mathematics MDPI

Article

On Graded S-Primary Ideals

Azzh Saad Alshehry

Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; asalshihry@pnu.edu.sa

Abstract: Let R be a commutative graded ring with unity, S be a multiplicative subset of homogeneous elements of R and P be a graded ideal of R such that $P\bigcap S=\emptyset$. In this article, we introduce the concept of graded S-primary ideals which is a generalization of graded primary ideals. We say that P is a graded S-primary ideal of R if there exists $s\in S$ such that for all $x,y\in h(R)$, if $xy\in P$, then $sx\in P$ or $sy\in Grad(P)$ (the graded radical of P). We investigate some basic properties of graded S-primary ideals.

Keywords: graded prime ideals; graded primary ideals; graded S-prime ideals; graded S-primary ideals

Citation: Alshehry, A.S. On Graded S-Primary Ideals. *Mathematics* **2021**, *9*, 2637. https://doi.org/10.3390/math9202637

Academic Editor: Takayuki Hibi

Received: 13 September 2021
Accepted: 17 October 2021
Published: 19 October 2021

1. Introduction

Throughout this article, G will be a group with the identity of e and R will be a commutative ring with a nonzero unity of 1. Then R is called G-graded if $R=\bigoplus\limits_{g\in G}R_g$ with $R_gR_h\subseteq R_{gh}$ for all $g,h\in G$ where R_g is an additive subgroup of R. The elements of R_g are called homogeneous of degree g. If $a\in R$, then a can be written uniquely as $\sum\limits_{g\in G}a_g$, where a_g is the component of a in R_g. The component R_e is a subring of R and $1\in R_e$. The set of all homogeneous elements of R is $h(R)=\bigcup\limits_{g\in G}R_g$. Let P be an ideal of a graded ring R. Then P is called a graded ideal if $P=\bigoplus\limits_{g\in G}(P\cap R_g)$, i.e., for $a\in P$, $a=\sum\limits_{g\in G}a_g$ where $a_g\in P$ for all $g\in G$. It is not necessary that every ideal of a graded ring is a graded ideal. For more details and terminology, look at [1,2].

Let P be a proper graded ideal of R. Then the graded radical of P is denoted by $Grad(P)$ and it is defined as written below:

$$Grad(P)=\left\{x=\sum_{g\in G}x_g\in R:\ \text{for all } g\in G,\ \text{there exists } n_g\in\mathbb{N} \text{ such that } x_g^{n_g}\in P\right\}.$$

Note that $Grad(P)$ is always a graded ideal of R (check [3]).

A proper graded ideal P of R is said to be a graded prime if $xy\in P$ implies that $x\in P$ or $y\in P$ where $x,y\in h(R)$ [3]. Graded prime ideals play a very important role in the Commutative Graded Rings Theory. There are several ways to generalize the concept of a graded prime ideal, for example, Refai and Al-Zoubi in [4] introduced the concept of graded primary ideals, a proper graded ideal P of R is said to be a graded primary ideal whenever $ab\in P$ where $a,b\in h(R)$, then either $a\in P$ or $b\in Grad(P)$.

Let $S\subseteq R$ be a multiplicative set and P be an ideal of R such that $P\bigcap S=\emptyset$. In [5], P is said to be a S-primary ideal of R if $s\in S$ exists such that for all $x,y\in R$, if $xy\in P$, then $sx\in P$ or sy is in the radical of P.

Let R be a graded ring, $S\subseteq h(R)$ be a multiplicative set and P be a graded ideal of R such that $P\bigcap S=\emptyset$. In [6], P is said to be a graded S-prime ideal of R if $s\in S$ exists such that if $xy\in P$, then $sx\in P$ or $sy\in P$ where $x,y\in h(R)$. Also, several properties of graded S-prime ideals have been examined and investigated in [7]. In this article, motivated by [5], we introduce the concept of graded S-primary ideals. We say that P is a graded S-primary ideal of R if there exists $s \in S$ such that for all $x, y \in h(R)$, if $xy \in P$, then $sx \in P$ or $sy \in Grad(P)$. Clearly, every S-primary ideal is graded S-primary, we prove that the converse is not necessarily true (Example 1). It is also evident that every graded primary ideal that is disjoint with S is graded S-primary, we prove that the converse is not necessarily true (Example 2). Note that if S consists of units of $h(R)$, then the notions of graded S-primary and graded primary ideal coincide. We investigate some basic properties of graded S-primary ideals. Indeed, our results are motivated by the interesting results proved in [5–7].

2. Graded S-Primary Ideals

In this section, we introduce the concept of graded S-primary ideals. We investigate some basic properties of graded S-primary ideals.

Definition 1. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and P be a graded ideal of R such that $P \bigcap S = \emptyset$. We say that P is a graded S-primary ideal of R if there exists $s \in S$ such that for all $x, y \in h(R)$, if $xy \in P$, then $sx \in P$ or $sy \in Grad(P)$.*

Clearly, every S-primary ideal is graded S-primary, but the converse is not necessarily true, check the following example that is raised from ([7], Example 2.2):

Example 1. *Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the graded ideal $I = 5R$ of R. We show that I is a graded prime ideal of R. Let $xy \in I$ for some $x, y \in h(R)$.*

Case (1): $x, y \in R_0$. In this case, $x, y \in \mathbb{Z}$ such that 5 divides xy, and then either 5 divides x or 5 divides y as 5 is a prime, which implies that either $x \in I$ or $y \in I$.

Case (2): $x, y \in R_1$. In this case, $x = ia$ and $y = ib$ for some $a, b \in \mathbb{Z}$ such that 5 divides $xy = -ab$, and then 5 divides ab in $\mathbb{Z}$, and again either 5 divides a or 5 divides b, which implies that either 5 divides $x = ia$ or 5 divides $y = ib$, and hence either $x \in I$ or $y \in I$.

Case (3): $x \in R_0$ and $y \in R_1$. In this case, $x \in \mathbb{Z}$ and $y = ib$ for some $b \in \mathbb{Z}$ such that 5 divides $xy = ixb$ in R, that is $ixb = 5(\alpha + i\beta)$ for some $\alpha, \beta \in \mathbb{Z}$, which gives that $xb = 5\beta$, that is 5 divides xb in $\mathbb{Z}$, and again either 5 divides x or 5 divides b, and then either 5 divides x or 5 divides $y = ib$ in R, and hence either $x \in I$ or $y \in I$.

So, I is a graded prime ideal of R. Consider the graded ideal $P = 10R$ of R and the multiplicative subset $S = \{2^n : n \text{ is a non-negative integer}\}$ of $h(R)$. We show that P is a graded S-prime ideal of R. Note that $P \bigcap S = \emptyset$. Let $xy \in P$ for some $x, y \in h(R)$. Then 10 divides xy in R. Then $xy \in I$, and then $x \in I$ or $y \in I$ as I is graded prime, which implies that $2x \in P$ or $2y \in P$. Therefore, P is a graded S-prime ideal of R, and hence P is a graded S-primary ideal of R.

On the other hand, P is not an S-primary ideal of R since $3 - i, 3 + i \in R$ with $(3 - i)(3 + i) \in P$, $(s(3 - i))^n \notin P$ and $(s(3 + i))^n \notin P$ for each $s \in S$ and positive integer n.

It is obvious that every graded primary ideal that is disjoint with S is graded S-primary, but the converse is not necessarily true, check the next example. In fact, if S consists of units of $h(R)$, then the notions of graded primary and graded S-primary ideals coincide. The next example is motivated by ([7], Example 2.3).

Example 2. *Consider $R = \mathbb{Z}[X]$ and $G = \mathbb{Z}$. Then R is G-graded by $R_j = \mathbb{Z}X^j$ for $j \geq 0$ and $R_j = \{0\}$ otherwise. Consider the graded ideal $P = 9XR$ of R and the multiplicative subset $S = \{9^n : n \text{ is a non-negative integer}\}$ of $h(R)$. We show that P is a graded S-prime ideal of R. Note that $P \bigcap S = \emptyset$. Let $f(X)g(X) \in P$ for some $f(X), g(X) \in h(R)$. Then X divides $f(X)g(X)$, and X divides $f(X)$ or X divides $g(X)$, which implies that $9f(X) \in P$ or $9g(X) \in P$. Therefore, P is a graded S-prime ideal of R, hence that P is a graded S-primary ideal of R. On the other hand, P is not a graded primary ideal of R since $9, X \in h(R)$ with $9.X \in P$, $9^n \notin P$ and $X^n \notin P$ for each positive integer n.*

Proposition 1. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicative set. If P is a graded S-primary ideal of R, then $Grad(P)$ is a graded S-prime ideal of R.*

Proof. Since $P \bigcap S = \emptyset$, $Grad(P) \bigcap S = \emptyset$. Let $x, y \in h(R)$ such that $xy \in Grad(P)$. Then $(xy)^n = x^n y^n \in P$ for some positive integer n, and then there exists $s \in S$ such that $sx^n \in P$ or $sy^n \in Grad(P)$, which implies that $sx \in Grad(P)$ or $sy \in Grad(Grad(P)) = Grad(P)$. Therefore, $Grad(P)$ is a graded S-prime ideal of R. $\square$

The next lemma is inspired by Example 2.

Lemma 1. *Let R be an integral domain. Suppose that R is a graded ring, $a, b \in h(R)$ such that Ra is a nonzero graded prime ideal of R and Rb is a graded primary ideal of R. If $Rb \nsubseteq Ra$ and $S = \{b^n : n$ is a non-negative integer$\}$. Then $P = Rab$ is a graded S-prime ideal of R which is not graded primary.*

Proof. Firstly, we show that $a \notin P$. If $a \in P$, then $a = rab$ for some $r \in R$, and then $a(1 - rb) = 0$, which implies that $a = 0$ or $1 = rb$, and then $Ra = \{0\}$ or b is a unit, which is a contradiction in both cases. Secondly, we show that $P \bigcap S = \emptyset$. If $x \in P \bigcap S$, then $x = b^n \in Ra$ for some non-negative integer n, and then $b \in Ra$ as Ra is graded prime, and so $Rb \subseteq Ra$, which is a contradiction. Now, let $x, y \in h(R)$ such that $xy \in P$. Then $xy \in Ra$, and then $x \in Ra$ or $y \in Ra$, so $s = b \in S$ such that $sx \in P$ or $sy \in P$. Therefore, P is a graded S-prime ideal of R. On the other hand, $a, b \in h(R)$ such that $ab \in P$ and $a \notin P$. If $b \in Grad(P)$, then $b^n \in P$ for some positive integer n, which yields that $b^n \in P \bigcap S$, which is a contradiction. Therefore, P is not a graded primary ideal of R. $\square$

Remark 1. *In Example 2, $\langle X \rangle$ is a nonzero graded prime ideal of R and $\langle 9 \rangle$ is a graded primary ideal of R with $\langle 9 \rangle \nsubseteq \langle X \rangle$. So, by Lemma 1, $P = \langle 9X \rangle = 9XR$ is a graded S-prime ideal of R which is not graded primary, where $S = \{9^n : n$ is a non-negative integer$\}$.*

Proposition 2. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and P be a graded ideal of R such that $P \bigcap S = \emptyset$. Then P is a graded S-primary ideal of R if and only if $(P : s)$ is a graded primary ideal of R for some $s \in S$.*

Proof. Suppose that P is a graded S-primary ideal of R. Then there exists $s \in S$ such that whenever $x, y \in h(R)$ with $xy \in P$, then either $sx \in P$ or $sy \in Grad(P)$. We show that $Grad((P : s)) = Grad((P : s^n))$ for all positive integer n. Let n be a positive integer. Then $(P : s) \subseteq (P : s^n)$, and then $Grad((P : s)) \subseteq Grad((P : s^n))$. Let $x \in Grad((P : s^n))$. Then $x_g \in Grad((P : s^n))$ for all $g \in G$ as the graded radical is a graded ideal, and then there exists a positive integer k such that $x_g^k s^n \in P$ for all $g \in G$. If $s^{n+1} \in Grad(P)$, then $s^{(n+1)m} \in P \bigcap S$ for some positive integer m, which is a contradiction. So, $sx_g^k \in P$ for all $g \in G$, and hence $x_g \in Grad((P : s))$ for all $g \in G$, so $x \in Grad((P : s))$. Therefore, $Grad((P : s)) = Grad((P : s^n))$. Now, let $x, y \in h(R)$ such that $xy \in (P : s)$. Then $sxy \in P$, and then $s^2x \in P$ or $sy \in Grad(P)$. If $s^2x \in P$, then as $s^3 \notin Grad(P)$, we have $sx \in P$, which means that $x \in (P : s)$. If $sy \in Grad(P)$, then $(sy)^n = s^n y^n \in P$ for some positive integer n, and then $y \in Grad((P : s^n)) = Grad((P : s))$. Hence, $(P : s)$ is a graded primary ideal of R. Conversely, assume that $(P : s)$ is a graded primary ideal of R for some $s \in S$. Let $x, y \in h(R)$ such that $xy \in P \subseteq (P : s)$. Then $x \in (P : s)$ or $y \in Grad((P : s))$. Therefore, either $sx \in P$ or $sy \in Grad(P)$. This shows that P is a graded S-primary ideal of R. $\square$

Proposition 3. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicative set. Suppose that P is a graded primary ideal of R with $P \bigcap S = \emptyset$. Then for any $s \in S$, sP is a graded S-primary ideal of R. Moreover, If $P \neq \{0\}$ and $\bigcap_{n=1}^{\infty} Rs^n = \{0\}$, then sP is not a graded primary ideal of R.*

Proof. Let $s \in S$ and $I = sP$. As $I \subseteq P$ and $P \bigcap S = \emptyset$, it follows that $I \bigcap S = \emptyset$. Since P is a graded primary ideal of R with $Grad(P) \bigcap S = \emptyset$, we get that $(I : s) = P$. Consequently, $(I : s)$ is a graded primary ideal of R. Therefore, we obtain from Proposition 2 that $I = sP$ is a graded S-primary ideal of R. Moreover, assume that $P \neq \{0\}$ and $\bigcap_{n=1}^{\infty} Rs^n = \{0\}$. if $P = sP$, then $P = s^n P$ for each $n \geq 1$. From $\bigcap_{n=1}^{\infty} Rs^n = \{0\}$, it follows that $P = \{0\}$, which is a contradiction. In consequence, $P \neq sP$. So, there exists $x \in P - sP$, and then $x_g \notin sP$ for some $g \in G$. Note that $x_g \in P$ as P is a graded ideal. Hence, $sx_g \in sP = I$ with $x_g \notin I$ and $s \notin Grad(I)$. Therefore, $I = sP$ is not a graded primary ideal of R. □

Proposition 4. *Allow R to be a graded ring and $S \subseteq h(R)$ be a multiplicative set. Suppose that $n \geq 1$, $i \in \{1, ..., n\}$ and P_i is a graded ideal of R with $P_i \bigcap S = \emptyset$. If P_i is a graded S-primary ideal of R for each i with $Grad(P_i) = Grad(P_j)$ for all $i, j \in \{1, ..., n\}$, then $\bigcap_{i=1}^{n} P_i$ is a graded S-primary ideal of R.*

Proof. Since P_i is a graded S-primary ideal of R, there exists $s_i \in S$ to this extent for all $x, y \in h(R)$ with $xy \in P_i$, we have either $s_i x \in P_i$ or $s_i y \in Grad(P_i)$. Let $s = \prod_{i=1}^{n} s_i$. Then $s \in S$. Assume that $x, y \in h(R)$ in such a way $xy \in \bigcap_{i=1}^{n} P_i$ and $sx \notin \bigcap_{i=1}^{n} P_i$. Then $sx \notin P_k$ for some $1 \leq k \leq n$, and then $s_k x \notin P_k$. Seeing as $xy \in P_k$, $s_k y \in Grad(P_k)$. Therefore, $sy \in Grad(P_k)$. By assumption, $Grad(P_1) = Grad(P_i)$ for all $1 \leq i \leq n$. Thus $sy \in Grad(P_1) = \bigcap_{i=1}^{n} Grad(P_i) = Grad\left(\bigcap_{i=1}^{n} P_i\right)$. Therefore, $\bigcap_{i=1}^{n} P_i$ is a graded S-primary ideal of R. □

Recall that if R is a G-graded ring and $S \subseteq h(R)$ is a multiplicative set, then $S^{-1}R$ is a G-graded ring with $(S^{-1}R)_g = \left\{\frac{a}{s}, a \in R_h, s \in S \cap R_{hg^{-1}}\right\}$ for all $g \in G$. In addition, if I is a graded ideal of R, then $S^{-1}I$ is a graded ideal of $S^{-1}R$ [2].

Lemma 2. *Let R be a graded ring and P be a graded ideal of R. If P is a graded prime ideal of R, then $S^{-1}P$ is a graded prime ideal of $S^{-1}R$.*

Proof. Let $x, y \in h(R)$ and $s_1, s_2 \in S$ in such wise $\frac{x}{s_1}\frac{y}{s_2} \in S^{-1}P$. Then there exists $s_3 \in S$ such that $s_3 xy \in P$, and $s_3 x \in P$ or $y \in P$. If $s_3 x \in P$, subsequently $\frac{x}{s_1} = \frac{s_3 x}{s_3 s_1} \in S^{-1}P$. If $y \in P$, then $\frac{y}{s_2} \in S^{-1}P$. Thereupon, $S^{-1}P$ is a graded prime ideal of $S^{-1}R$. □

By ([4], Lemma 1.8), if P is a graded primary ideal of R, then $Q = Grad(P)$ is a graded prime ideal of R, and we say that P is a graded Q-primary ideal of R.

Lemma 3. *Allow R to be a graded ring and P be a graded ideal of R. If P is a graded Q-primary ideal of R, then $S^{-1}P$ is a graded $S^{-1}Q$-primary ideal of $S^{-1}R$.*

Proof. Let $x, y \in h(R)$ and $s_1, s_2 \in S$ such that $\frac{x}{s_1}\frac{y}{s_2} \in S^{-1}P$. Then there exists $s_3 \in S$ such that $s_3 xy \in P$, then $s_3 x \in P$ or $y \in Grad(P)$. If $s_3 x \in P$, then $\frac{x}{s_1} = \frac{s_3 x}{s_3 s_1} \in S^{-1}P$. If $y \in Grad(P)$, then $\frac{y}{s_2} \in S^{-1}Grad(P) = Grad\left(S^{-1}P\right)$ by ([8], Proposition 3.11 (v)). Therefore, $S^{-1}P$ is a graded primary ideal of $S^{-1}R$. Note that, $Grad(S^{-1}P) = S^{-1}Grad(P) = S^{-1}Q$ which is a graded prime ideal of $S^{-1}R$ by Lemma 2. Thereupon, $S^{-1}P$ is a graded $S^{-1}Q$-primary ideal of $S^{-1}R$. □

Proposition 5. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicative set. Suppose that P is a graded ideal of R with $P \bigcap S = \emptyset$. Then P is a graded S-primary ideal of R if and only if $S^{-1}P$ is a graded primary ideal of $S^{-1}R$ and $SP = (P : s)$ for some $s \in S$.*

Proof. Suppose that P is a graded S-primary ideal of R. Then there exists $s \in S$ in such a manner for all $x, y \in h(R)$ with $xy \in P$, we have either $sx \in P$ or $sy \in Grad(P)$. Considering $P \bigcap S = \emptyset$, $S^{-1}P \neq S^{-1}R$ by ([8], Proposition 3.11 (ii)). Allow $x, y \in h(R)$ and $s_1, s_2 \in S$ such that $\frac{x}{s_1}\frac{y}{s_2} \in S^{-1}P$. Then there exists $s_3 \in S$ such that $s_3xy \in P$, and then $ss_3x \in P$ or $sy \in Grad(P)$. If $ss_3x \in P$, then $\frac{x}{s_1} = \frac{ss_3x}{ss_3s_1} \in S^{-1}P$. If $sy \in Grad(P)$, then $\frac{y}{s_2} = \frac{sy}{ss_2} \in S^{-1}Grad(P) = Grad\big(S^{-1}P\big)$ by ([8], Proposition 3.11 (v)). Thus, $S^{-1}P$ is a graded primary ideal of $S^{-1}R$. Now, by Proposition 2, $(P : s)$ is a graded primary ideal of R for some $s \in S$. Clearly, $(P : s) \bigcap S = \emptyset$. On that account, $S((P : s)) = (P : s)$ by Lemma 3. Also, by ([8], Corollary 3.15), $S^{-1}(P : s) = (S^{-1}P :_{S^{-1}R} \frac{s}{1})$. Since $\frac{s}{1} \in U(S^{-1}R)$, $S^{-1}(P : s) = S^{-1}P$, and $S((P : s)) = SP$, accordingly $SP = (P : s)$. Contrarily, if $S^{-1}P$ is a graded $S^{-1}Q$-primary ideal of $S^{-1}R$, then SP is a graded Q-primary ideal of R. Hence, we get that $(P : s)$ is a graded primary ideal of R for some $s \in S$. Thence, we obtain by Proposition 2 that P is a graded S-primary ideal of R. □

Theorem 1. *Let R be a graded ring, S be a multiplicative subset of $h(R)$ and P be a graded ideal of R such that $P \bigcap S = \emptyset$. Thus the following statements are equivalent:*

1. *P is a graded S-primary ideal of R.*
2. *$(P : s)$ is a graded primary ideal of R for some $s \in S$.*
3. *$S^{-1}P$ is a graded primary ideal of $S^{-1}R$ and $SP = (P : s)$ for some $s \in S$.*

Proof. It follows from Propositions 2 and 5. □

Proposition 6. *Let R be a graded ring, S be a multiplicative subset of $h(R)$ and P be a graded ideal of R such that $P \bigcap S = \emptyset$. If P is a graded S-primary ideal of R, then the ascending sequence of graded ideals $(P : sr) \subseteq (P : sr^2) \subseteq (P : sr^3) \subseteq ...$ is stationary for some $s \in S$ and for all $r \in h(R)$.*

Proof. By Proposition 2, $(P : s)$ is a graded primary ideal of R for some $s \in S$. Let $r \in h(R)$. Suppose that $r \notin Grad((P : s))$. As $(P : s)$ is a graded primary ideal of R, it follows that for all positive integer n, $(P : sr^n) = (P : s)$. Assume that $r \in Grad((P : s))$. Then $sr^k \in P$ for some positive integer k. Hence, for all $j \geq k$, $(P : sr^j) = R$. □

Proposition 7. *Let R be a graded ring, S be a multiplicative subset of $h(R)$ and P be a graded ideal of R such that $P \bigcap S = \emptyset$. If P is a graded S-primary ideal of R, then the ascending sequence of graded ideals $(P : r) \subseteq (P : r^2) \subseteq (P : r^3) \subseteq ...$ is S-stationary for all $r \in h(R)$.*

Proof. Let $r \in h(R)$. Now, there exists positive integer n such that for all $j \geq n$, $(P : sr^j) = (P : sr^n)$ for some $s \in S$ by Proposition 6. Let $j \geq n$ and $a \in (P : r^j)$. Then $sar^j \in P$ so, $a \in (P : sr^j) = (P : sr^n)$. This implies that $sa \in (P : r^n)$. This proves that $s(P : r^j) \subseteq (P : r^n)$ for all $j \geq n$. Wherefore, the ascending sequence of graded ideals $(P : r) \subseteq (P : r^2) \subseteq (P : r^3) \subseteq ...$ is S-stationary for all $r \in h(R)$. □

Remark 2. *Let R be a graded ring that is not graded local, $S = U(R)$, X_1, X_2 be two distinct graded maximal ideals of R and $P = X_1 \bigcap X_2$. Presume $r \in h(R)$. Then for any positive integer n, $(P : r^n) = (X_1 : r^n) \bigcap (X_2 : r^n)$. For $i = 1, 2$, if $r \in X_i$, then $(X_i : r^n) = R$ for all positive integer n, and if $r \notin X_i$, then $(X_i : r^n) = X_i$ for all positive integer n. As a result, the ascending sequence of graded ideals $(P : r) \subseteq (P : r^2) \subseteq (P : r^3) \subseteq ...$ is stationary, but P is not a graded primary ideal of R.*

Let R be a G-graded ring. Then R is said to be a graded von Neumann regular ring if for each $a \in R_g$ ($g \in G$), there exists $x \in R_{g^{-1}}$ such that $a = a^2x$ [9].

Proposition 8. *Let R be a graded von Neumann regular ring and I be a graded ideal of R. Then $Grad(I) = I$.*

Proof. Clearly, $I \subseteq Grad(I)$. Let $a \in Grad(I)$. Then $a_g \in Grad(I)$ for all $g \in G$ as $Grad(I)$ is a graded ideal. Suppose that $g \in G$. Then $a_g^n \in I$ for some positive integer n. Since R is graded von Neumann regular, there exists $x \in R_{g^{-1}}$ such that $a_g = a_g^2 x$. Hence, $Ra_g = Ra_g^2$. So, $Ra_g = Ra_g^n \subseteq I$ and so, $a_g \in I$ for all $g \in G$, and hence $a \in I$. This proves that $Grad(I) \subseteq I$ and so, $I = Grad(I)$. □

Corollary 1. *Let R be a graded von Neumann regular ring and I be a graded ideal of R. Then I is a graded prime ideal of R if and only if I is a graded primary ideal of R.*

Proof. Apply Proposition 8. □

Theorem 2. *Let R be a graded ring, S be a multiplicative subset of $h(R)$ and P be a graded ideal of R such that $P \bigcap S = \emptyset$. Suppose that $S^{-1}R$ is graded von Neumann regular. Then P is a graded S-prime ideal of R if and only if P is a graded S-primary ideal of R.*

Proof. Suppose that P is a graded S-primary ideal of R. By Proposition 5, $S^{-1}P$ is a graded primary ideal of $S^{-1}R$ and $SP = (P : s)$ for some $s \in S$. Since $S^{-1}R$ is graded von Neumann regular, we get that $S^{-1}P$ is a graded prime ideal of $S^{-1}R$ by Corollary 1. Thereupon, SP is a graded prime ideal of R. As $SP = (P : s)$, we obtain that $(P : s)$ is a graded prime ideal of R for some $s \in S$. Therefore, it follows from ([7], Proposition 2.4) that P is a graded S-prime ideal of R. The converse is clear. □

3. Graded Strongly S-Primary Ideals

In this section, we introduce and study the concept of graded strongly S-primary ideals. We examine some basic properties of graded strongly S-primary ideals.

Definition 2.

1. *Let R be a graded ring and P be a graded primary ideal of R. Then P is said to be a graded strongly primary ideal of R if $(Grad(P))^n \subseteq P$ for some $n \in \mathbb{N}$.*
2. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and P be a graded S-primary ideal of R. Then P is said to be a graded strongly S-primary ideal of R if there exist $s' \in S$ and $n \in \mathbb{N}$ such that $s'(Grad(P))^n \subseteq P$.*

Proposition 9. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicative set. If P is a graded S-prime ideal of R, then P is a graded strongly S-primary ideal of R.*

Proof. Since P is a graded S-prime ideal of R, $(P : s)$ is a graded prime ideal of R for some $s \in S$ by ([7], Proposition 2.4), and then $s(Grad(P)) \subseteq s(Grad((P : s))) = s(P : s) \subseteq P$. Therefore, P is a graded strongly S-primary ideal of R. □

Proposition 10. *Allow R to be a graded ring, $S \subseteq h(R)$ be a multiplicative set and P be a graded ideal of R such that $P \bigcap S = \emptyset$. Then P is a graded strongly S-primary ideal of R if and only if $(P : s)$ is a graded strongly primary ideal of R for some $s \in S$.*

Proof. Suppose that P is a graded strongly S-primary ideal of R. Then there exist $s, s' \in S$ and $n \in \mathbb{N}$ such that for all $x, y \in h(R)$ with $xy \in P$, we have either $sx \in P$ or $sy \in Grad(P)$ and $s'(Grad(P))^n \subseteq P$. Note that $ss' \in S$, for all $x, y \in h(R)$ with $xy \in P$, we have either $ss'x \in P$ or $ss'y \in Grad(P)$ and $ss'(Grad(P))^n \subseteq P$. Hence, on replacing s, s' by ss', we can assume without loss of generality that $s = s'$. Now, $(P : s)$ is a graded primary ideal of R by Proposition 2. Let $r \in Grad((P:s))$. Then $sr^m \in P$ for some $m \in \mathbb{N}$. Hence, $sr \in Grad(P)$. This implies that $s.Grad((P:s)) \subseteq Grad(P)$. Take that $I = (P:s)$. Then $s^{n+1}(Grad(I))^n \subseteq s(Grad(P))^n \subseteq P \subseteq (P:s)$. As $s^{n+1} \notin Grad((P:s))$ and $(P:s)$ is a graded primary ideal of R, we get that $(Grad(I))^n \subseteq (P:s) = I$. This proves that $(P:s)$ is a graded strongly primary ideal of R. Contrariwise, take that $I = (P:s)$. Now, P is a graded S-primary ideal of R by Proposition 2 and there exists $n \in \mathbb{N}$ such that $(Grad(I))^n \subseteq I = (P:s)$. As $P \subseteq I$, we get that $(Grad(P))^n \subseteq (Grad(I))^n \subseteq (P:s)$. This implies that $s(Grad(P))^n \subseteq P$ and so, P is a graded strongly S-primary ideal of R. □

Proposition 11. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicative set. Suppose that $n \geq 1$, $i \in \{1, ..., n\}$ and P_i is a graded ideal of R with $P_i \bigcap S = \emptyset$. If P_i is a graded strongly S-primary ideal of R for each i with $Grad(P_i) = Grad(P_j)$ for all $i, j \in \{1, ..., n\}$, then $\bigcap_{i=1}^{n} P_i$ is a graded strongly S-primary ideal of R.*

Proof. It is already verified that $\bigcap_{i=1}^{n} P_i$ is a graded S-primary ideal of R by Proposition 4. Now, for each $i \in \{1, ..., n\}$, there exist $s_i \in S$ and a positive integer k_i such that $s_i(Grad(P_i))^{k_i} \subseteq P_i$. As $Grad\left(\bigcap_{i=1}^{n} P_i\right) = Grad(P_j)$ for all $j \in \{1, ..., n\}$, it follows that $s(Grad(I))^k \subseteq I$, where $s = \prod_{i=1}^{n} s_i$, $I = \bigcap_{i=1}^{n} P_i$ and $k = max\{t_1, ..., t_n\}$. This proves that $\bigcap_{i=1}^{n} P_i$ is a graded strongly S-primary ideal of R. □

Proposition 12. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicative set. Intend that P is a graded ideal of R with $P \bigcap S = \emptyset$. Then P is a graded strongly S-primary ideal of R if and only if $S^{-1}P$ is a graded strongly primary ideal of $S^{-1}R$ and $SP = (P:s)$ for some $s \in S$.*

Proof. Suppose that P is a graded strongly S-primary ideal of R. Then there exist $s \in S$ and $n \in \mathbb{N}$ such that for all $x, y \in h(R)$ with $xy \in P$, we have either $sx \in P$ or $sy \in Grad(P)$ and $s(Grad(P))^n \subseteq P$. It is already verified that $S^{-1}P$ is a graded primary ideal of $S^{-1}R$ and $SP = (P:s)$ for some $s \in S$ by Proposition 5. Now, as $\frac{s}{1} \in U(S^{-1}R)$, it follows from ([8], Proposition 3.11 (v)) that $(Grad(S^{-1}P))^n = S^{-1}(s(Grad(P))^n) \subseteq S^{-1}P$. Hence, $S^{-1}P$ is a graded strongly primary ideal of $S^{-1}R$. Again, if $S^{-1}P$ is a graded strongly $S^{-1}Q$-primary ideal of $S^{-1}R$, then SP is a graded strongly Q-primary ideal of R. Hence, we get that $(P:s)$ is a graded strongly primary ideal of R for some $s \in S$. Therefore, we obtain by Proposition 10 that P is a graded strongly S-primary ideal of R. □

Theorem 3. *Allow R to be a graded ring, S to be a multiplicative subset of $h(R)$ and P to be a graded ideal of R such that $P \bigcap S = \emptyset$. Then the following statements are equivalent:*

1. *P is a graded strongly S-primary ideal of R.*
2. *$(P:s)$ is a graded strongly primary ideal of R for some $s \in S$.*
3. *$S^{-1}P$ is a graded strongly primary ideal of $S^{-1}R$ and $SP = (P:s)$ for some $s \in S$.*

Proof. It follows from Propositions 10 and 12. □

4. Conclusions

In this study, we introduced the concept of graded S-primary ideals which is a generalization of graded primary ideals. Furthermore, we introduced the concept of graded strongly S-primary ideals. We investigated some basic properties of graded S-primary ideals and graded strongly S-primary ideals. As a proposal to further the work on the topic, we are going to study the concepts of graded S-absorbing and graded S-absorbing primary ideals as a generalization of the concepts of graded absorbing and graded absorbing primary ideals.

Funding: This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author gratefully thank the referees for the constructive comments, corrections and suggestions which definitely help to improve the readability and quality of the article.

Conflicts of Interest: The author declares that there is no conflict of interest.

References

1. Hazrat, R. *Graded Rings and Graded Grothendieck Groups*; Cambridge University Press: Cambridge, UK, 2016.
2. Nastasescu, C.; Oystaeyen, F. *Methods of Graded Rings, Lecture Notes in Mathematics, 1836*; Springer: Berlin, Germany, 2004.
3. Refai, M.; Hailat, M.; Obiedat, S. Graded radicals and graded prime spectra. *Far East J. Math. Sci.* **2000**, 59–73.
4. Refai, M.; Al-Zoubi, K. On graded primary ideals. *Turk. J. Math.* **2004**, *28*, 217–229.
5. Visweswaran, S. Some results on S-primary ideals of a commutative ring. *Beiträge zur Algebra Und Geom. Algebra Geom.* **2021**. [CrossRef]
6. Saber, H.; Alraqad, T.; Abu-Dawwas, R. On graded s-prime submodules. *Aims Math.* **2020**, *6*, 2510–2524. [CrossRef]
7. Saber, H.; Alraqad, T.; Abu-Dawwas, R.; Shtayat, H.; Hamdan, M. On graded weakly S-Prime ideals. *Preprints* **2021**, 2021080479, submitted. [CrossRef]
8. Atiyah, M.F.; Macdonald, I.G. *Introduction to Commutative Algebra*; Addison-Wesley: Reading, MA, USA, 1969.
9. Refai, M.; Abu-Dawwas, R.; Tekir, Ü.; Koç, S.; Awawdeh, R.; Yıldız, E. On graded ϕ-1-absorbing prime ideals. *arXiv* **2021**, arXiv:2107.04659v2.