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Multigranulation Roughness of Intuitionistic Fuzzy Sets by Soft Relations and Their Applications in Decision Making

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Abstract: Multigranulation rough set (MGRS) based on soft relations is a very useful technique to describe the objectives of problem solving. This MGRS over two universes provides the combination of multiple granulation knowledge in a multigranulation space. This paper extends the concept of fuzzy set Shabir and Jamal in terms of an intuitionistic fuzzy set (IFS) based on multi-soft binary relations. This paper presents the multigranulation roughness of an IFS based on two soft relations over two universes with respect to the aftersets and foresets. As a result, two sets of IF soft sets with respect to the aftersets and foresets are obtained. These resulting sets are called lower approximations and upper approximations with respect to the aftersets and with respect to the foresets. Some properties of this model are studied. In a similar way, we approximate an IFS based on multi-soft relations and discuss their some algebraic properties. Finally, a decision-making algorithm has been presented with a suitable example.

Keywords: intuitionistic fuzzy set; soft relation; multigranulation roughness; decision making

MSC: 03E72; 20F10

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1. Introduction

In our real world, many problems naturally involve uncertainty. This uncertainty can be observed in several fields, such as environmental science, medical science, economics and engineering. Researchers are active and interested to address uncertainty. In this respect, many theories have been presented, such as the probability theory, fuzzy set (FS) theory, rough set (RS) theory, intuitionistic fuzzy set (IFS) theory and soft set (SS) theory etc.

Fuzzy set (FS) proposed by Zadeh in [1] is a framework to address partial truth, uncertainty and impreciseness. Zadeh's FS is a very crucial, innovative and ingenious set because of its importance in multiple research dimensions. Often, we are encountered by ill-defined situations which are addressed through quantitative expressions. To evaluate better results from these critical situations, the FS is much useful by using qualitative expressions due to its degree of membership. The FS represents degree of membership for each element of the universe of a discourse to a subset of it, and later on, Attanassov presented intuitionistic fuzzy set (IFS) [2] which avails the opportunity to model the problem precisely based on the observations and treat more accurately to uncertainty quantification. Attanassov discussed the literature based on theory and fundamentals of IFSs in [3]. An IFS is a very useful concept with its applications in many different fields, such as electoral system, market prediction, machine learning, pattern recognition, career determination and medical diagnosis [4]. The description in terms of membership degree only in many cases is insufficient because the presence of non-membership degree is helpful to deal with uncertainty in good manner.

Molodtsov [5] presented an untraditional approach known as soft set (SS) theory for handling the vagueness and uncertainty. A collection of approximate descriptions of an element in terms of parameters by a set-valued map is known as a soft set. This theory has become a successful approach to different problems in different fields due to its rich operations. In decision-making problems, this is an applicable tool using the RSs [6]. Many researchers hybridized the models of SSs with different applicable theories [7–9]. Maji et al. defined Fuzzy SS (FSS) and Intuitionistic fuzzy soft set (IFSS) [10,11]. After that, several extensions of SSs have been presented, such as the vague SS [12], the soft RS (SRS) [13,14], the generalized FSS [15], the trapezoidal FSS [16], interval-valued FSS [17]. Agarwal built a framework of the generalized IFSS [18]. Feng et al. [19] pointed out some errors in generalized IFSS [18] and rebuilt the generalized IFSS. Many authors combined the concepts of IFSs and fuzzy RSs (FRSs). Samanta and Mondal [20] presented the IF rough set (IFRS) model. In [21], the combination of RS and FS has been studied. To overcome the unnaturalness of FRSs, Sang et al. [22] proposed a newly defined IFRS model.

Pawlak presented rough sets (RSs) to deal with incomplete data, vagueness and uncertainty [6,23]. To solve different problems based on incomplete data, many researchers showed interest in RSs. The RS theory is an untraditional technique to discuss data investigation, representation of vague or inexact data and reasoning based reduction of vague data [24]. Recently, researchers have investigated RSs in the light of dataset features, varieties of remedy and procedure control. Many extensions of RSs have been presented for many requirements, such as the RS model based on reflexive relations, equivalence relations and tolerance relations, FRS model, SRS model and rough FS (RFS) model [25–27]. As it is commonly known, many problems have different universes of discourse [28], such as the objections of customers and their solutions in enterprise management, the characteristics of customers and the features of products in personalized marketing, the mechanical defects and their solutions in machine diagnosis, the symptoms of diseases and drugs in diseased diagnosis. To formalize these problems, the RS models have been generalized over two universes [29–31]. Pie et al. [32] built a framework of the RS model on algebraic characteristics over two universes. According to the inter-relationship between two universes, Liu et al. [33] connected the graded RS with appropriate parameters. Ma and Sun [34] proposed a framework of probability RS to deal with impreciseness [19,35].

Granular computing is very useful to describe the objectives of a problem solving through multiple binary relations [36]. Under a single granulation, a set is characterized by lower and upper approximations in the light of granular computing. By using multiple equivalence relations, multigranulation rough set (MGRS) approximations have been investigated. Rauszer [37] presented a framework of a multi-agent system based on an equivalence relation where each agent has its own knowledge base. Khan and Banerjee [38,39] considered the agent as a "source" in a more general setting but many other scholars considered "agent" as granulation [40–42]. Khan et al. [38,39] presented two approximation operators in terms of multiple source approximation system [35]. In the same sense, Qian et al. [43] presented the MGRS theory. There are two types of MGRS, named optimistic MGRS (OMGRS) and pessimistic MGRS (PMGRS) [41]. Later, many extensions of MGRS have been presented by different authors [44,45]. The real dataset involving multiple and overlapping knowledge has been dealt with by presenting MGRS and covering RSs. These two special models have been generalized as many hybrid models such as covering MGRS [43] and MGRS based on multiple equivalence relations [41] etc. Liu et al. [46] used a covering approximation space and presented four types of covering MGRS models. Later on, Xu et al. presented a covering MGRS based on order relations [47], fuzzy compatible relations [48] and generalized relations [49] by relaxing the conditions of equivalence relations. After that, many researchers proposed different MGRS models according to different dataset needs. Dou et al. [50] presented a useful MGRS model for variable-precision and discussed its properties. Ju et al. [51] proposed a heuristic algorithm using their newly defined variable-precision MGRS model for computing reduction. Feng et al. [52] presented a three-way decision-based type-1 variable-precision multigranulation decision-theoretic fuzzy rough set [53]. In management science and various professional fields, the MGRS showed its importance and extensions of MGRS have made their role with respect to the nature of problems [54–58]. Qian et al. discussed the risk attitudes by presenting OMGRS and PMGRS models using multiple binary relations. Huang et al. [59] combined MGRS and IFS and presented an IFMGRS model. Pang et al. [60] combined MGRS with three-way decision making and proposed a multi-criteria decision-making (MCGDM) model. Different experts have different experiences and expertises to solve different decision-making problems. Better decisions can be made by taking opinions of multiple experts compared with taking one expert's opinion only. In view of this logic, Zadeh [36] introduced granular computing knowledge through multiple relations. Sun and Ma [61] proposed the MGRS model over two universes. For selective dataset approximation, Tan et al. [35] presented the MGRS model with granularity selection algorithm [56]. Xu et al. [62] combined Pawlak RS model, FRS model and MGRS model in terms of granular computing and proposed multi-granulation fuzzy rough set (MGFRS) model. Recently, Shabir et al. [63] proposed a useful model of MGRS with multi-soft binary relations. After that, Shabir et al. [64] extended that optimistic MGRS in terms of FS which is called optimistic multigranulation fuzzy rough set (OMGFRS). The existing MGRS models have obvious disadvantages regarding FSs.

1. The existing MGRS models with FSs are unable to manage the real life situations where only degree of membership is discussed;
2. Decision experts have hesitation to make better decision due to no consideration of their own subjective consciousness.

To manage these above critical situations, we extended the model of MGFRS based on soft binary relations in [64] in terms of IFS. We used IFSs instead of FSs to present an optimistic multigranulation intuitionistic fuzzy rough set (OMGIFRS) model.

The organization of the remaining paper is as follows. In Section 2, some basic definitions and fundamental concepts of FS, IFS, RS, SS, FRS, IFSS, MGRS, and soft relation are given. Section 3 presents the optimistic granulation roughness of an IFS based on two soft relations over two universes with their basic properties and examples. OMGIFRS over two universes and their properties are discussed in Section 4. Section 5 presents the decision making algorithm with a practical example about decision making problems. In Section 6, we made a comparison of our proposed model with other existing theories. Finally, we conclude our research work in Section 7.

2. Preliminaries and Basic Concepts

In this section, some fundamental notions about IFS, RS, MGRS, SS, soft binary relation, and IFSS are given. Throughout this paper, U_1 and U_2 represent two non-empty finite sets unless stated otherwise.

Definition 1 ([2]). *Let U be a non-empty universe. An IFS B in the universe U is an object having the form $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in U\}$, where $\mu_B : U \to [0,1]$ and $\gamma_B : U \to [0,1]$ satisfying $0 \le \mu_B(x) + \gamma_B(x) \le 1$ for all $x \in U$. The values $\mu_B(x)$ and $\gamma_B(x)$ are called degree of membership and degree of non-membership of $x \in U$ to B, respectively. The number $\pi_B(x) = 1 - \mu_B(x) - \gamma_B(x)$ is called the degree of hesitancy of $x \in U$ to B. The collection of all IFSs in U is denoted by $IF(U)$. In the remaining paper, we shall write an IFS by $B = \langle \mu_B, \gamma_B\rangle$ instead of $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in U\}$. Let $B = \langle \mu_B, \gamma_B\rangle$ and $B_1 = \langle \mu_{B_1}, \gamma_{B_1}\rangle$ be two IFSs in U. Then, $B \subseteq B_1$ if and only if $\mu_B(x) \le \mu_{B_1}(x)$ and $\gamma_{B_1}(x) \le \gamma_B(x)$ for all $x \in U$. Two IFSs B and B_1 are said to be equal if and only if $B \subseteq B_1$ and $B_1 \subseteq B$.*

Definition 2 ([2]). *The union and intersection of two IFSs B and B_1 in U are denoted and defined by $B \cup B_1 = \langle \mu_B \cup \mu_{B_1}, \gamma_B \cap \gamma_{B_1}\rangle$ and $B \cap B_1 = \langle \mu_B \cap \mu_{B_1}, \gamma_B \cup \gamma_{B_1}\rangle$ where $(\mu_B \cup \mu_{B_1})(x) = \sup\{\mu_B(x), \mu_{B_1}(x)\}$, $(\gamma_B \cap \gamma_{B_1})(x) = \inf\{\gamma_B(x), \gamma_{B_1}(x)\}$, $(\mu_B \cap \mu_{B_1})(x) = \inf\{\mu_B(x), \mu_{B_1}(x)\}$, $(\gamma_B \cup \gamma_{B_1})(x) = \sup\{\gamma_B(x), \gamma_{B_1}(x)\}$, for all $x \in U$.*

Next, we define two special types of IFSs as:

The IF universe set $U = 1_U =< 1, 0 >$ and IF empty set $\Phi = 0_U =< 0, 1 >$, where $1(x) = 1$ and $0(x) = 0$ for all $x \in U$. The complement of an IFS $B =< \mu, \gamma >$ is denoted and defined as $B^c =< \gamma, \mu >$. See Table 1 for acronyms.

Table 1. List of acronyms.

Acronyms	Representations
FSs	Fuzzy sets
IFSs	Intuitionistic fuzzy sets
RSs	Rough sets
SSs	Soft sets
FSSs	Fuzzy soft sets
IFSSs	Intuitionistic fuzzy soft sets
SRSs	Soft rough sets
IFRSs	Intuitionistic fuzzy rough sets
FRSs	Fuzzy rough sets
RFSs	Rough fuzzy sets
MGRS	Multigranulation rough set
OMGRS	Optimistic multigranulation rough set
PMGRS	Pessimistic multigranulation rough set
IFMGRS	Intuitionistic fuzzy multigranulation rough set
MCGDM	Multi-critria group decision making
OMGFRS	Optimistic multigranulation fuzzy rough set
OMGIFRS	Optimistic multigranulation intuitionistic fuzzy rough set
IFN	Intuitionistic fuzzy number
IFV	Intuitionistic fuzzy value

For a fixed $x \in U$, the pair $(\mu_B(x), \gamma_B(x))$ is called intuitionistic fuzzy value (IFV) or intuitionistic fuzzy number (IFN). In order to define the order between two IFNs, Chen and Tan [65] introduced the score function as $S(x) = \mu_B(x) - \gamma_B(x)$ and Hong and Choi [66] defined the accuracy function as $H(x) = \mu_B(x) + \gamma_B(x)$, where $x \in U$. Xu [62,67] used both the score and accuracy functions to form the order relation between any pair of IFVs (x, y) as given below:

(a) if $S(x) > S(y)$, then $x > y$;
(b) if $S(x) = S(y)$, then
  (1) if $H(x) = H(y)$, then $x = y$;
  (2) if $H(x) < H(y)$, then $x < y$.

Definition 3 ([6]). *Let σ be an equivalence relation on a universe U. For any $M \subseteq U$, the Pawlak lower and upper approximations of M with respect to σ are defined by*

$$\begin{aligned} \underline{\sigma}(M) &= \{u \in U : [u]_\sigma \subseteq M\}, \\ \overline{\sigma}(M) &= \{u \in U : [u]_\sigma \cap M \neq \emptyset\}, \end{aligned}$$

where $[u]_\sigma$ is the equivalence class of u with respect to σ. The set $BM_\sigma = \overline{\sigma}(M) - \underline{\sigma}(M)$ is the boundary region of $M \subseteq U$. If $BM_\sigma(M) = \emptyset$, then M is defineable (exact), otherwise, M is rough with respect to σ.

Qian et al. [40] extended the Pawlak rough set model to the MGRS model, where the set approximations are defined by using multi-equivalence relations on a universe.

Definition 4 ([40])**.** *Let $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ be n equivalence relations on a universe U. For any $M \subseteq U$, the Pawlak lower and upper approximations of M are defined by*

$$\begin{aligned} \underline{M}_{\sum_{i=1}^{n}\sigma_i} &= \{u \in U : [u]_{\sigma_i} \subseteq M \text{ for some } i,\ 1 \leq i \leq n\}, \\ \overline{M}^{\sum_{i=1}^{n}\sigma_i} &= (\underline{M^c}_{\sum_{i=1}^{n}\sigma_i})^c, \end{aligned}$$

where $[u]_{\sigma_i}$ is the equivalence class of u with respect to σ_i.

Definition 5 ([5])**.** *A pair (F, A) is called an SS over U if F is a mapping given by $F : A \to P(U)$, where A is a subset of E (the set of parameters) and $P(U)$ is the power set of U. Thus, $F(e)$ is a subset of U for all $e \in A$. Hence, a SS over U is a parameterized collection of subsets of U.*

Definition 6 ([68])**.** *Let (σ, A) be an SS over $U_1 \times U_1$. Then, (σ, A) is called a soft binary relation on U_1. In fact, (σ, A) is a parameterized collection of binary relations on U_1, that is, we have a binary relation $\sigma(e)$ on U_1 for each parameter $e \in A$.*

Li et al. [69] presented the generalization of the soft binary relation over U_1 to U_2, as follows.

Definition 7 ([69])**.** *A soft binary relation (σ, A) from U_1 to U_2 is an SS over $U_1 \times U_2$, that is, $\sigma : A \to P(U_1 \times U_2)$, where A is a subset of the set of parameters E.*

Of course, (σ, A) is a parameterized collection of binary relations from U_1 to U_2. That is, for each $e \in A$, we have a binary relation $\sigma(e)$ from U_1 to U_2.

Definition 8 ([11])**.** *A pair (F, A) is called an IFSS over U if F is a mapping given by $F : A \to IF(U)$ and A is a subset of E (the set of parameters). Thus, $F(e)$ is an IFS in U for all $e \in A$. Hence, an IFSS over U is a parameterized collection of IF sets in U.*

Definition 9 ([11])**.** *For two IFSSs (F, A) and (G, B) over a common universe U, we say that (F, A) is an IF soft subset of (G, B) if (1) $A \subseteq B$ and (2) $F(e)$ is an IF subset of $G(e)$ for all $e \in A$. Two IFSSs (F, A) and (G, B) over a common universe U are said to be IF soft equal if (F, A) is an IF soft subset of (G, B) and (G, B) is an IF soft subset of (F, A). The union of two IFSSs (F, A) and (G, A) over the common universe U is the IFSS (H, A), where $H(e) = F(e) \cup G(e)$ for all $e \in A$. The intersection of two IFSSs (F, A) and (G, A) over the common universe U is the IFSS (K, A), where $K(e) = F(e) \cap G(e)$ for all $e \in A$.*

Definition 10 ([70])**.** *Let (σ, A) be a soft binary relation from U_1 to U_2 and $B = \langle \mu_B, \gamma_B \rangle$ be an IFS in U_2. Then, lower approximation $\underline{\sigma}^B = (\underline{\sigma}^{\mu_B}, \underline{\sigma}^{\gamma_B})$ and upper approximation $\overline{\sigma}^B = (\overline{\sigma}^{\mu_B}, \overline{\sigma}^{\gamma_B})$ of $B = \langle \mu_B, \gamma_B \rangle$ with respect to aftersets are defined as follows:*

$$\underline{\sigma}^{\mu_B}(e)(u_1) = \begin{cases} \bigwedge_{a \in u_1\sigma(e)} \mu_B(a) & \text{if } u_1\sigma(e) \neq \emptyset; \\ 1 & \text{if } u_1\sigma(e) = \emptyset; \end{cases}$$

$$\underline{\sigma}^{\gamma_B}(e)(u_1) = \begin{cases} \bigvee_{a \in u_1\sigma(e)} \gamma_B(a) & \text{if } u_1\sigma(e) \neq \emptyset; \\ 0 & \text{if } u_1\sigma(e) = \emptyset; \end{cases}$$

and

$$\overline{\sigma}^{\mu_B}(e)(u_1) = \begin{cases} \bigvee_{a \in u_1\sigma(e)} \mu_B(a) & \text{if } u_1\sigma(e) \neq \emptyset; \\ 0 & \text{if } u_1\sigma(e) = \emptyset; \end{cases}$$

$$\overline{\sigma}^{\gamma_B}(e)(u_1) = \begin{cases} \bigwedge_{a \in u_1\sigma(e)} \gamma_B(a) & \text{if } u_1\sigma(e) \neq \emptyset; \\ 1 & \text{if } u_1\sigma(e) = \emptyset, \end{cases}$$

where $u_1\sigma(e) = \{a \in U_2 : (u_1, a) \in \sigma(e)\}$ and is called the afterset of u_1 for $u_1 \in U_1$ and $e \in A$.

- $\underline{\sigma}^{\mu_B}(e)(u_1)$ indicates the degree to which u_1 definitely has the property e;
- $\underline{\sigma}^{\gamma_B}(e)(u_1)$ indicates the degree to which u_1 probably does not have the property e;
- $\overline{\sigma}^{\mu_B}(e)(u_1)$ indicates the degree to which u_1 probably has the property e;
- $\overline{\sigma}^{\gamma_B}(e)(u_1)$ indicates the degree to which u_1 definitely does not have the property e.

Definition 11 ([70]). *Let (σ, A) be a soft binary relation from U_1 to U_2 and $B = \langle \mu_B, \gamma_B \rangle$ be an IFS in U_1. Then, lower approximation ${}^B\underline{\sigma} = ({}^{\mu_B}\underline{\sigma}, {}^{\gamma_B}\underline{\sigma})$ and upper approximation ${}^B\overline{\sigma} = ({}^{\mu_B}\overline{\sigma}, {}^{\gamma_B}\overline{\sigma})$ of $B = \langle \mu_B, \gamma_B \rangle$ with respect to foresets are defined as follows:*

$$ {}^{\mu_B}\underline{\sigma}(e)(u_2) = \begin{cases} \wedge_{a \in \sigma(e)u_2} \mu_B(a) & \text{if } \sigma(e)u_2 \neq \emptyset; \\ 1 & \text{if } \sigma(e)u_2 = \emptyset; \end{cases} $$

$$ {}^{\gamma_B}\underline{\sigma}(e)(u_2) = \begin{cases} \vee_{a \in \sigma(e)u_2} \gamma_B(a) & \text{if } \sigma(e)u_2 \neq \emptyset; \\ 0 & \text{if } \sigma(e)u_2 = \emptyset; \end{cases} $$

and

$$ {}^{\mu_B}\overline{\sigma}(e)(u_2) = \begin{cases} \vee_{a \in \sigma(e)u_2} \mu_B(a) & \text{if } \sigma(e)u_2 \neq \emptyset; \\ 0 & \text{if } \sigma(e)u_2 = \emptyset; \end{cases} $$

$$ {}^{\gamma_B}\overline{\sigma}(e)(u_2) = \begin{cases} \wedge_{a \in \sigma(e)u_2} \gamma_B(a) & \text{if } \sigma(e)u_2 \neq \emptyset; \\ 1 & \text{if } \sigma(e)u_2 = \emptyset, \end{cases} $$

where $\sigma(e)u_2 = \{a \in U_1 : (a, u_2) \in \sigma(e)\}$ and is called the foreset of u_2 for $u_2 \in U_2$ and $e \in A$.

Of course, $\underline{\sigma}^B : A \to IF(U_1)$, $\overline{\sigma}^B : A \to IF(U_1)$ and ${}^B\underline{\sigma} : A \to IF(U_2)$, ${}^B\overline{\sigma} : A \to IF(U_2)$.

Theorem 1 ([70]). *Let (σ, A) be a soft binary relation from U_1 to U_2, that is $\sigma : A \to P(U_1 \times U_2)$. For any IFSs $B = \langle \mu_B, \gamma_B \rangle$, $B_1 = \langle \mu_{B_1}, \gamma_{B_1} \rangle$ and $B_2 = \langle \mu_{B_2}, \gamma_{B_2} \rangle$ of U_2, the following are true:*

(1) If $B_1 \subseteq B_2$ then $\underline{\sigma}^{B_1} \subseteq \underline{\sigma}^{B_2}$;
(2) If $B_1 \subseteq B_2$ then $\overline{\sigma}^{B_1} \subseteq \overline{\sigma}^{B_2}$;
(3) $\underline{\sigma}^{B_1} \cap \underline{\sigma}^{B_2} = \underline{\sigma}^{B_1 \cap B_2}$;
(4) $\overline{\sigma}^{B_1} \cap \overline{\sigma}^{B_2} \supseteq \overline{\sigma}^{B_1 \cap B_2}$;
(5) $\underline{\sigma}^{B_1} \cup \underline{\sigma}^{B_2} \subseteq \underline{\sigma}^{B_1 \cup B_2}$;
(6) $\overline{\sigma}^{B_1} \cup \overline{\sigma}^{B_2} = \overline{\sigma}^{B_1 \cup B_2}$;
(7) $\underline{\sigma}^{1_{U_2}} = 1U_1$ if $u_1\sigma(e) \neq \emptyset$;
(8) $\overline{\sigma}^{1_{U_2}} = 1U_1$ if $u_1\sigma(e) \neq \emptyset$;
(9) $\underline{v}^B = \left(\overline{\sigma}^{B^c}\right)^c$ if $u_1\sigma(e) \neq \emptyset$;
(10) $\overline{\sigma}^B = \left(\underline{\sigma}^{B^c}\right)^c$ if $u_1\sigma(e) \neq \emptyset$;
(11) $\underline{\sigma}^{0_{U_2}} = 0_{U_1} = \overline{\sigma}^{0_{U_2}}$ if $u_1\sigma(e) \neq \emptyset$.

Theorem 2 ([70]). *Let (σ, A) be a soft binary relation from U_1 to U_2, that is $\sigma : A \to P(U_1 \times U_2)$. For any IFSs $B = \langle \mu_B, \gamma_B \rangle$, $B_1 = \langle \mu_{B_1}, \gamma_{B_1} \rangle$ and $B_2 = \langle \mu_{B_2}, \gamma_{B_2} \rangle$ of U_1, the following are true:*

(1) If $B_1 \subseteq B_2$, then ${}^{B_1}\underline{\sigma} \subseteq {}^{B_2}\underline{\sigma}$;
(2) If $B_1 \subseteq B_2$, then ${}^{B_1}\overline{\sigma} \subseteq {}^{B_2}\overline{\sigma}$;
(3) ${}^{B_1}\underline{\sigma} \cap {}^{B_2}\underline{\sigma} = {}^{B_1 \cap B_2}\underline{\sigma}$;
(4) ${}^{B_1}\overline{\sigma} \cap {}^{B_2}\overline{\sigma} \supseteq {}^{B_1 \cap B_2}\overline{\sigma}$;
(5) ${}^{B_1}\underline{\sigma} \cup {}^{B2}\underline{\sigma} \subseteq {}^{B_1 \cup B_2}\underline{\sigma}$;
(6) ${}^{B_1}\overline{\sigma} \cup {}^{B_2}\overline{\sigma} = {}^{B_1 \cup B_2}\overline{\sigma}$;
(7) $\underline{\sigma}^{1_{U_1}} = 1_{U_2}$ if $u_2\sigma(e) \neq \emptyset$;
(8) $\overline{\sigma}^{1_{U_1}} = 1_{U_2}$ if $u_2\sigma(e) \neq \emptyset$;

(9) $^{B}\underline{\sigma} = \left(^{B^c}\overline{\sigma}\right)^c$ *if* $u_2\sigma(e) \neq \emptyset$;

(10) $^{B}\overline{\sigma} = \left(^{B^c}\underline{\sigma}\right)^c$ *if* $u_2\sigma(e) \neq \emptyset$;

(11) $^{0_{U_1}}\underline{\sigma} = 0_{U_2} = ^{0_{U_1}}\overline{\sigma}$.

3. Roughness of an Intuitionistic Fuzzy Set by Two Soft Relations

In this section, we discuss the optimistic roughness of an IFS by two soft binary relations from U_1 to U_2. We approximate an IFS of universe U_2 in universe U_1 and an IFS of U_1 in U_2 by using aftersets and foresets of soft binary relations, respectively. In this way, we obtain two IFSSs corresponding to IFSs in $U_2(U_1)$. We also study some properties of these approximations.

Definition 12. *Let U_1 and U_2 be two non-empty sets, (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2 and $B = \langle \mu_B, \gamma_B \rangle$ be an IFS in U_2. Then, the optimistic lower approximation $\underline{\sigma_1 + \sigma_2}_o^B = \left(\underline{\sigma_1 + \sigma_2}_o^{\mu_B}, \underline{\sigma_1 + \sigma_2}_o^{\gamma_B}\right)$ and the upper approximation $^o\overline{\sigma_1 + \sigma_2}^B = \left(^o\overline{\sigma_1 + \sigma_2}^{\mu_B}, ^o\overline{\sigma_1 + \sigma_2}^{\gamma_B}\right)$ of $B = \langle \mu_B, \gamma_B \rangle$ are IF soft sets over U_1 and are defined as:*

$$\underline{\sigma_1 + \sigma_2}_o^{\mu_B}(e)(u_1) = \begin{cases} \wedge\{\mu_B(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\}, & \text{if } u_1\sigma_1(e) \cup u_1\sigma_2(e) \neq \emptyset; \\ 1 & \text{otherwise;} \end{cases}$$

$$\underline{\sigma_1 + \sigma_2}_o^{\gamma_B}(e)(u_1) = \begin{cases} \vee\{\gamma_B(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\}, & \text{if } u_1\sigma_1(e) \cap u_1\sigma_2(e) \neq \emptyset; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$^o\overline{\sigma_1 + \sigma_2}^{\mu_B}(e)(u_1) = \begin{cases} \vee\{\mu_B(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\}, & \text{if } u_1\sigma_1(e) \cap u_1\sigma_2(e) \neq \emptyset; \\ 0 & \text{otherwise;} \end{cases}$$

$$^o\overline{\sigma_1 + \sigma_2}^{\gamma_B}(e)(u_1) = \begin{cases} \wedge\{\gamma_B(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\}, & \text{if } u_1\sigma_1(e) \cup u_1\sigma_2(e) \neq \emptyset; \\ 1 & \text{otherwise;} \end{cases}$$

for all $u_1 \in U_1$, where $u_1\sigma_1(e) = \{u_2 \in U_2 : (u_1, u_2) \in \sigma_1(e)\}$ and $u_1\sigma_2(e) = \{u_2 \in U_2 : (u_1, u_2) \in \sigma_2(e)\}$ are called the aftersets of u_1 for $u_1 \in U_1$ and $e \in A$. Obviously, $(\underline{\sigma_1 + \sigma_2}_o^B(e)), A)$ and $(^o\overline{\sigma_1 + \sigma_2}^B(e))$ are two IFS soft sets over U_1.

Definition 13. *Let U_1 and U_2 be two non-empty sets, (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2 and $B = \langle \mu_B, \gamma_B \rangle$ be an IFS in U_1. Then, the optimistic lower approximation $^B\underline{\sigma_1 + \sigma_2}_o = \left(^{\mu_B}\underline{\sigma_1 + \sigma_2}_o, ^{\gamma_B}\underline{\sigma_1 + \sigma_2}_o\right)$ and the optimistic upper approximation $^B\overline{\sigma_1 + \sigma_2} = \left(^{\mu_B}\overline{\sigma_1 + \sigma_2}^o, ^{\gamma_B}\overline{\sigma_1 + \sigma_2}^o\right)$ of $B = \langle \mu_B, \gamma_B \rangle$ are IF soft sets over U_2 and are defined as:*

$$^{\mu_B}\underline{\sigma_1 + \sigma_2}_o(e)(u_2) = \begin{cases} \wedge\{\mu_B(u_1) : u_1 \in (\sigma_1 u_2(e) \cup \sigma_2 u_2(e))\}, & \text{if } \sigma_1 u_2(e) \cup \sigma_2 u_2(e) \neq \emptyset; \\ 1 & \text{otherwise;} \end{cases}$$

$$^{\gamma_B}\underline{\sigma_1 + \sigma_2}_o(e)(u_2) = \begin{cases} \vee\{\gamma_B(u_1) : u_1 \in (\sigma_1 u_2(e) \cap \sigma_2 u_2(e))\}, & \text{if } \sigma_1 u_2(e) \cap \sigma_2 u_2(e) \neq \emptyset; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$^{\mu_B}\overline{\sigma_1 + \sigma_2}^o(e)(u_2) = \begin{cases} \vee\{\mu_B(u_1) : u_1 \in (\sigma_1 u_2(e) \cap \sigma_2 u_2(e))\}, & \text{if } \sigma_1 u_2(e) \cap \sigma_2 u_2(e) \neq \emptyset; \\ 0 & \text{otherwise;} \end{cases}$$

$$^{\gamma_B}\overline{\sigma_1 + \sigma_2}^o(e)(u_2) = \begin{cases} \wedge\{\gamma_B(u_1) : u_1 \in (\sigma_1 u_2(e) \cup \sigma_2 u_2(e))\}, & \text{if } \sigma_1 u_2(e) \cup \sigma_2 u_2(e) \neq \emptyset; \\ 1 & \text{otherwise;} \end{cases}$$

for all $u_2 \in U_2$ *where* $\sigma_1(e)u_2 = \{u_1 \in U_1 : (u_1, u_2) \in \sigma_1(e)\}$ *and* $\sigma_2(e)u_2 = \{u_1 \in U_1 : (a, u_2) \in \sigma_2(e)\}$ *are called the foresets of* u_2 *for* $u_2 \in U_2$ *and* $e \in A$*. Obviously,* $({}^{B}\underline{\sigma_1 + \sigma_2}_{o}(e)), A)$ *and* $({}^{B}\overline{\sigma_1 + \sigma_2}^{o}(e))$ *are two IFS soft sets over* U_2*.*

Of course, $\underline{\sigma_1 + \sigma_2}_{o}^{B}(e) : A \to IF(U_1)$, ${}^{o}\overline{\sigma_1 + \sigma_2}^{B}(e) : A \to IF(U_1)$ and ${}^{B}\underline{\sigma_1 + \sigma_2}_{o}(e) : A \to IF(U_2)$, ${}^{B}\overline{\sigma_1 + \sigma_2}^{o}(e) : A \to IF(U_2)$.

The following example explains the above definitions.

Example 1. *Let* $U_1 = \{1, 2, 3\}, U_2 = \{a, b, c\}$ *and* $A = \{e_1, e_2\}$*, and* (σ_1, A) *and* (σ_2, A) *be soft binary relations from* U_1 *to* U_2 *defined by*

$$\begin{aligned} \sigma_1(e_1) &= \{(1,a),(1,b),(2,a)\}, \sigma_1(e_2) = \{(2,b),(3,a)\}, \\ \sigma_2(e_1) &= \{(2,b),(2,c),(3,a)\} \text{ and } \sigma_2(e_2) = \{(1,c),(3,b),(3,c)\}. \end{aligned}$$

Then, their aftersets and foresets are

$$\begin{aligned} 1\sigma_1(e_1) &= \{a,b\},\ 2\sigma_1(e_1) = \{a\},\ \ 3\sigma_1(e_1) = \emptyset, \\ 1\sigma_1(e_2) &= \emptyset,\ \ 2\sigma_1(e_2) = \{b\},\ \ 3\sigma_1(e_2) = \{a\} \text{ and} \\ 1\sigma_2(e_1) &= \emptyset,\ \ 2\sigma_2(e_1) = \{b,c\},\ 3\sigma_2(e_1) = \{a\}, \\ 1\sigma_2(e_2) &= \{c\},\ \ 2\sigma_2(e_2) = \emptyset,\ \ 3\sigma_2(e_2) = \{b,c\}, \end{aligned}$$

$$\begin{aligned} \sigma_1(e_1)a &= \{1,2\},\ \sigma_1(e_1)b = \{1\},\ \sigma_1(e_1)c = \emptyset, \\ \sigma_1(e_2)a &= \{3\},\ \ \sigma_1(e_2)b = \{2\},\ \sigma_1(e_2)c = \emptyset \text{and} \\ \sigma_2(e_1)a &= \{3\},\ \ \sigma_2(e_1)b = \{2\},\ \sigma_2(e_1)c = \{2\}, \\ \sigma_2(e_2)a &= \emptyset,\ \ \sigma_2(e_2)b = \{3\},\ \sigma_2(e_2)c = \{1,3\}. \end{aligned}$$

(1) Define $B_1 = \langle \mu_{B_1}, \gamma_{B_1} \rangle : U_2 \to [0, 1]$ (given in Table 2).

Table 2. Intuitionistic fuzzy set B_1.

B_1	a	b	c
μ_{B_1}	0.5	0.4	0.3
γ_{B_1}	0.4	0.5	0.7

The optimistic multigranulation lower and upper approximations of B_1 with respect to the aftersets are given in Tables 3 and 4.

Table 3. Optimistic multigranulation lower approximation of B_1.

	1	**2**	**3**
$\underline{\sigma_1 + \sigma_2}_{o}^{\mu_{B_1}}(e_1)$	0.4	0.3	0.5
$\underline{\sigma_1 + \sigma_2}_{o}^{\gamma_{B_1}}(e_1)$	0.5	0.7	0.4
$\underline{\sigma_1 + \sigma_2}_{o}^{\mu_{B_1}}(e_2)$	0.3	0.4	0.3
$\underline{\sigma_1 + \sigma_2}_{o}^{\gamma_{B_1}}(e_2)$	0.7	0.5	0.7

Table 4. Optimistic multigranulation upper approximation of B_1.

	1	2	3
${}^{o}\overline{\sigma_1+\sigma_2}^{\mu_{B_1}}(e_1)$	0	0	0
${}^{o}\overline{\sigma_1+\sigma_2}^{\gamma_{B_1}}(e_1)$	1	1	1
${}^{o}\overline{\sigma_1+\sigma_2}^{\mu_{B_1}}(e_2)$	0	0	0
${}^{o}\overline{\sigma_1+\sigma_2}^{\gamma_{B_1}}(e_2)$	1	1	1

(2) Define $B = \langle \mu_B, \gamma_B \rangle : U_1 \to [0,1]$ as given in Table 5.

Table 5. Intuitionistic fuzzy set B.

B	1	2	3
μ_B	0.3	0.7	0.6
γ_B	0.6	0.3	0.2

The optimistic multigranulation lower and upper approximations of B with respect to the foresets are given in Tables 6 and 7.

Table 6. Optimistic multigranulation lower approximation of B.

	a	*b*	*c*
${}^{\mu_B}\underline{\sigma_1+\sigma_2}_o(e_1)$	0.3	0.3	0.7
${}^{\gamma_B}\underline{\sigma_1+\sigma_2}_o(e_1)$	0.6	0.6	0.3
${}^{\mu_B}\underline{\sigma_1+\sigma_2}_o(e_2)$	0.6	0.6	0.3
${}^{\gamma_B}\underline{\sigma_1+\sigma_2}_o(e_2)$	0.2	0.2	0.6

Table 7. Optimistic multigranulation upper approximation of B.

	a	*b*	*c*
${}^{\mu_B}\overline{\sigma_1+\sigma_2}^{o}(e_1)$	0	0	0
${}^{\gamma_B}\overline{\sigma_1+\sigma_2}^{o}(e_1)$	1	1	1
${}^{\mu_B}\overline{\sigma_1+\sigma_2}^{o}(e_2)$	0	0	0
${}^{\gamma_B}\overline{\sigma_1+\sigma_2}^{o}(e_2)$	1	1	1

Proposition 1. *Let $(\sigma_1, A), (\sigma_2, A)$ be two soft relations from U_1 to U_2, that is $\sigma_1 : A \to P(U_1 \times U_2)$ and $\sigma_2 : A \to P(U_1 \times U_2)$ and $B \in IF(U_2)$. Then, the following hold with respect to the aftersets:*

1. $\underline{\sigma_1+\sigma_2}_o^{B} \leqslant \underline{\sigma_1}^{B} \vee \underline{\sigma_2}^{B}$;
2. ${}^{o}\overline{\sigma_1+\sigma_2}^{B} \leqslant \overline{\sigma_1}^{B} \wedge \overline{\sigma_2}^{B}$.

Proof. (1) Let $u_1 \in U_1$. Then, $\underline{\sigma_1+\sigma_2}_o^{\mu_B}(e)(u_1) = \wedge\{\mu_B(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} \leqslant (\wedge\{\mu_B(u_2) : u_2 \in (u_1\sigma_1(e)\}) \vee (\wedge\{\mu_B(u_2) : u_2 \in (u_1\sigma_2(e)\}) = \underline{\sigma_1}^{\mu_B}(e)(u_1) \vee \underline{\sigma_2}^{\mu_B}(e)(u_1)$.

Similarly, let $u_1 \in U_1$. Then $\underline{\sigma_1+\sigma_2}_o^{\gamma_B}(e)(u_1) = \vee\{\gamma_B(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\} \geqslant (\vee\{\gamma_B(u_2) : u_2 \in (u_1\sigma_1(e)\}) \wedge (\vee\{\gamma_B(u_2) : u_2 \in (u_1\sigma_2(e)\}) = \underline{\sigma_1}^{\gamma_B}(e)(u_1) \wedge \underline{\sigma_2}^{\gamma_B}(e)(u_1)$.

Hence, $\underline{\sigma_1+\sigma_2}_o^{B} \leqslant \underline{\sigma_1}^{B} \vee \underline{\sigma_2}^{B}$.

(2) The properties can be proved similarly to (1). □

Proposition 2. *Let $(\sigma_1, A), (\sigma_2, A)$ be two soft relations from U_1 to U_2, that is $\sigma_1 : A \to P(U_1 \times U_2)$ and $\sigma_2 : A \to P(U_1 \times U_2)$ and $B \in IF(U_1)$. Then, the following hold with respect to the foresets:*

1. ${}^{B}\underline{\sigma_1+\sigma_2}_o \leqslant^{B} \underline{\sigma_1} \vee^{B} \underline{\sigma_2}$;
2. ${}^{B}\overline{\sigma_1+\sigma_2}^{o} \leqslant^{B} \overline{\sigma_1} \wedge^{B} \overline{\sigma_2}$.

Proof. The proof is similar to the proof of Proposition 1. □

For the converse, we have the following example.

Example 2 (Continued from Example 1)**.** *According to Example 1, we have the following:*

$$\begin{aligned}
\underline{\sigma_1}^{\mu_{B_1}}(e_1)(2) &= 0.5 \text{ and } \underline{\sigma_1}^{\gamma_{B_1}}(e_1)(2) = 0.4;\\
\underline{\sigma_2}^{\mu_{B_1}}(e_1)(2) &= 0.3 \text{ and } \underline{\sigma_2}^{\gamma_{B_1}}(e_1)(2) = 0.7;\\
\overline{\sigma_1}^{\mu_{B_1}}(e_1)(2) &= 0.5 \text{ and } \overline{\sigma_1}^{\mu_{B_1}}(e_1)(2) = 0.4;\\
\overline{\sigma_2}^{\mu_{B_1}}(e_1)(2) &= 0.4 \text{ and } \overline{\sigma_2}^{\gamma_{B_1}}(e_1)(2) = 0.5.
\end{aligned}$$

Hence,

$$\begin{aligned}
\underline{\sigma_1+\sigma_2}_o^{\mu_{B_1}}(e_1)(2) &= 0.3 \ngeq 0.5 = \underline{\sigma_1}^{\mu_{B_1}}(e_1)(2) \vee \underline{\sigma_2}^{\mu_{B_1}}(e_1)(2);\\
\underline{\sigma_1+\sigma_2}_o^{\gamma_{B_1}}(e_1)(2) &= 0.7 \nleq 0.4 = \underline{\sigma_1}^{\gamma_{B_1}}(e_1)(2) \wedge \underline{\sigma_2}^{\gamma_{B_1}}(e_1)(2);\\
{}^o\overline{\sigma_1+\sigma_2}^{\mu_{B_1}}(e_1)(2) &= 0 \ngeq 0.4 = \overline{\sigma_1}^{\mu_{B_1}}(e_1)(2) \wedge \overline{\sigma_2}^{\mu_{B_1}}(e_1)(2);\\
{}^o\overline{\sigma_1+\sigma_2}^{\gamma_{B_1}}(e_1)(2) &= 1 \nleq 0.5 = \overline{\sigma_1}^{\gamma_{B_1}}(e_1)(2) \vee \overline{\sigma_1}^{\gamma_{B_1}}(e_1)(2).
\end{aligned}$$

In addition,

$$\begin{aligned}
{}^{\mu_B}\underline{\sigma_1}(e_1)(a) &= 0.3 \text{ and } {}^{\gamma_B}\underline{\sigma_1}(e_1)(a) = 0.6\\
{}^{\mu_B}\underline{\sigma_2}(e_1)(a) &= 0.6 \text{ and } {}^{\gamma_B}\underline{\sigma_2}(e_1)(a) = 0.2\\
{}^{\mu_B}\overline{\sigma_1}(e_1)(a) &= 0.7 \text{ and } {}^{\gamma_B}\overline{\sigma_1}(e_1)(a) = 0.3\\
{}^{\mu_B}\overline{\sigma_2}(e_1)(a) &= 0.6 \text{ and } {}^{\gamma_B}\overline{\sigma_2}(e_1)(a) = 0.2
\end{aligned}$$

Hence,

$$\begin{aligned}
{}^{\mu_B}\underline{\sigma_1+\sigma_2}_o(e_1)(a) &= 0.3 \ngeq 0.7 = {}^{\mu_B}\underline{\sigma_1}(e_1)(a) \vee {}^{\mu_B}\underline{\sigma_2}(e_1)(a);\\
{}^{\gamma_B}\underline{\sigma_1+\sigma_2}_o(e_1)(a) &= 0.3 \nleq 0.2 = {}^{\gamma_B}\underline{\sigma_1}(e_1)(a) \wedge {}^{\gamma_B}\underline{\sigma_2}(e_1)(a);\\
{}^{\mu_B}\overline{\sigma_1+\sigma_2}^o(e_1)(a) &= 0 \ngeq 0.6 = {}^{\mu_B}\overline{\sigma_1}(e_1)(a) \wedge {}^{\mu_B}\overline{\sigma_2}(e_1)(a);\\
{}^{\gamma_B}\overline{\sigma_1+\sigma_2}^o(e_1)(a) &= 1 \nleq 0.3 = {}^{\gamma_B}\overline{\sigma_1}(e_1)(a) \vee {}^{\gamma_B}\overline{\sigma_2}(e_1)(a).
\end{aligned}$$

Proposition 3. *Let $(\sigma_1, A), (\sigma_2, A)$ be two soft relations from U_1 to U_2, that is $\sigma_1 : A \to P(U_1 \times U_2)$ and $\sigma_2 : A \to P(U_1 \times U_2)$ and $B \in IF(U_2)$. Then, the following hold:*

(1) $\underline{\sigma_1+\sigma_2}_o^{1_{U_2}} = 1_{U_1}$ for all $e \in A$;

(2) ${}^o\overline{\sigma_1+\sigma_2}^{1_{U_2}} = 1_{U_1}$ if $u_1\sigma_1(e) \cap u_1\sigma_2(e) \neq \emptyset$ and $u_1\sigma_1(e) \cup u_1\sigma_2(e) \neq \emptyset$;

(3) $\underline{\sigma_1+\sigma_2}_o^{0_{U_2}} = 0_{U_1}$ if $u_1\sigma_1(e) \cup u_1\sigma_2(e) \neq \emptyset$ and $u_1\sigma_1(e) \cap u_1\sigma_2(e) \neq \emptyset$;

(4) ${}^o\overline{\sigma_1+\sigma_2}^{0_{U_2}} = 0_{U_1}$ for all $e \in A$.

Proof. (1) Let $u_1 \in U_1$ and $1_{U_2} = \langle 1, 0\rangle$ be the universal set of U_2. If $u_1\sigma_1(e) \cup u_1\sigma_2(e) = \emptyset$, then $\underline{\sigma_1+\sigma_2}_o^1(e)(u_1) = 1$ and $\underline{\sigma_1+\sigma_2}_o^0(e)(u_1) = 0$.

If $u_1\sigma_1(e) \cup u_1\sigma_2(e) \neq \emptyset$, then $\underline{\sigma_1+\sigma_2}_o^1(e)(u_1) = \wedge\{1(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} = \wedge\{1 : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} = 1$,

and $\underline{\sigma_1+\sigma_2}_o^0(e)(u_1) = \vee\{0(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\} = \vee\{0 : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\} = 0$.

(2) The properties can be proved similarly to (1).

(3) Let $u_1 \in U_1$ and $0_{U_2} = \langle 0, 1\rangle$ be the universal set of U_2. If $u_1\sigma_1(e) \cup u_1\sigma_2(e) \neq \emptyset$, then $\underline{\sigma_1+\sigma_2}_o^0(e)(u_1) = \wedge\{0(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} = \wedge\{0 : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} = 0$,

and $\underline{\sigma_1+\sigma_2}_o^1(e)(u_1) = \vee\{1(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\} = \vee\{1 : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\} = 1$.

(4) The properties can be proved similarly to (3). □

Proposition 4. *Let $(\sigma_1, A), (\sigma_2, A)$ be two soft relations from U_1 to U_2, that is $\sigma_1 : A \to P(U_1 \times U_2)$ and $\sigma_2 : A \to P(U_1 \times U_2)$ and and $B \in IF(U_1)$. Then, the following hold:*

(1) ${}^{1_{U_1}}\underline{\sigma_1 + \sigma_2}_o = 1_{U_2}$ *for all* $e \in A$*;*

(2) ${}^{1_{U_1}}\overline{\sigma_1 + \sigma_2}^o = 1_{U_2}$ *for all* $e \in A$*,for all* $e \in A$*, if* $\sigma_1(e)u_2 \cap \sigma_2(e)u_2 \neq \emptyset$ *and* $\sigma_1(e)u_2 \cup \sigma_2(e)u_2 \neq \emptyset$*;*

(3) ${}^{0_{U_1}}\underline{\sigma_1 + \sigma_2}_o = 0_{U_2}$ *for all* $e \in A$*,for all* $e \in A$*, if* $\sigma_1(e)u_2 \cup \sigma_2(e)u_2 \neq \emptyset$ *and* $\sigma_1(e)u_2 \cap \sigma_2(e)u_2 \neq \emptyset$*;*

(4) ${}^{0_{U_1}}\overline{\sigma_1 + \sigma_2}^o$ *for all* $e \in A$*.*

Proof. The proof is similar to the proof of Proposition 3. □

Proposition 5. *Let $(\sigma_1, A), (\sigma_2, A)$ be two soft relations from U_1 to U_2, that is $\sigma_1 : A \to P(U_1 \times U_2)$ and $\sigma_2 : A \to P(U_1 \times U_2)$ and $B, B_1, B_2 \in IF(U_2)$. Then, the following properties hold:*

(1) If $B_1 \subseteq B_2$ *then* $\underline{\sigma_1 + \sigma_2}_o^{B_1} \subseteq \underline{\sigma_1 + \sigma_2}_o^{B_2}$*;*

(2) If $B_1 \subseteq B_2$ *then* ${}^o\overline{\sigma_1 + \sigma_2}^{B_1} \subseteq^o \overline{\sigma_1 + \sigma_2}^{B_2}$*;*

(3) $\underline{\sigma_1 + \sigma_2}_o^{B_1 \cap B_2} = \underline{\sigma_1 + \sigma_2}_o^{B_1} \cap \underline{\sigma_1 + \sigma_2}_o^{B_2}$*;*

(4) $\underline{\sigma_1 + \sigma_2}_o^{B_1 \cup B_2} \supseteq \underline{\sigma_1 + \sigma_2}_o^{B_1} \cup \underline{\sigma_1 + \sigma_2}_o^{B_2}$*;*

(5) ${}^o\overline{\sigma_1 + \sigma_2}^{B_1 \cup B_2} =^o \overline{\sigma_1 + \sigma_2}^{B_1} \cup^o \overline{\sigma_1 + \sigma_2}^{B_2}$*;*

(6) ${}^o\overline{\sigma_1 + \sigma_2}^{B_1 \cap B_2} \subseteq^o \overline{\sigma_1 + \sigma_2}^{B_1} \cap^o \overline{\sigma_1 + \sigma_2}^{B_2}$*.*

Proof. (1) Since $B_1 \subseteq B_2$ so $\mu_{B_1} \leqslant \mu_{B_2}$ and $\gamma_{B_1} \geqslant \gamma_{B_2}$. Thus $\underline{\sigma_1 + \sigma_2}_o^{\mu_{B_1}}(e)(u_1) = \wedge\{\mu_{B_1}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} \leqslant \wedge\{\mu_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} = \underline{\sigma_1 + \sigma_2}_o^{\mu_{B_2}}(e)(u_1)$,

and $\underline{\sigma_1 + \sigma_2}_o^{\gamma_{B_1}}(e)(u_1) = \vee\{\gamma_{B_1}(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\} \geqslant \vee\{\gamma_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\} = \underline{\sigma_1 + \sigma_2}_o^{\gamma_{B_2}}(e)(u_1)$.

(2) The properties can be proved similarly to (1).

(3) Let $u_1 \in U_1$. If $u_1\sigma_1(e) \cup u_1\sigma_2(e) = \emptyset$, then $\underline{\sigma_1 + \sigma_2}_o^{\mu_{B_1 \cap B_2}}(e)(u_1) = 1 = \underline{\sigma_1 + \sigma_2}_o^{\mu_{B_1}}(e)$
$(u_1) \cap \underline{\sigma_1 + \sigma_2}_o^{\mu_{B_2}}(e)(u_1)$

and $\underline{\sigma_1 + \sigma_2}_o^{\gamma_{B_1 \cap B_2}}(e)(u_1) = 0 = \underline{\sigma_1 + \sigma_2}_o^{\gamma_{B_1}}(e)(u_1) \cup \underline{\sigma_1 + \sigma_2}_o^{\gamma_{B_2}}(e)(u_1)$.

If $u_1\sigma_1(e) \cup u_1\sigma_2(e) \neq \emptyset$,

then $\underline{\sigma_1 + \sigma_2}_o^{\mu_{B_1 \cap B_2}}(e)(u_1) = \wedge\{(\mu_{B_1} \wedge \mu_{B_2})(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} = \wedge\{\mu_{B_1}$
$(u_2) \wedge \mu_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\}$

$= (\wedge\{\mu_{B_1}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\}) \wedge (\wedge\{\mu_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\})$

$= \underline{\sigma_1 + \sigma_2}_o^{\mu_{B_1}}(e)(u_1) \cap \underline{\sigma_1 + \sigma_2}_o^{\mu_{B_2}}(e)(u_1)$.

In addition, $\underline{\sigma_1 + \sigma_2}_o^{\gamma_{B_1 \cap B_2}}(e)(u_1) = \vee\{(\mu_{B_1} \vee \mu_{B_2})(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} =$
$\vee\{\mu_{B_1}$
$(u_2) \vee \mu_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\}$

$= (\vee\{\mu_{B_1}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\}) \vee (\vee\{\mu_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\})$

$= \underline{\sigma_1 + \sigma_2}_o^{\mu_{B_1}}(e)(u_1) \cup \underline{\sigma_1 + \sigma_2}_o^{\mu_{B_2}}(e)(u_1)$.

This shows that $\underline{\sigma_1 + \sigma_2}_o^{B_1 \cap B_2} = \underline{\sigma_1 + \sigma_2}_o^{B_1} \cap \underline{\sigma_1 + \sigma_2}_o^{B_2}$.

(4) The properties can be proved similarly to (3).

(5) Let $u_1 \in U_1$. If $u_1\sigma_1(e) \cap u_1\sigma_2(e) = \emptyset$, then ${}^o\overline{\sigma_1 + \sigma_2}^{\mu_{B_1 \cup B_2}}(e)(u_1) = 0 =^o \overline{\sigma_1 + \sigma_2}^{\mu_{B_1}}$
$(e)(u_{1)} \cup^o \overline{\sigma_1 + \sigma_2}^{\mu_{B_2}}(e)(u_{1)}$

and ${}^o\overline{\sigma_1 + \sigma_2}^{\gamma_{B_1 \cup B_2}}(e)(u_1) = 1 =^o \overline{\sigma_1 + \sigma_2}^{\mu_{B_1}}(e)(u_{1)} \cap^o \overline{\sigma_1 + \sigma_2}^{\mu_{B_2}}(e)(u_{1)}$.

If $u_1\sigma_1(e) \cap u_1\sigma_2(e) \neq \emptyset$,

then ${}^o\overline{\sigma_1 + \sigma_2}^{\mu_{B_1 \cup B_2}}(e)(u_1) = \vee\{(\mu_{B_1} \vee \mu_{B_2})(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\} = \vee\{\mu_{B_1}$
$(u_2) \vee \mu_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\}$

$= (\vee\{\mu_{B_1}(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\}) \vee (\vee\{\mu_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cap u_1\sigma_2(e))\})$

$=^o \overline{\sigma_1 + \sigma_2}^{\mu_{B_1}}(e)(u_1) \cup^o \overline{\sigma_1 + \sigma_2}^{\mu_{B_2}}(e)(u_1)$.

In addition, ${}^o\overline{\sigma_1 + \sigma_2}^{\gamma_{B_1 \cup B_2}}(e)(u_1) = \wedge\{(\gamma_{B_1} \wedge \gamma_{B_2})(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\} =$
$\wedge\{\gamma_{B_1}(u_2) \wedge \gamma_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\}$

$= (\wedge\{\gamma_{B_1}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\}) \wedge (\wedge\{\gamma_{B_2}(u_2) : u_2 \in (u_1\sigma_1(e) \cup u_1\sigma_2(e))\})$
$=^o \overline{\sigma_1+\sigma_2}^{\gamma_{B_1}}(e)(u_1) \cap^o \overline{\sigma_1+\sigma_2}^{\gamma_{B_2}}(e)(u_1)$.
This shows that ${}^o\overline{\sigma_1+\sigma_2}^{B_1\cup B_2} =^o \overline{\sigma_1+\sigma_2}^{B_1} \cup^o \overline{\sigma_1+\sigma_2}^{B_2}$.
(6) The properties can be proved similarly to (5). □

Proposition 6. *Let $(\sigma_1, A), (\sigma_2, A)$ be two soft relations from U_1 to U_2, that is $\sigma_1 : A \to P(U_1 \times U_2)$ and $\sigma_2 : A \to P(U_1 \times U_2)$ and $B, B_1, B_2 \in IF(U_2)$. Then, the following properties hold:*
(1) If $B_1 \subseteq B_2$ then ${}^{B_1}\underline{\sigma_1+\sigma_2}_o \subseteq^{B_2} \underline{\sigma_1+\sigma_2}_o$;
(2) If $B_1 \subseteq B_2$ then ${}^{B_1}\overline{\sigma_1+\sigma_2}^o \subseteq^{B_2} \overline{\sigma_1+\sigma_2}^o$;
(3) ${}^{B_1\cap B_2}\underline{\sigma_1+\sigma_2}_o =^{B_1} \underline{\sigma_1+\sigma_2}_o \cap^{B_2} \underline{\sigma_1+\sigma_2}_o$;
(4) ${}^{B_1\cup B_2}\underline{\sigma_1+\sigma_2}_o \supseteq^{B_1} \underline{\sigma_1+\sigma_2}_o \cup^{B_2} \underline{\sigma_1+\sigma_2}_o$;
(5) ${}^{B_1\cup B_2}\overline{\sigma_1+\sigma_2}^o =^{B_1} \overline{\sigma_1+\sigma_2}^o \cup^{B_2} \overline{\sigma_1+\sigma_2}^o$;
(6) ${}^{B_1\cap B_2}\overline{\sigma_1+\sigma_2}^o \subseteq^{B_1} \overline{\sigma_1+\sigma_2}^o \cap^{B_2} \overline{\sigma_1+\sigma_2}^o$.

Proof. The proof is similar to the proof of Proposition 5. □

4. Roughness of an Intuitionistic Fuzzy Set over Two Universes by Multi-Soft Relations

In this section, we discuss the optimistic roughness of an IFS by multi-soft binary relations from U_1 to U_2 and approximate an IFS of universe U_2 in universe U_1 and an IFS U_1 in U_2 by using aftersets and foresets of soft binary relations, respectively. In this way, we obtain two intuitionistic fuzzy soft sets corresponding to IFSs in $U_2(U_1)$. We also study some properties of these approximations.

Definition 14. *Let U_1 and U_2 be two non-empty finite universes, π be a family of soft binary relations from U_1 to U_2. Then, we say (U_1, U_2, π) the multigranulation generalized soft approximation space over two universes.*

Definition 15. *Let (U_1, U_2, π) be the multigranulation generalized soft approximation space over two universes U_1 and U_2, where $\pi = \sigma_1, \sigma_2, \sigma_3, ...\sigma_m$ and $B = \langle\mu_B, \gamma_B\rangle$ be an IFS in U_2. Then, the optimistic lower approximation $\underline{\Sigma_{i=1}^m \sigma_i}_o^B = \left(\underline{\Sigma_{i=1}^m \sigma_i}_o^{\mu_B}, \underline{\Sigma_{i=1}^m \sigma_i}_o^{\gamma_B}\right)$ and the optimistic upper approximation ${}^o\overline{\Sigma_{i=1}^m \sigma_i}^B = \left({}^o\overline{\Sigma_{i=1}^m \sigma_i}^{\mu_B}, {}^o\overline{\Sigma_{i=1}^m \sigma_i}^{\gamma_B}\right)$ of $B = \langle\mu_B, \gamma_B\rangle$ are IF soft sets over U_1 with respect to the aftersets of soft relations $(\sigma_i, A) \in \pi$ and are defined as:*

$$\underline{\Sigma_{i=1}^m \sigma_i}_o^{\mu_B}(e)(u_1) = \begin{cases} \wedge\{\mu_B(u_2) : u_2 \in \cup_{i=1}^m u_1\sigma_i(e)\}, & \text{if } \cup_{i=1}^m u_1\sigma_i(e) \neq \emptyset; \\ 1 & \text{otherwise;} \end{cases}$$

$$\underline{\Sigma_{i=1}^m \sigma_i}_o^{\gamma_B}(e)(u_1) = \begin{cases} \vee\{\gamma_B(u_2) : u_2 \in \cap_{i=1}^m u_1\sigma_i(e)\}, & \text{if } \cap_{i=1}^m u_1\sigma_i(e) \neq \emptyset; \\ 0 & \text{otherwise;} \end{cases}$$

and

$${}^o\overline{\Sigma_{i=1}^m \sigma_i}^{\mu_B}(e)(u_1) = \begin{cases} \vee\{\mu_B(u_2) : u_2 \in \cap_{i=1}^m u_1\sigma_i(e)\}, & \text{if } \cap_{i=1}^m u_1\sigma_i(e) \neq \emptyset; \\ 0 & \text{otherwise;} \end{cases}$$

$${}^o\overline{\Sigma_{i=1}^m \sigma_i}^{\gamma_B}(e)(u_1) = \begin{cases} \wedge\{\gamma_B(u_2) : u_2 \in \cup_{i=1}^m u_1\sigma_i(e)\}, & \text{if } \cup_{i=1}^m u_1\sigma_i(e) \neq \emptyset; \\ 1 & \text{otherwise,} \end{cases}$$

where $u_1\sigma_i(e) = \{u_2 \in U_2 : (u_1, u_2) \in \sigma_i(e)\}$ are called the aftersets of u_1 for $u_1 \in U_1$ and $e \in A$. Obviously, $(\underline{\Sigma_{i=1}^m \sigma_i}_o^B, A)$ and $({}^o\overline{\Sigma_{i=1}^m \sigma_i}^B, A)$ are two IFS soft sets over U_1.

Definition 16. *Let (U_1, U_2, π) be the multigranulation generalized soft approximation space over two universes U_1 and U_2, where $\pi = \sigma_1, \sigma_2, \sigma_3, ...\sigma_m$ and $B = \langle\mu_B, \gamma_B\rangle$ be an IFS in U_1. Then, the optimistic lower approximation ${}^B\underline{\Sigma_{i=1}^m \sigma_i}_o = \left({}^{\mu_B}\underline{\Sigma_{i=1}^m \sigma_i}_o, {}^{\gamma_B}\underline{\Sigma_{i=1}^m \sigma_i}_o\right)$ and the optimistic upper* *approximation* ${}^{B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o} = \left({}^{\mu_B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o}, {}^{\gamma_B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o}\right)$ *of* $B = \langle \mu_B, \gamma_B \rangle$ *are IF soft sets over* U_2 *with respect to the foresets of soft relations* $(\sigma_i, A) \in \pi$ *and are defined as:*

$$ {}^{\mu_B}\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}(e)(u_2) = \begin{cases} \wedge\{\mu_B(u_1) : u_1 \in \cup_{i=1}^{m}\sigma_i(e)u_2\}, & \text{if } \cup_{i=1}^{m}\sigma_i(e)u_2 \neq \emptyset; \\ 1 & \text{otherwise;} \end{cases} $$

$$ {}^{\gamma_B}\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}(e)(u_2) = \begin{cases} \vee\{\gamma_B(u_1) : u_1 \in \cap_{i=1}^{m}\sigma_i(e)u_2\}, & \text{if } \cap_{i=1}^{m}\sigma_i(e)u_2 \neq \emptyset; \\ 0 & \text{otherwise;} \end{cases} $$

and

$$ {}^{\mu_B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o}(e)(u_2) = \begin{cases} \vee\{\mu_B(u_1) : u_1 \in \cap_{i=1}^{m}\sigma_i(e)u_2\}, & \text{if } \cap_{i=1}^{m}\sigma_i(e)u_2 \neq \emptyset; \\ 0 & \text{otherwise;} \end{cases} $$

$$ {}^{\gamma_B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o}(e)(u_2) = \begin{cases} \wedge\{\gamma_B(u_1) : u_1 \in \cup_{i=1}^{m}\sigma_i(e)u_2\}, & \text{if } \cup_{i=1}^{m}\sigma_i(e)u_2 \neq \emptyset; \\ 1 & \text{otherwise,} \end{cases} $$

where $\sigma_1(e)u_2 = \{u_1 \in U_1 : (u_1, u_2) \in \sigma_i(e)\}$ *are called the foresets of* u_2 *for* $u_2 \in U_2$ *and* $e \in A$. *Obviously,* $({}^{B}\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}, A)$ *and* $({}^{B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o}, A)$ *are two IFS soft sets over* U_2.

Moreover, $\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}^{B} : A \to IF(U_1), {}^{o}\overline{\Sigma_{i=1}^{m}\sigma_i}^{B} : A \to IF(U_1)$ and ${}^{B}\underline{\Sigma_{i=1}^{m}\sigma_i}_{o} : A \to IF(U_2), {}^{B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o} : A \to IF(U_2)$.

Proposition 7. *Let* (U_1, U_2, π) *be the multigranulation generalized soft approximation space over two universes* U_1 *and* U_2 *and* $B = \langle \mu_B, \gamma_B \rangle$ *be an IFS in* U_2. *Then, the following properties for* $\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}^{B}, {}^{o}\overline{\Sigma_{i=1}^{m}\sigma_i}^{B}$ *hold:*

(1) $\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}^{B} \subseteq \vee_{i=1}^{m}\underline{\sigma_i}^{B}$;

(2) ${}^{o}\overline{\Sigma_{i=1}^{m}\sigma_i}^{B} \subseteq \wedge_{i=1}^{m}\overline{\sigma_i}^{B}$.

Proof. The proof is similar to the proof of Proposition 1. □

Proposition 8. *Let* (U_1, U_2, π) *be the multigranulation generalized soft approximation space over two universes* U_1 *and* U_2 *and* $B = \langle \mu_B, \gamma_B \rangle$ *be an IFS in* U_1. *Then, the following properties for* ${}^{B}\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}, {}^{B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o}$ *hold:*

(1) ${}^{B}\underline{\Sigma_{i=1}^{m}\sigma_i}_{o} \subseteq \vee_{i=1}^{m} {}^{B}\underline{\sigma_i}$;

(2) ${}^{B}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o} \subseteq \wedge_{i=1}^{m} {}^{B}\overline{\sigma_i}$.

Proof. The proof is similar to the proof of Proposition 2. □

Proposition 9. *Let* (U_1, U_2, π) *be the multigranulation generalized soft approximation space over two universes* U_1 *and* U_2. *Then, the following properties with respect to the aftersets hold:*

(1) $\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}^{1_{U_2}} = 1_{U_1}$ *for all* $e \in A$;

(2) ${}^{o}\overline{\Sigma_{i=1}^{m}\sigma_i}^{1_{U_2}} = 1_{U_1}$ *if* $\cap_{i=1}^{m}u_1\sigma_i(e) \neq \emptyset$ *and* $\cup_{i=1}^{m}u_1\sigma_i(e) \neq \emptyset$, *for some* $i \leq m$;

(3) $\underline{\Sigma_{i=1}^{m}\sigma_i}_{o}^{0_{U_2}} = 0_{U_1}$ *if* $\cup_{i=1}^{m}u_1\sigma_i(e) \neq \emptyset$ *and* $\cap_{i=1}^{m}u_1\sigma_i(e) \neq \emptyset$, *for some* $i \leq m$;

${}^{o}\overline{\Sigma_{i=1}^{m}\sigma_i}^{0_{U_2}} = 0_{U_1}$ *for all* $e \in A$.

Proof. The proof is similar to the proof of Proposition 3. □

Proposition 10. *Let* (U_1, U_2, π) *be the multigranulation generalized soft approximation space over two universes* U_1 *and* U_2. *Then, the following properties with respect to the foresets hold:*

(1) ${}^{1_{U_1}}\underline{\Sigma_{i=1}^{m}\sigma_i}_{o} = 1_{U_2}$ *for all* $e \in A$;

(2) ${}^{1_{U_1}}\overline{\Sigma_{i=1}^{m}\sigma_i}^{o} = 1_{U_2}$ *if* $\cap_{i=1}^{m}\sigma_i(e)u_2 \neq \emptyset$ *and* $\cup_{i=1}^{m}\sigma_i(e)u_2 \neq \emptyset$, *for some* $i \leq m$;

(3) ${}^{0_{U_1}}\underline{\underline{\Sigma_{i=1}^{m}\sigma_i}}_o = 0_{U_2}$ *if* $\cup_{i=1}^{m}\sigma_i(e)u_2 \neq \emptyset$ *and* $\cap_{i=1}^{m}\sigma_i(e)u_2 \neq \emptyset$, *for some* $i \leq m$;
(4) ${}^{0_{U_1}}\overline{\Sigma_{i=1}^{m}\sigma_i}^o = 0_{U_2}$ *for all* $e \in A$.

Proof. The proof is similar to the proof of Proposition 4. □

Proposition 11. *Let* (U_1, U_2, π) *be the multigranulation generalized soft approximation space over two universes* U_1 *and* U_2 *and* $B, B_1, B_2 \in IF(U_2)$. *Then, the following properties for* $\underline{\Sigma_{i=1}^{m}\sigma_i}_o^B$, ${}^o\overline{\Sigma_{i=1}^{m}\sigma_i}^B$ *with respect the aftersets hold:*
(1) *If* $B_1 \subseteq B_2$ *then* $\underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_1} \subseteq \underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_2}$;
(2) *If* $B_1 \subseteq B_2$ *then* ${}^o\overline{\Sigma_{i=1}^{m}\sigma_i}^{B_1} \subseteq^o \overline{\Sigma_{i=1}^{m}\sigma_i}^{B_2}$;
(3) $\underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_1\cap B_2} = \underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_1} \cap \underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_2}$;
(4) $\underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_1\cup B_2} \supseteq \underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_1} \cup \underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_2}$;
(5) ${}^o\overline{\Sigma_{i=1}^{m}\sigma_i}^{B_1\cup B_2} =^o \overline{\Sigma_{i=1}^{m}\sigma_i}^{B_1} \cup^o \overline{\Sigma_{i=1}^{m}\sigma_i}^{B_2}$;
(6) ${}^o\overline{\Sigma_{i=1}^{m}\sigma_i}^{B_1\cap B_2} \subseteq^o \overline{\Sigma_{i=1}^{m}\sigma_i}^{B_1} \cap^o \overline{\Sigma_{i=1}^{m}\sigma_i}^{B_2}$.

Proof. The proof is similar to the proof of Proposition 5. □

Proposition 12. *Let* (U_1, U_2, π) *be the multigranulation generalized soft approximation space over two universes* U_1 *and* U_2 *and* $B, B_1, B_2 \in IF(U_1)$. *Then, the following properties for* ${}^B\underline{\Sigma_{i=1}^{m}\sigma_i}_o$, ${}^B\overline{\Sigma_{i=1}^{m}\sigma_i}^o$ *with respect the foresets hold:*
(1) *If* $B_1 \subseteq B_2$ *then* ${}^{B_1}\underline{\Sigma_{i=1}^{m}\sigma_i}_o \subseteq^{B_2} \underline{\Sigma_{i=1}^{m}\sigma_i}_o$;
(2) *If* $B_1 \subseteq B_2$ *then* ${}^{B_1}\overline{\Sigma_{i=1}^{m}\sigma_i}^o \subseteq^{B_2} \overline{\Sigma_{i=1}^{m}\sigma_i}^o$;
(3) ${}^{B_1\cap B_2}\underline{\Sigma_{i=1}^{m}\sigma_i}_o =^{B_1} \underline{\Sigma_{i=1}^{m}\sigma_i}_o \cap^{B_2} \underline{\Sigma_{i=1}^{m}\sigma_i}_o$;
(4) ${}^{B_1\cup B_2}\underline{\Sigma_{i=1}^{m}\sigma_i}_o \supseteq^{B_1} \underline{\Sigma_{i=1}^{m}\sigma_i}_o \cup^{B_2} \underline{\Sigma_{i=1}^{m}\sigma_i}_o$;
(5) ${}^{B_1\cup B_2}\overline{\Sigma_{i=1}^{m}\sigma_i}^o =^{B_1} \overline{\Sigma_{i=1}^{m}\sigma_i}^o \cup^{B_2} \overline{\Sigma_{i=1}^{m}\sigma_i}^o$;
(6) ${}^{B_1\cap B_2}\overline{\Sigma_{i=1}^{m}\sigma_i}^o \subseteq^{B_1} \overline{\Sigma_{i=1}^{m}\sigma_i}^o \cap^{B_2} \overline{\Sigma_{i=1}^{m}\sigma_i}^o$.

Proof. The proof is similar to the proof of Proposition 6. □

Proposition 13. *Let* (U_1, U_2, π) *be the multigranulation generalized soft approximation space over two universes* U_1 *and* U_2 *and* $B_1, B_2, B_3, \ldots B_n \in IF(U_2)$, *and* $B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots \subseteq B_n$. *Then, the following properties with respect the aftersets hold:*
(1) $\underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_1} \subseteq \underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_2} \subseteq \underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_3} \subseteq \ldots\ldots \subseteq \underline{\Sigma_{i=1}^{m}\sigma_i}_o^{B_n}$;
(2) ${}^o\overline{\Sigma_{i=1}^{m}\sigma_i}^{B_1} \subseteq^o \overline{\Sigma_{i=1}^{m}\sigma_i}^{B_2} \subseteq^o \overline{\Sigma_{i=1}^{m}\sigma_i}^{B_3} \subseteq \ldots\ldots \subseteq^o \overline{\Sigma_{i=1}^{m}\sigma_i}^{B_n}$.

Proof. The proof is similar to the proof of Proposition 5. □

Proposition 14. *Let* (U_1, U_2, π) *be the multigranulation generalized soft approximation space over two universes* U_1 *and* U_2 *and* $B_1, B_2, B_3, \ldots B_n \in IF(U_1)$, *and* $B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots \subseteq B_n$. *Then, the following properties with respect the foresets hold:*
(1) ${}^{B_1}\underline{\Sigma_{i=1}^{m}\sigma_i}_o \subseteq^{B_2} \underline{\Sigma_{i=1}^{m}\sigma_i}_o \subseteq^{B_3} \underline{\Sigma_{i=1}^{m}\sigma_i}_o \subseteq \ldots\ldots \subseteq^{B_n} \underline{\Sigma_{i=1}^{m}\sigma_i}_o$;
(2) ${}^{B_1}\overline{\Sigma_{i=1}^{m}\sigma_i}^o \subseteq^{B_2} \overline{\Sigma_{i=1}^{m}\sigma_i}^o \subseteq^{B_1} \overline{\Sigma_{i=1}^{m}\sigma_i}^o \subseteq \ldots\ldots \subseteq^{B_n} \overline{\Sigma_{i=1}^{m}\sigma_i}^o$.

Proof. The proof is similar to the proof of Proposition 6. □

5. Application in Decision-Making Problem

Decision making is a major area of study in almost all types of data analysis. To select effective alternatives from aspirants is the process of decision making. Since our environment is becoming changeable and complicated day by day and the decision-making process proposed by a single expert is no longer good, therefore, a decision-making algorithm based on consensus by using collective wisdom is a better approach. From imprecise multi-observer data, Maji et al. [71] proposed a useful technique of object recognition. Feng et al. [72] pointed out errors in Maji et al. [71] and rebuilt a framework correctly. Shabir et al. [63] presented MGRS model based on soft relations by using crisp sets and proposed a decision-making algorithm. Jamal and Shabir [64] presented a decision-making algorithm by using the OMGRS model in terms of FS based on soft relations. This paper extends Jamal's OMGFRS model and presents the decision-making method based on multi-soft relations by use of OMGIFRS.

The lower and upper approximations are the closest to approximated subsets of a universe. We obtain two corresponding values $\underline{\Sigma_{i=1}^{m}\sigma_i}_o^B(e_j)(x_k)$ and ${}^o\overline{\Sigma_{i=1}^{m}\sigma_i}^B(e_j)(x_k)$ with respect to the afterset to the decision alternative $x_k \in U_1$ by the IF soft lower and upper approximations of an IF $B \in IF(U_2)$.

We present Algorithms 1 and 2 for our proposed model here.

Algorithm 1: Aftersets for decision-making problem

(1) Compute the optimistic multigranulation lower IF soft set approximation $\underline{\Sigma_{i=1}^{m}\sigma_i}_o^B$ and optimistic multigranulation upper IF soft set approximation ${}^o\overline{\Sigma_{i=1}^{m}\sigma_i}^B$ of an IF set $B = \langle \mu_B, \gamma_B \rangle$ with respect to the aftersets;
(2) Compute the score values for each of the entries of the $\underline{\Sigma_{i=1}^{m}\sigma_i}_o^B$ and ${}^o\overline{\Sigma_{i=1}^{m}\sigma_i}^B$ and denote them by $\underline{S}_{ij}(x_i, e_j)$ and $\overline{S}_{ij}(x_i, e_j)$ for all i, j;
(3) Compute the aggregated score $\underline{S}(x_i) = \sum_{j=1}^{n} \underline{S}_{ij}(x_i, e_j)$ and $\overline{S}(x_i) = \sum_{j=1}^{n} \overline{S}_{ij}(x_i, e_j)$;
(4) Compute $S(x_i) = \underline{S}(x_i) + \overline{S}(x_i)$;
(5) The best decision is $x_k = \max_i S(x_i)$;
(6) If k has more than one value, say k_1, k_2, then we calculate the accuracy values $\underline{H}_{ij}(x_i, e_j)$ and $\overline{H}_{ij}(x_i, e_j)$ for only those x_k for which $S(x_k)$ are equal;
(7) Compute $H(x_k) = \sum_{j=1}^{n} \underline{H}_{kj}(x_k, e_j) + \sum_{j=1}^{n} \overline{H}_{kj}(x_k, e_j)$ for $k = k_1, k_2$;
(8) If $H(x_{k_1}) > H(x_{k_2})$, then we select x_{k_1};
(9) If $H(x_{k_1}) = H(x_{k_2})$, then select any one of x_{k_1} and x_{k_2}.

Algorithm 2: Foresets for decision-making problem

(1) Compute the optimistic multigranulation lower IF soft set approximation ${}^B\underline{\Sigma_{i=1}^{m}\sigma_i}_o^B$ and upper multigranulation IF soft set approximation ${}^B\overline{\Sigma_{i=1}^{m}\sigma_i}^o$ of an IF set $B = \langle \mu_B, \gamma_B \rangle$ with respect to the foresets;
(2) Compute the score values for each of the entries of the ${}^B\underline{\Sigma_{i=1}^{m}\sigma_i}_o^B$ and ${}^B\overline{\Sigma_{i=1}^{m}\sigma_i}^o$ and denote them by $\underline{S}_{ij}(x_i, e_j)$ and $\overline{S}_{ij}(x_i, e_j)$ for all i, j;
(3) Compute the aggregated score $\underline{S}(x_i) = \sum_{j=1}^{n} \underline{S}_{ij}(x_i, e_j)$ and $\overline{S}(x_i) = \sum_{j=1}^{n} \overline{S}_{ij}(x_i, e_j)$;
(4) Compute $S(x_i) = \underline{S}(x_i) + \overline{S}(x_i)$;
(5) The best decision is $x_k = \max_i S(x_i)$;
(6) If k has more than one value, say k_1, k_2, then we calculate the accuracy values $\underline{H}_{ij}(x_i, e_j)$ and $\overline{H}_{ij}(x_i, e_j)$ for only those x_k for which $S(x_k)$ are equal;
(7) Compute $H(x_k) = \sum_{j=1}^{n} \underline{H}_{kj}(x_k, e_j) + \sum_{j=1}^{n} \overline{H}_{kj}(x_k, e_j)$ for $k = k_1, k_2$;
(8) If $H(x_{k_1}) > H(x_{k_2})$ then we select x_{k_1};
(9) If $H(x_{k_1}) = H(x_{k_2})$ then select any one of x_{k_1} and x_{k_2}.

Now, we show the proposed approach of decision making step by step by using following example. The following example discusses an algorithm to make a wise decision for the selection of a car.

Example 3. *Suppose a multi-national company wants to select a best officer and there are 10 short-listed applicants which are categorized in two groups, platinum and diamond. The set* $U_1 = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ *represents the applicants of platinum group and* $U_2 = \{c_1, c_2, c_3, c_4\}$ *represents the applicants of diamond group. Let* $A = \{e_1, e_2, e_3\}$ *={*e_1*=education,*e_2*=experience,* e_3*=computer knowledge} be the set of parameters. Let two different teams of interviewers analyze and compare the competencies of these applicants.*

We have $\sigma_1 : A \to P(U_1 \times U_2)$ represent the comparison of the first-interviewer team defined by

$$\begin{aligned}
\sigma_1(e_1) &= \{(m_1,c_1),(m_1,c_2),(m_2,c_2),(m_2,c_4),(m_4,c_2),(m_4,c_3),(m_5,c_3),(m_5,c_4),(m_6,c_1)\}, \\
\sigma_1(e_2) &= \{(m_1,c_1),(m_2,c_3),(m_4,c_1),(m_5,c_1),(m_6,c_2),(m_6,c_3)\}, \\
\text{and } \sigma_1(e_3) &= \{(m_1,c_1),(m_2,c_4),(m_3,c_1),(m_3,c_3),(m_4,c_1),(m_5,c_3),(m_5,c_4)\},
\end{aligned}$$

where $\sigma_1(e_1)$ compares the education of applicants, $\sigma_1(e_2)$ compares the experience of applicants, $\sigma_1(e_3)$ compares the computer knowledge of applicants.

Similarly, $\sigma_2 : A \to P(U_1 \times U_2)$ represent the comparison of the second-interviewer team defined by

$$\begin{aligned}
\sigma_2(e_1) &= \{(m_1,c_1),(m_1,c_2),(m_2,c_3),(m_3,c_4),(m_4,c_2),(m_5,c_2),(m_6,c_3)\}, \\
\sigma_2(e_2) &= \{(m_1,c_1),(m_3,c_2),(m_4,c_1),(m_6,c_4)\}, \\
\text{and } \sigma_2(e_3) &= \{(m_1,c_1),(m_1,c_3),(m_2,c_2),(m_2,c_3),(m_4,c_1),(m_5,c_4),(m_6,c_4)\},
\end{aligned}$$

where $\sigma_2(e_1)$ compares the education of applicants, $\sigma_2(e_2)$ compares the experience of applicants, $\sigma_2(e_3)$ compares the computer knowledge of applicants.

From these comparisons, we obtain two soft relations from U_1 to U_2. Now, the aftersets

$$\begin{aligned}
m_1\sigma_1(e_1) &= \{c_1,c_2\},\ m_2\sigma_1(e_1) = \{c_2,c_4\},\ m_3\sigma_1(e_1) = \emptyset, \\
m_4\sigma_1(e_1) &= \{c_2,c_3\},\ m_5\sigma_1(e_1) = \{c_3,c_4\},\ m_6\sigma_1(e_1) = \{c_1\} \text{ and} \\
m_1\sigma_1(e_2) &= \{c_1\},\quad m_2\sigma_1(e_2) = \{c_3\},\quad m_3\sigma_1(e_2) = \emptyset, \\
m_4\sigma_1(e_2) &= \{c_1\},\quad m_5\sigma_1(e_2) = \{c_1\},\quad m_6\sigma_1(e_2) = \{c_2,c_3\}, \text{ and} \\
m_1\sigma_1(e_3) &= \{c_1\},\quad m_2\sigma_1(e_3) = \{c_4\},\quad m_3\sigma_1(e_3) = \{c_1,c_3\}, \\
m_4\sigma_1(e_3) &= \{c_1\},\quad m_5\sigma_1(e_3) = \{c_3,c_4\},\ m_6\sigma_1(e_3) = \emptyset, \text{ and} \\
m_1\sigma_2(e_1) &= \{c_1,c_2\},\ m_2\sigma_2(e_1) = \{c_3\},\quad m_3\sigma_2(e_1) = \{c_4\}, \\
m_4\sigma_2(e_1) &= \{c_2\},\quad m_5\sigma_2(e_1) = \{c_2\},\quad m_6\sigma_2(e_1) = \{c_3\} \text{ and} \\
m_1\sigma_2(e_2) &= \{c_1\},\quad m_2\sigma_2(e_2) = \emptyset,\quad m_3\sigma_2(e_2) = \{c_2\}, \\
m_4\sigma_2(e_2) &= \{c_1\},\quad m_5\sigma_2(e_2) = \emptyset,\quad m_6\sigma_2(e_2) = \{c_4\}, \text{ and} \\
m_1\sigma_2(e_3) &= \{c_1,c_3\},\ m_2\sigma_2(e_3) = \{c_2,c_3\},\ m_3\sigma_2(e_3) = \emptyset, \\
m_4\sigma_2(e_3) &= \{c_1\},\quad m_5\sigma_2(e_3) = \{c_4\},\quad m_6\sigma_2(e_3) = \{c_4\},
\end{aligned}$$

where $m_i\sigma_j(e_1)$ represents all those applicants of the diamond group whose education is equal to m_i, $m_i\sigma_j(e_2)$ represents all those applicants of the diamond group whose experience is equal to m_i and $m_i\sigma_j(e_3)$ represents all those applicants of the diamond group whose computer knowledge is equal to m_i. In addition, foresets

$$\begin{aligned}
\sigma_1(e_1)c_1 &= \{m_1,m_6\}, \quad \sigma_1(e_1)c_2=\{m_1,m_2,m_4\},\ \sigma_1(e_1)c_3=\{m_4,m_5\},\ \sigma_1(e_1)c_4=\{m_2,m_5\}, \text{ and}\\
\sigma_1(e_2)c_1 &= \{m_1,m_4,m_5\},\ \sigma_1(e_2)c_2=\{m_6\}, \quad \sigma_1(e_2)c_3=\{m_2,m_6\},\ \sigma_1(e_2)c_4=\varnothing, \text{ and}\\
\sigma_1(e_3)c_1 &= \{m_1,m_3,m_4\},\ \sigma_1(e_3)c_2=\varnothing, \quad \sigma_1(e_3)c_3=\{m_3,m_5\}, \sigma_1(e_3)c_4=\{m_2,m_5\}.\\
\sigma_2(e_1)c_1 &= \{m_1\}, \quad \sigma_2(e_1)c_2=\{m_1,m_4,m_5\},\ \sigma_2(e_1)c_3=\{m_2,m_6\},\ \sigma_2(e_1)c_4=\{m_3\}, \text{ and}\\
\sigma_2(e_2)c_1 &= \{m_1,m_4\}, \quad \sigma_2(e_2)c_2=\{m_3\}, \quad \sigma_2(e_2)c_3=\varnothing, \quad \sigma_2(e_2)c_4=\{m_6\}, \text{ and}\\
\sigma_2(e_3)c_1 &= \{m_1,m_4\}, \quad \sigma_2(e_3)c_2=\{m_2\}, \quad \sigma_1(e_3)c_3=\{m_1,m_2\},\ \sigma_2(e_3)c_4=\{m_5,m_6\},
\end{aligned}$$

where $\sigma_j(e_1)c_i$ represents all those applicants of the platinum group whose education is equal to c_i, $\sigma_j(e_2)c_i$ represents all those applicants of the platinum group whose experience is equal to c_i and $\sigma_j(e_3)c_i$ represents all those applicants of the platinum group whose computer knowledge is equal to c_i.

$$\begin{aligned}
\text{Define } B &= \langle\mu_B,\gamma_B\rangle : U_2\to[0,1] \text{ which represents the preference of applicants given by a multi-national company such that}\\
\mu_B(c_1) &= 0.9,\ \mu_B(c_2)=0.8,\ \mu_B(c_3)=0.4,\ \mu_B(c_4)=0 \text{ and}\\
\gamma_B(c_1) &= 0.0,\ \gamma_B(c_2)=0.2,\ \gamma_B(c_3)=0.5,\ \gamma_B(c_4)=0.8.\\
\text{Define } B_1 &= \langle\mu_{B_1},\gamma_{B_1}\rangle : U_1\to[0,1] \text{ which represents the preference of applicants given by a multi-national company such that}\\
\mu_{B_1}(m_1) &= 1,\ \mu_{B_1}(m_2)=0.7,\ \mu_{B_1}(m_3)=0.5,\ \mu_{B_1}(m_4)=0.1,\\
\mu_{B_1}(m_5) &= 0,\ \mu_{B_1}(m_6)=0.4 \text{ and}\\
\gamma_{B_1}(m_1) &= 0,\ \gamma_{B_1}(m_2)=0.2,\ \gamma_{B_1}(m_3)=0.5,\ \gamma_{B_1}(m_4)=0.7,\\
\gamma_{B_1}(m_5) &= 1,\ \gamma_{B_1}(m_6)=0.5.
\end{aligned}$$

Therefore, the optimistic multigranulation lower and upper approximations (with respect to the aftersets as well as with respect to the foresets) are given in Tables 8 and 9.

$$\begin{aligned}
\underline{\sigma_1+\sigma_2}_o^{B} &= (\underline{\sigma_1+\sigma_2}_o^{\mu_B}, \underline{\sigma_1+\sigma_2}_o^{\gamma_B}),\\
{}^{o}\overline{\sigma_1+\sigma_2}^{B} &= ({}^{o}\overline{\sigma_1+\sigma_2}^{\mu_B}, {}^{o}\overline{\sigma_1+\sigma_2}^{\gamma_B}).
\end{aligned}$$

Table 8. Optimistic multigranulation lower approximations of B.

$\underline{\sigma_1+\sigma_2}_o^{\mu_B}, \underline{\sigma_1+\sigma_2}_o^{\gamma_B}$	m_1	m_2	m_3	m_4	m_5	m_6
$\underline{\sigma_1+\sigma_2}_o^{\mu_B}(e_1)$	0.8	0	0	0.4	0	0.4
$\underline{\sigma_1+\sigma_2}_o^{\mu_B}(e_2)$	0.9	0.4	0.8	0.9	0.9	0
$\underline{\sigma_1+\sigma_2}_o^{\mu_B}(e_3)$	0.4	0	0.4	1	0	0
$\underline{\sigma_1+\sigma_2}_o^{\gamma_B}(e_1)$	0.2	0.8	0.8	0.5	0.8	0.5
$\underline{\sigma_1+\sigma_2}_o^{\gamma_B}(e_2)$	0	0.5	0.2	0	0	0.8
$\underline{\sigma_1+\sigma_2}_o^{\gamma_B}(e_3)$	0.5	0.8	0.5	0	0.8	0.8

Table 8 shows the exact degree of competency of applicant m_i to B in education, experience and computer knowledge.

Table 9 shows the possible degree of competency of applicant m_i to B in education, experience and computer knowledge.

In Table 10, $S(m_1) = S(m_4)$, so we calculate accuracy values for m_1 and m_4, as shown in Table 11.

Table 9. Optimistic multigranulation upper approximations of B.

${}^{o}\overline{\sigma_1+\sigma_2}^{\mu_B}, {}^{o}\overline{\sigma_1+\sigma_2}^{\gamma_B}$	m_1	m_2	m_3	m_4	m_5	m_6
${}^{o}\overline{\sigma_1+\sigma_2}^{\mu_B}(e_1)$	0.9	0	0	0.8	0	0
${}^{o}\overline{\sigma_1+\sigma_2}^{\mu_B}(e_2)$	0.9	0	0	0.9	0	0
${}^{o}\overline{\sigma_1+\sigma_2}^{\mu_B}(e_3)$	0.9	0	0	0	0	0
${}^{o}\overline{\sigma_1+\sigma_2}^{\gamma_B}(e_1)$	0	1	1	0.2	1	1
${}^{o}\overline{\sigma_1+\sigma_2}^{\gamma_B}(e_2)$	0	1	1	0	1	1
${}^{o}\overline{\sigma_1+\sigma_2}^{\gamma_B}(e_3)$	0	1	1	1	0.8	1

Table 10. Values of score function of applicants.

	$\underline{S}_{ij}(e_1)$	$\underline{S}_{ij}(e_2)$	$\underline{S}_{ij}(e_3)$	$\overline{S}_{ij}(e_1)$	$\overline{S}_{ij}(e_2)$	$\overline{S}_{ij}(e_3)$	$\underline{S}(x_i)$	$\overline{S}(x_i)$	$S(x_i)$
m_1	0.6	0.9	−0.1	0.9	0.9	0.9	1.4	2.7	4.1
m_2	−0.8	−0.1	−0.8	−1	−1	−1	−1.7	−3	−4.7
m_3	−0.8	0.6	−0.1	−1	−1	−1	−0.3	−3	−3.3
m_4	−0.1	0.9	0.9	0.6	0.9	0.9	1.7	2.4	4.1
m_5	−0.8	0.9	−0.8	−1	−1	−0.8	−0.7	−2.8	−3.5
m_6	−0.1	−0.8	−0.8	−1	−1	−1	1.7	−3	−4.7

Table 11. Values of accuracy function.

	$\underline{H}_{ij}(e_1)$	$\underline{H}_{ij}(e_2)$	$\underline{H}_{ij}(e_3)$	$\overline{H}_{ij}(e_1)$	$\overline{H}_{ij}(e_2)$	$\overline{H}_{ij}(e_3)$	H
m_1	1	0.9	0.9	0.9	0.9	0.9	5.5
m_6	0.9	0.9	1	1	0.9	1	5.7

It is shown in Table 11 that $H(m_6) = 5.7$ is maximum. Therefore, a multi-national company will select applicant m_6.

Therefore, the optimistic multigranulation lower and upper approximations (with respect to the foresets) are given in Tables 12 and 13.

$$\begin{aligned} {}^{B}\underline{\sigma_1+\sigma_2}_{o} &= ({}^{\mu_B}\underline{\sigma_1+\sigma_2}_{o}, {}^{\gamma_B}\underline{\sigma_1+\sigma_2}_{o}), \\ {}^{B}\overline{\sigma_1+\sigma_2}^{o} &= ({}^{\mu_B}\overline{\sigma_1+\sigma_2}^{o}, {}^{\gamma_B}\overline{\sigma_1+\sigma_2}^{o}). \end{aligned}$$

Table 12. Optimistic multigranulation lower approximations of B.

${}^{\mu_B}\underline{\sigma_1+\sigma_2}_{o}, {}^{\gamma_B}\underline{\sigma_1+\sigma_2}_{o}$	c_1	c_2	c_3	c_4
${}^{\mu_B}\underline{\sigma_1+\sigma_2}_{o}(e_1)$	0.4	0	0	0
${}^{\mu_B}\underline{\sigma_1+\sigma_2}_{o}(e_2)$	0	0.4	0.4	0.4
${}^{\mu_B}\underline{\sigma_1+\sigma_2}_{o}(e_3)$	0.1	0.7	0	0
${}^{\gamma_B}\underline{\sigma_1+\sigma_2}_{o}(e_1)$	0.5	1	1	1
${}^{\gamma_B}\underline{\sigma_1+\sigma_2}_{o}(e_2)$	1	0.5	0.5	0.5
${}^{\gamma_B}\underline{\sigma_1+\sigma_2}_{o}(e_3)$	0.7	0.2	1	1

Table 12 shows the exact degree of competency of applicant c_i to B in education, experience and computer knowledge.

Table 13 shows the possible degree of competency of applicant c_i to B in education, experience and computer knowledge.

Table 13. Optimistic multigranulation upper approximations of B.

$\mu_B\overline{\sigma_1+\sigma_2}^o, \gamma_B\overline{\sigma_1+\sigma_2}^o$	c_1	c_2	c_3	c_4
$\mu_B\overline{\sigma_1+\sigma_2}^o(e_1)$	1	0.1	0	0
$\mu_B\overline{\sigma_1+\sigma_2}^o(e_2)$	1	0	0	0
$\mu_B\overline{\sigma_1+\sigma_2}^o(e_3)$	1	0	0	0
$\gamma_B\overline{\sigma_1+\sigma_2}^o(e_1)$	0	0.7	1	1
$\gamma_B\overline{\sigma_1+\sigma_2}^o(e_2)$	0	1	1	1
$\gamma_B\overline{\sigma_1+\sigma_2}^o(e_3)$	0	1	1	1

It is shown in Table 14 that $S(c_1) = 1.3$ is maximum. Therefore, a multi-national company will select applicant c_1.

Table 14. Values of score function of colors of car.

	$\underline{S}_{ij}(e_1)$	$\underline{S}_{ij}(e_2)$	$\underline{S}_{ij}(e_3)$	$\overline{S}_{ij}(e_1)$	$\overline{S}_{ij}(e_2)$	$\overline{S}_{ij}(e_3)$	$\underline{S}(x_i)$	$\overline{S}(x_i)$	$S(x_i)$
c_1	−0.1	−1	−0.6	1	1	1	−1.7	3	1.3
c_2	−1	−0.1	0.5	−0.6	−1	−1	−0.6	−2.6	−3.2
c_3	−1	−0.1	−1	−1	−1	−1	−2.1	−3	−5.1
c_4	−1	−0.1	−1	−1	−1	−1	−2.1	−3	−5.1

6. Comparison

The RS describes a target set by a lower and upper approximation based on single granulation. However, the multiple granulation with approximations of a target set is needed in many real world problems as well. For example, Qian et al. [41,42] built a framework of OMGRS and PMGRS by getting inspiration of multi-source datasets and multiple granulation is needed by multi-scale data for set approximations [73]. Many things are different when comparing our work with existing theories. Mainly, we make a note on the differences of our work and existing ones, such as angle of thinking, MGRS environment and research objective. Our research with respect to the angle of thinking is different from other existing theories. For a comparative study, our proposed model transforms decision-making systems into a formal decision context. Our study is different from the existing ones in [41,63,74] in terms of MGRS because our work is about IFSs which are useful in dealing with uncertainty. In [63], Shabir et al. used crisp sets to present MGRS model based on soft relations. Later, they used a FS instead of a crisp set and presented OMGFRS [64]. We extended the OMGFRS model in terms of IFS and proposed OMGIFRS model based on soft binary relations to make better decision in decision making-problems. An IFS is better than a crisp set or a FS to discuss the uncertainty. In IFS, an element is described with membership degree as well as non-membership degree but in FS, an element is described with membership degree only. That is why our proposed model has more capability to reveal the uncertainty because of IFS. Furthermore, we have used soft relations which have many applications in dealing with uncertainty because of its parameterized collection of binary relations.

7. Conclusions

This paper proposes the MGRS model in terms of IFS based on soft binary relations over two universes. First of all, we defined granulation roughness based on two soft binary relations using IFSs with respect to the aftersets and foresets. In this way, we obtain two

IFSSs with respect to the aftersets and foresets. Some properties of OMGIFRS have been studied. Then, we generalized these concepts to granulation roughness of an IFS based on multi-soft relations and discussed their properties. We presented a decision-making algorithm regarding the aftersets and foresets with an example in practical decision-making problem. In IFS, the sum of membership degree, non-membership degree and hesitant degree of an element is less than or equal to 1. However, in some decision-making problems, the sum of membership degree, non-membership degree and hesitant degree of an element may be greater than 1. In this case, the Pythagorean fuzzy set which is an extension of IFS is the better set to deal with uncertainty. The Pythagorean fuzzy set extension makes better improvement in applicability and flexibility of IFS. Further work may be discussed about investigation of pessimistic MGRS of an IFS based on soft relations. Other OMGIFRS models with interval valued IFSs, uncertain linguistic FSs, basic uncertain information SSs, linguistic Z-number FSSs and Pythagorean FSs may be discussed in future.

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