_a^{4l}} \chi(\xi_a^l t_1, \xi_a^l t_2, \cdots, \xi_a^l t_m), \delta\right)$$
$$\geq F(\chi(\xi_a^l t_1, \xi_a^l t_2, \cdots, \xi_a^l t_m), \xi_a^{4l}\delta),$$

for all $\delta > 0$ and all $t_1, t_2, \cdots, t_m \in E$. Then, by the same method as in Theorem 3, we obtain that the mapping $Q_4 : E \to W$ is quartic. As $\gamma(\Psi_a\phi, \phi) \leq L^{1-a}$, it follows from (iii) that $\gamma(\phi, Q_4) \leq \frac{L^{1-a}}{1-L}$, which means (28).

Finally, we show that Q_4 is unique. Let $R_4 : E \to W$ be another quartic mapping fulfilling (28). Since $Q_4(2^l t) = 2^{4l} Q_4(t)$ and $R_4(2^l t) = 2^{4l} R_4(t)$ for all $t \in E$ and all $l \in \mathbb{N}$, we obtain

$$\begin{aligned}
&G(Q_4(t) - R_4(t), \delta) \\
&= G\left(\frac{Q_4(2^l t)}{2^{4l}} - \frac{R_4(2^l t)}{2^{4l}}, \delta\right) \\
&= G\left(\frac{Q_4(2^l t)}{2^{4l}} - \frac{\phi(2^l t)}{2^{4l}} + \frac{\phi(2^l t)}{2^{4l}} - \frac{R_4(2^l t)}{2^{4l}}, \delta\right) \\
&\geq \min\left\{G\left(\frac{Q_4(2^l t)}{2^{4l}} - \frac{\phi(2^l t)}{2^{4l}}, \frac{\delta}{2}\right), G\left(\frac{\phi(2^l t)}{2^{4l}} - \frac{R_4(2^l t)}{2^{4l}}, \frac{\delta}{2}\right)\right\} \\
&\geq F\left(\frac{L^{1-a}}{1-L}\varphi(2^l t), \frac{2^{4l}\delta}{2}\right).
\end{aligned}$$

By (25), we have

$$\lim_{l\to\infty} F\left(\frac{L^{1-a}}{1-L}\varphi(2^l t), \frac{2^{4l}\delta}{2}\right) = 1.$$

Consequently, $G(Q_4(t) - R_4(t), \delta) = 1$ for all $t \in E$ and $\delta > 0$. So $Q_4(t) = R_4(t)$ for all $t \in E$, which ends the proof. □

Now, we provide a corollary.

Corollary 5. *Assume that an even mapping $\phi : E \to W$ with $\phi(0) = 0$ fulfills*

$$G(D\phi(t_1, t_2, \cdots, t_m), \delta) \geq \begin{cases} F(\theta, \delta), \\ F\left(\theta \sum_{i=1}^m \|t_i\|^s, \delta\right), \\ F\left(\theta\left(\sum_{i=1}^m \|t_i\|^{ms} + \prod_{i=1}^m \|t_i\|^s\right), \delta\right), \end{cases}$$

for all $t_1, t_2, \cdots, t_m \in E$ and $\delta > 0$, where $\theta > 0$ and $s > 0$ are constants. Then, there exists a unique quartic mapping $Q_4 : E \to W$ such that

$$G(\phi(t) - Q_4(t), \delta) \geq \begin{cases} F(\theta, 15|\psi|\delta), & \\ F\left(\theta\|t\|^s, |2^4 - 2^s||\psi|\delta\right), & s \neq 4, \\ F\left(\theta\|t\|^{ms}, |2^4 - 2^{ms}||\psi|\delta\right), & s \neq \frac{4}{m}, \end{cases}$$

for all $t \in E$ and all $\delta > 0$.

Proof. We take

$$\chi(t_1, t_2, \cdots, t_m) = \begin{cases} \theta, \\ \theta \sum_{i=1}^m \|t_i\|^s, \\ \theta\left(\sum_{i=1}^m \|t_i\|^{ms} + \prod_{i=1}^m \|t_i\|^s\right), \end{cases}$$

for all $t_1, t_2, \cdots, t_m \in E$. Then,

$$F\Big(\chi\Big(\xi_a^l t_1, \xi_a^l t_2, \cdots, \xi_a^l t_m\Big), \xi_a^{4l}\delta\Big)$$
$$= \begin{cases} F\Big(\theta, \xi_a^{4l}\delta\Big), \\ F\Big(\theta \sum_{i=1}^m \|t_i\|^s, \xi_a^{(4-s)l}\delta\Big), \\ F\Big(\theta\big(\prod_{i=1}^m \|t_i\|^s + \sum_{i=1}^m \|t_i\|^{ms}\big), \xi_a^{(4-ms)l}\delta\Big), \end{cases} \tag{32}$$
$$\to \begin{cases} 1, \text{ if } a = 0, \\ 1, \text{ if } (a = 0 \text{ and } s < 4) \text{ or } (a = 1 \text{ and } s > 4), \\ 1, \text{ if } (a = 0 \text{ and } sm < 4) \text{ or } (a = 1 \text{ and } sm > 4). \end{cases}$$

Letting

$$\varphi(t) = \frac{1}{\psi}\chi\Big(0, \frac{t}{2}, 0, \cdots, 0\Big),$$

we then have

$$F\Big(\frac{1}{\xi_a^4}\varphi(\xi_a t), \delta\Big) = \begin{cases} F\big(\xi_a^{-4}\varphi(t), \delta\big), \\ F\big(\xi_a^{s-4}\varphi(t), \delta\big), \\ F\big(\xi_a^{ms-4}\varphi(t), \delta\big), \end{cases} \tag{33}$$

and

$$\begin{aligned} F(\varphi(t), \delta) \quad &= F\Big(\chi\Big(0, \frac{t}{2}, 0, \cdots, 0\Big), |\psi|\delta\Big) \\ &= \begin{cases} F(\theta, |\psi|\delta), \\ F\Big(\frac{\theta}{2^s}\|t\|^s, |\psi|\delta\Big), \\ F\Big(\frac{\theta}{2^{ms}}\|t\|^{ms}, |\psi|\delta\Big). \end{cases} \end{aligned} \tag{34}$$

Using (32)–(34) and applying Theorem 5, we consider the following cases.
Case (i): $L = \frac{1}{2^4}$ for $a = 0$;

$$G(\phi(t) - Q_4(t), \delta) \geq F\Big(\frac{2^{-4}}{1 - 2^{-4}}\varphi(t), \delta\Big) = F(\theta, 15|\psi|\delta).$$

Case (ii): $L = 2^{s-4}$ for $(a = 0, s < 4)$;

$$G(\phi(t) - Q_4(t), \delta) \geq F\Big(\frac{2^{s-4}}{1 - 2^{s-4}}\varphi(t), \delta\Big) = F\Big(\theta\|t\|^s, (2^4 - 2^s)|\psi|\delta\Big).$$

Case (iii): $L = 2^{4-s}$ for $(a = 1, s > 4)$;

$$G(\phi(t) - Q_4(t), \delta) \geq F\Big(\frac{1}{1 - 2^{4-s}}\varphi(t), \delta\Big) = F\Big(\theta\|t\|^s, (2^s - 2^4)|\psi|\delta\Big).$$

Case (iv): $L = 2^{ms-4}$ for $(a = 0, s < \frac{4}{m})$;

$$\begin{aligned} G(\phi(t) - Q_4(t), \delta) &\geq F\Big(\frac{2^{ms-4}}{1 - 2^{ms-4}}\varphi(t), \delta\Big) \\ &= F\Big(\theta\|t\|^{ms}, (2^4 - 2^{ms})|\psi|\delta\Big). \end{aligned}$$

Case (v): $L = 2^{4-ms}$ for $(a = 1, s > \frac{4}{m})$;

$$G(\phi(t) - Q_4(t), \delta) \geq F\left(\frac{1}{1 - 2^{4-ms}}\varphi(t), \delta\right)$$
$$= F\left(\theta\|t\|^{ms}, (2^{ms} - 2^4)|\psi|\delta\right).$$

Hence, the proof is completed. □

5. Conclusions

In this work, we have introduced a new type of quartic functional equation and have derived its general solution. Mainly, we have showed its Hyers–Ulam stability by means of direct and fixed point techniques in fuzzy normed spaces. As a byproduct, we have obtained that if the control function is the fuzzy norm of products of powers of norms, then the quartic functional equation is hyperstable.

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