*mathematics*

*Article*

# The Proof of a Conjecture Related to Divisibility Properties of $z(n)$

**Eva Trojovská * and Kandasamy Venkatachalam**

Department of Mathematics, Faculty of Science, University of Hradec Králové, 500 03 Hradec Králové, Czech Republic; venkatachalam.kandasamy@uhk.cz
* Correspondence: eva.trojovska@uhk.cz; Tel.: +42-049-333-2859

**Abstract:** The order of appearance of $n$ (in the Fibonacci sequence) $z(n)$ is defined as the smallest positive integer $k$ for which $n$ divides the $k$—the Fibonacci number $F_k$. Very recently, Trojovský proved that $z(n)$ is an even number for almost all positive integers $n$ (in the natural density sense). Moreover, he conjectured that the same is valid for the set of integers $n \geq 1$ for which the integer 4 divides $z(n)$. In this paper, among other things, we prove that for any $k \geq 1$, the number $z(n)$ is divisible by $2^k$ for almost all positive integers $n$ (in particular, we confirm Trojovský's conjecture).

**Keywords:** order of appearance; fibonacci numbers; parity; natural density; prime numbers

check for updates

**Citation:** Trojovská, E.; Venkatachalam, K. The Proof of a Conjecture Related to Divisibility Properties of $z(n)$. *Mathematics* **2021**, *9*, 2638. https://doi.org/10.3390/math9202638

Academic Editors: Iwona Wloch and Yang-Hui He

Received: 23 August 2021
Accepted: 18 October 2021
Published: 19 October 2021



## 1. Introduction

Let $(F_n)_n$ be the Fibonacci sequence which is defined by the binary recurrence $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$ and $F_1 = 1$. For any integer $n \geq 1$, the order of appearance of $n$ (in the Fibonacci sequence), denoted by $z(n)$ as $z(n) := \min\{k \geq 1 : n \mid F_k\}$. The arithmetic function $z : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ is well defined (see [1] (p. 300)) and $z(n) \leq 2n$ is the sharpest upper bound (as proved by Sallé [2]). We refer the reader to [3–9] for more (recent) results on $z(n)$. The first few values of $z(n)$ (for $n \in [1, 20]$) are (see sequence A001177 in OEIS [10]):

$$1, 3, 4, 6, 5, 12, 8, 6, 12, 15, 10, 12, 7, 24, 20, 12, 9, 12, 18, 30.$$

Recall that the natural density of $\mathcal{A} \subseteq \mathbb{Z}_{>0}$ is the following limit (if it exists):

$$\delta(\mathcal{A}) := \lim_{x\to\infty} \frac{\#\mathcal{A}(x)}{x},$$

where $\mathcal{A}(x) := \mathcal{A} \cap [1, x]$ for $x > 0$. Recently, Trojovský [11] showed that the set $\{n \geq 1 : z(n) < \epsilon n\}$ has natural density equal to 1 for all previously fixed $\epsilon > 0$ (this led to a generalized result about $\liminf_{n\to\infty} z(n)/n$, see [12]).

Here, we are interested in some arithmetic properties of $z : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$. For that, for an integer $m \geq 2$, we denote $\mathcal{E}_z^{(m)}$ as the set of all $n \in \mathbb{Z}_{\geq 1}$ for which $z(n)$ is a multiple of $m$ (i.e., $\mathcal{E}_z^{(m)} := \{n \geq 1 : z(n) \equiv 0 \pmod{m}\}$).

We know that $1/3$ of Fibonacci numbers are even (because $2 \mid F_n$ if and only if $3 \mid n$). However, Trojovský [13] (Theorem 2) showed that the situation is quite different if we replace $F_n$ by $z(n)$. Indeed, he proved that $z(n)$ is an even number for almost all positive integers $n$. In other words, the natural density of $\mathcal{E}_z^{(2)}$ is equal to 1. He also posed the following conjecture regarding the size of $\mathcal{E}_z^{(4)}$:

**Conjecture 1** (Conjecture 1 of [13])**.** *The natural density of $\mathcal{E}_z^{(4)}$ is equal to* 1*.*

Therefore, the aim of this paper is to study this conjecture from a more general viewpoint. We start by providing an infinite family of prime numbers (lying in an arithmetic progression) belonging to some desired sets. More precisely, we prove the following:

**Theorem 1.** *Let $k \geq 2$ be an integer with $k \not\equiv 0 \pmod 4$. If $p \equiv 2^k - 1 \pmod{2^k \cdot 5}$ is a prime number, then*

$$z(p) \equiv \begin{cases} 2 \pmod 4, & \text{if } k \equiv 1 \pmod 4; \\ 0 \pmod{2^k}, & \text{if } k \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

*In particular, if $k \equiv 2$ or $3 \pmod 4$, then all prime numbers $p \equiv 2^k - 1 \pmod{2^k \cdot 5}$ belong to $\mathcal{E}_z^{(2^k)}$.*

**Remark 1.** *We remark that if $4 \mid k$, then $\gcd(2^k - 1, 5) = 5$ (actually $4$ is the order of $2$ modulo $5$) and so no numbers of the form $2^k - 1 + 2^k \cdot 5\ell$ can be a prime number (for $k \geq 2$). Moreover, the condition $k \not\equiv 0 \pmod 4$ ensures, by the Dirichlet's theorem on arithmetic progressions, the existence of infinitely many primes $p \equiv 2^k - 1 \pmod{2^k \cdot 5}$.*

Now, let us observe the following table 1:

**Table 1.** Proportion of arguments for which $z(n)$ is divisible by 4, 8, 16, and 32, respectively.

| $x$ | **10** | **20** | **50** | **100** | **200** | **500** | **1000** | **3000** |
|---|---|---|---|---|---|---|---|---|
| $\frac{\#\mathcal{E}_z^{(4)}(x)}{x}$ | 0.4 | 0.45 | 0.56 | 0.58 | 0.585 | 0.606 | 0.623 | 0.643 |
| $\frac{\#\mathcal{E}_z^{(8)}(x)}{x}$ | 0.1 | 0.1 | 0.22 | 0.23 | 0.24 | 0.26 | 0.277 | 0.298 |
| $\frac{\#\mathcal{E}_z^{(16)}(x)}{x}$ | 0 | 0 | 0.02 | 0.03 | 0.045 | 0.052 | 0.06 | 0.072 |
| $\frac{\#\mathcal{E}_z^{(32)}(x)}{x}$ | 0 | 0 | 0 | 0 | 0.01 | 0.02 | 0.026 | 0.032 |

Table 1 suggests that $\#\mathcal{E}_z^{(2^k)}(x)/x$ (for $k \in [2,4]$) nondecreases as a function of $x$. Therefore, a natural question arises:

**Question 1.** *Is $\delta(\mathcal{E}_z^{(2^k)}) = 1$ for all $k \geq 1$?*

Clearly, Theorem 1 of [13] solves the case $k = 1$, while Conjecture 1 asks about the case $k = 2$.

The next result shows that the answer for Question 1 is yes (in particular, it solves Conjecture 1). More precisely, we have the following:

**Theorem 2.** *Let $k \geq 2$ be an integer. Then there exists a positive effective computable constant $c$ such that*

$$\#\mathcal{E}_z^{(2^k)}(x) \geq x - \frac{cx}{(\log x)^{1/2^{k+6}}},$$

*for all $x > e^{125 \cdot 8^{k+5}}$. In particular, the natural density of $\mathcal{E}_z^{(2^k)}$ is equal to $1$ for all $k \geq 2$.*

The proof of both theorems combines Diophantine properties of $z(n)$ with analytical tools concerning primes in arithmetic progressions.

## 2. Auxiliary Results

In this section, we present some results which will be essential tools in the proof. The first ingredient is related to the value of $z(p^k)$ for a prime number $p$ and $k \geq 1$:

**Lemma 1** (Theorem 2.4 of [14]). *We have that $z(2^k) = 3 \cdot 2^{k-1}$ for all $k \geq 2$, and $z(3^k) = 4 \cdot 3^{k-1}$ for all $k \geq 1$. In general, it holds that*

$$z(p^k) = p^{\max\{k-e(p),0\}} z(p),$$

*where $e(p) := \max\{k \geq 0 : p^k \mid F_{z(p)}\}$.*

The next lemma provides the largest arithmetic progression, which contains infinitely many prime numbers, belonging completely to $\mathcal{E}_z^{(2)}$.

**Lemma 2** (Theorem 1 of [13]). *The number $z(4n+3)$ is even for all integers $n \geq 0$.*

Another well-known arithmetic function related to Fibonacci numbers is the *Pisano period* $\pi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ for which $\pi(n)$ is the smallest period of $(F_k \pmod n)_k$. The first few values of $\pi(n)$ (for $n \in [1, 20]$) are (see sequence A001175 in OEIS):

$$1, 3, 8, 6, 20, 24, 16, 12, 24, 60, 10, 24, 28, 48, 40, 24, 36, 24, 18, 60.$$

Observe that $\pi(n)$ and $z(n)$ have similar definitions (these functions are strongly connected as can be seen in Lemma 4). However, they have a very distinct behavior related to their parity. Indeed, $\pi(n)$ is even for all $n \geq 3$, while $\mathbb{Z}_{\geq 1} \backslash \mathcal{E}_z^{(2)}$ is an infinite set (since $z(5^k) = 5^k$ is an odd number for all $k \geq 0$).

The next result provides some divisibility properties of the Pisano period for prime numbers.

**Lemma 3** (Theorem 2.2 of [14]). *Let $p$ be a prime number. We have that*

*(i)* *If $p \equiv \pm 1 \pmod 5$, then $\pi(p)$ divides $p-1$.*
*(ii)* *If $p \equiv \pm 2 \pmod 5$, then $\pi(p)$ divides $2(p+1)$. Furthermore, $\pi(p) = 2(p+1)/t$ for some odd number $t$.*

Observe that $F_{\pi(n)} \equiv F_0 \equiv 0 \pmod n$ and then $z(n)$ divides $\pi(n)$. Our next tool provides a characterization of the quotient $\pi(n)/z(n)$.

**Lemma 4** (Theorem 1 of [15]). *We have that $\pi(n)/z(n) \in \{1, 2, 4\}$ for all $n \geq 1$. Moreover, $\pi(n) = 4\, z(n)$ if and only if $z(n)$ is odd.*

The next tool is a kind of "formula" for $z(n)$ depending on $z(p^a)$ for all primes $p$ dividing $n$. The proof of this fact can be found in [16].

**Lemma 5** (Theorem 3.3 of [16]). *Let $n > 1$ be an integer with prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$. Then*

$$z(n) = lcm(z(p_1^{a_1}), \ldots, z(p_k^{a_k})).$$

*In general, it holds that*

$$z(lcm(m_1, \ldots, m_k)) = lcm(z(m_1), \ldots, z(m_k)).$$

In order to prove Theorem 2, we need an analytic tool related to the profusion of integers having factorization allowing only some classes of primes. The following notation will be used throughout this work: Let $\mathbb{P}$ be the set of prime numbers and for an integer $q \geq 2$, set $\mathbb{P}(a, q)$ as the set of all prime numbers of the form $a + kq$ for some integer

$k \geq 0$ (Dirichlet's theorem on arithmetic progressions ensures that $\mathbb{P}(a,q)$ is an infinite set whenever $\gcd(a,q) = 1$). Moreover, let $\mathcal{B}$ be the union of $B$ distinct reduced residue classes modulo $q$. Let $\mathcal{N}_{\mathcal{B}} = \{n \geq 1 : p \mid n \Rightarrow p \in \mathcal{B}\}$ be the set of all positive integers whose prime factors belong exclusively to $\mathcal{B}$. Additionally, denote $\beta := B/\phi(q)$ (where $\phi(n)$ is the Euler totient function) and

$$G_{\mathcal{B}}(s) := \zeta(s)^{-\beta} \prod_{p \in \mathcal{B}} (1 - p^{-s})^{-1},$$

which has an analytic continuation to a neighborhood of $s = 1$. Here, as usual, $\zeta(s)$ denotes the Riemann zeta function.

Our next auxiliary lemma is related to a work due to Chang and Martin [17]. More precisely,

**Lemma 6** (Theorem 3.4 of [17])**.** *For any integer $q \geq 3$, there exists a positive absolute constant $C$ such that uniformly for $q \leq (\log x)^{1/3}$, we have*

$$\#\{n \leq x : n \in \mathcal{N}_{\mathcal{B}}\} = \frac{x}{(\log x)^{1-\beta}} \left( \frac{G_{\mathcal{B}}(1)}{\Gamma(\beta)} + O\Big(C(\log x)^{-1/4}\Big) \right), \tag{1}$$

*where, as usual, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ denotes the Gamma function.*

Now, we are ready to deal with the proof of the theorems.

## 3. The Proofs

### *3.1. The Proof of Theorem 1*

The Case $k \equiv 1 \pmod 4$.

Note that if $k = 4\ell + 1$, then

$$p \equiv 2^k - 1 \equiv 2^{4\ell+1} - 1 \equiv 1 \pmod 5.$$

Thus, by Lemma 3 (i), we have that $\pi(p)$ divides $p - 1$. Since $p \equiv 2^k - 1 \equiv 3 \pmod 4$ (because $k \geq 2$), then there exist positive integers $r$ and $s$ with $s$ odd, such that $2s = p - 1 = \pi(p)r$. Moreover, since $\pi(p)$ is an even number (since $p \geq 3$), then $r$ must divide $s$. On the other hand, $p \equiv 3 \pmod 4$ and Lemma 2 yields that $z(p)$ is an even number. Hence, by Lemma 4, we have that

$$\pi(p) = z(p) \text{ or } \pi(p) = 2z(p).$$

Therefore

$$z(p) = \pi(p) = \tfrac{2s}{r} \text{ or } z(p) = \tfrac{\pi(p)}{2} = \tfrac{s}{r}.$$

Since $z(p)$ is even and $s/r$ is odd (because so is $s$), the possibility $z(p) = s/r$ is ruled out. Therefore $z(p) = 2s/r \equiv 2 \pmod 4$. $\square$

The case $k \equiv 2$ or $3 \pmod 4$.

If $k = 4\ell + 2$, then

$$p \equiv 2^k - 1 \equiv 2^{4\ell+2} - 1 \equiv 3 \pmod 5.$$

In addition, in the case $k = 4\ell + 3$, we have

$$p \equiv 2^k - 1 \equiv 2^{4\ell+3} - 1 \equiv 2 \pmod 5.$$

In any case, we can use Lemma 3 (ii) to deduce that $\pi(p) = 2(p+1)/t$ for some odd integer $t$. Again, we use that $z(p)$ is even (because of $p \equiv 3 \pmod 4$ and Lemma 2) to apply Lemma 4. Then, we obtain that

$$\pi(p) = z(p) \text{ or } \pi(p) = 2z(p).$$

That is,

$$z(p) = 2^i\pi(p) = \tfrac{2^{i+1}(p+1)}{t}$$

for some $i \in \{-1, 0\}$. On the other hand, $p \equiv 2^k - 1 \pmod{2^k \cdot 5}$ and so $p + 1$ is a multiple of $2^k$, say $p + 1 = 2^k r$ for some integer $r$. Thus

$$z(p) = 2^{i+1}\pi(p) = \frac{2^{i+1}(p+1)}{t} = \frac{2^{k+i+1}r}{t} \equiv 0 \pmod{2^k},$$

where we used that $i + 1 \geq 0$ and $t \equiv 1 \pmod 2$. The proof is complete. □

*3.2. The Proof of Theorem 2*

We have that

$$\#\mathcal{E}_z^{(2^k)}(x) = \#\{n \leq x : z(n) \equiv 0 \pmod{2^k}\}.$$

Set $t := 4\lfloor (k+3)/4 \rfloor + 2$ and note that

$$t = 4\left\lfloor \frac{k+3}{4} \right\rfloor + 2 > 4\left(\frac{k+3}{4} - 1\right) + 2 = k + 1.$$

Thus, $t > k$ which yields that $\mathcal{E}_z^{(2^t)} \subseteq \mathcal{E}_z^{(2^k)}$. In particular,

$$\#\mathcal{E}_z^{(2^k)}(x) \geq \#\mathcal{E}_z^{(2^t)}(x) = \#\{n \leq x : z(n) \equiv 0 \pmod{2^t}\}.$$

Let $\mathcal{B}_k$ be the set of the $2^{t+1} - 1 = \phi(2^t \cdot 5) - 1$ reduced residue classes modulo $2^t \cdot 5$ unless the class $2^t - 1 \pmod{2^t \cdot 5}$. Note that since $z(n)$ is a multiple of $z(p)$ for all prime $p$ in the factorization of $n$ (by Lemmas 1 and 5), then a sufficient condition for $z(n)$ to be divisible by $2^t$ is $n$ to have at least one prime factor of the form $2^t - 1 + 2^t \cdot 5\ell$ (since $2^t \mid z(2^t - 1 + 2^t \cdot 5\ell)$, by Theorem 1 and because $t \equiv 2 \pmod 4$). Therefore,

$$\begin{aligned}
\#\mathcal{E}_z^{(2^k)}(x) &\geq \#\mathcal{E}_z^{(2^t)}(x) && (2)\\
&\geq \#\{n \leq x : \exists p \mid n \text{ with } p \in \mathbb{P}(2^t - 1, 2^t \cdot 5)\} \\
&= x - \#\{n \leq x : p \mid n \Rightarrow p \in \mathcal{B}_k\}. && (3)
\end{aligned}$$

On the other hand,

$$\{n \leq x : p \mid n \Rightarrow p \in \mathcal{B}_k\} = \{n \leq x : x \in \mathcal{N}_{\mathcal{B}_k}\} \tag{4}$$

and we can apply Lemma 6 to obtain an upper bound for the size of the previous set. Thus, for $\beta := \#\mathcal{B}_k/\phi(2^t \cdot 5) = 1 - 1/2^{t+1}$, Lemma 6 implies in the existence of an absolute constant $C > 0$ such that

$$\#\{n \leq x : n \in \mathcal{N}_{\mathcal{B}_k}\} = \frac{x}{(\log x)^{1/2^{t+1}}}\left(\frac{G_{\mathcal{B}_k}(1)}{\Gamma(1 - 1/2^{t+1})} + O\Big(C(\log x)^{-1/4}\Big)\right), \tag{5}$$

for all $x > e^{125 \cdot 8^t}$. Moreover, we have that

$$G_{2^t-1,2^t\cdot 5}(s) := \zeta(s)^{-\frac{1}{2^{t+1}}} \prod_{p \in \mathbb{P}(2^t-1,2^t\cdot 5)} (1 - p^{-s})^{-1} \tag{6}$$

and

$$G_{3,4}(s) := \zeta(s)^{-\frac{2^{t+1}-1}{2^{t+1}}} \prod_{p \in \mathcal{B}_k} (1 - p^{-s})^{-1}. \tag{7}$$

Since both sets $\mathbb{P}(2^t-1, 2^t\cdot 5)$ and $\mathcal{B}_k\cap\mathbb{P}$ have nonzero density inside the set of all primes, then $G_{2^t-1,2^t\cdot 5}(s)\asymp G_{\mathcal{B}_k}(s)$ (i.e., $G_{2^t-1,2^t\cdot 5}(s)\ll G_{\mathcal{B}_k}(s)$ and $G_{\mathcal{B}_k}(s)\ll G_{2^t-1,2^t\cdot 5}(s)$). By multiplying (6) and (7), we obtain that

$$\begin{aligned} G_{2^t-1,2^t\cdot 5}(s)\, G_{\mathcal{B}_k}(s) &= \zeta(s)^{-1}\left(\prod_{p\in\mathbb{P}(2^t-1,2^t\cdot 5)}(1-p^{-s})^{-1}\right)\cdot\left(\prod_{p\in\mathcal{B}_k}(1-p^{-s})^{-1}\right) \\ &= \zeta(s)^{-1}\prod_{p\in\mathbb{P}(1,2)}(1-p^{-s})^{-1} \\ &= \left(\prod_{p\in\mathbb{P}}(1-p^{-s})\right)\cdot\left(\prod_{p\in\mathbb{P}(1,2)}(1-p^{-s})^{-1}\right) \\ &= 1-\frac{1}{2^s}, \end{aligned}$$

where we applied the Euler product $\zeta(s)=\prod_{p\in\mathbb{P}}(1-p^{-s})^{-1}$ (we refer to [18] (p. 39) and that $\mathbb{P}(1,2)=\mathbb{P}\backslash\{2\}$. Thus, one has that $G_{2^t-1,2^t\cdot 5}(1)\, G_{\mathcal{B}_k}(1)=1/2$ and so $G_{2^t-1,2^t\cdot 5}(1)$ is a constant. Therefore, there exists a positive constant $c$ such that (5) becomes

$$\#\{n\le x: n\in\mathcal{N}_{\mathcal{B}_k}\}\le\frac{cx}{(\log x)^{1/2^{t+1}}}. \tag{8}$$

Finally, we combine (3), (4), and (8) to infer that

$$\#\mathcal{E}_z^{(2^k)}(x)\ge x-\frac{cx}{(\log x)^{1/2^{k+6}}}$$

holds for all $x>e^{125\cdot 8^t}$ (we used that $t=4\lfloor(k+3)/4\rfloor+2\le k+5$).

We also obtain that

$$1\ge\delta(\mathcal{E}_z^{(2^k)})=\lim_{x\to\infty}\frac{\#\mathcal{E}_z^{(2^k)}(x)}{x}\ge\lim_{x\to\infty}\left(1-\frac{c}{(\log x)^{1/2^{k+6}}}\right)=1,$$

and we obtain that the natural density of $\mathcal{E}_z^{(2^k)}$ is equal to 1. The proof is then complete. □

## 4. Further Comments

We close this paper by making some comments about the two other questions which were raised in [13], namely,

**Question 2.** *Are there infinitely many prime numbers $p$ for which $\delta(\mathcal{E}_z^{(p)})=1$?*

**Question 3.** *Let $m\ge 1$ be an integer. Is it possible to provide an explicit positive lower bound for $\#\mathcal{E}_z^{(m)}(x)$?*

In a general scenario, in order to have $\delta(\mathcal{E}_z^{(m)})=1$ (by a mimic of the proof of Theorem 2), it suffices to prove the existence of positive coprime integers $a$ and $b$ such that $\mathbb{P}(a,b)\subseteq\mathcal{E}_z^{(m)}$. This does not seem to be an easy task, since it depends on a better knowledge of $z(p)$ for prime numbers $p$. However, we even do not know if $z(p)=p+1$ has infinitely many prime solutions. For this reason, Question 2 remains as an open problem.

On the other hand, Question 3 is too general (since nothing is required about this lower bound—we are assuming that it should be a nondecreasing function of $x$). In this case, we are able to answer this question reasonably as follows.

**Proposition 1.** *For any $m \geq 3$, we have that*

$$\#\mathcal{E}_z^{(m)}(x) \geq \left\lfloor \frac{x}{F_m} \right\rfloor \tag{9}$$

*holds for all $x \geq 2$.*

**Proof.** First, let us consider that $m \geq 13$. By the primitive divisor theorem (see [19] for the most general version), there exists a prime number $p$ such that $p \mid F_m$ and $p \nmid F_j$ for all $j \in [1, m-1]$. In particular, one has that $z(p) = m$. Now, if $n \in p\mathbb{Z}_{\geq 1}$, then $n = p^a r$ (for some $a$ and $r \geq 1$, where $\gcd(p, r) = 1$) and so, by Lemmas 1 and 5, we infer that

$$z(n) = \text{lcm}(z(p^a), z(r)) = \text{lcm}(p^{a-e(p)}z(p), z(r)) = \text{lcm}(p^{a-e(p)}m, z(r)) \equiv 0 \pmod{m}.$$

Thus $n \in \mathcal{E}_z^{(m)}$, yielding that $p\mathbb{Z}_{\geq 1} \subseteq \mathcal{E}_z^{(m)}$. In conclusion, we have

$$\#\mathcal{E}_z^{(m)}(x) \geq \#\{n \leq x : p \mid n\} = \left\lfloor \frac{x}{p} \right\rfloor \geq \left\lfloor \frac{x}{F_m} \right\rfloor,$$

where we used that $p \leq F_m$ (since $p$ divides $F_m$). For the case $m \in [3, 12]$, we obtain the inequality in (9) only by noting that $z(2) = 3$, $z(3) = 4$, $z(5) = 5$, $z(4) = 6$, $z(13) = 7$, $z(7) = 8$, $z(17) = 9$, $z(11) = 10$, $z(89) = 11$, and $z(6) = 12$. This completes the proof. □

**Remark 2.** *We still remark that the bound $p \leq F_m$ cannot be improved (in general), since the problem of the existence of infinitely many prime numbers in the Fibonacci sequence is still an unsolved question (which is expected to have an affirmative answer).*

We finish by raising the following conjecture which, in particular, solves the previous questions:

**Conjecture 2.** *For all positive integer $m$, there exist positive constants $c_m$ and $d_m$ such that*

$$\#\mathcal{E}_z^{(m)}(x) \geq x - c_m \frac{x}{(\log x)^{1+d_m}},$$

*for all sufficiently large $x$. In particular, $\delta(\mathcal{E}_z^{(m)}) = 1$ for all $m \geq 1$.*

## 5. Conclusions

In this paper, we work on a conjecture related to the arithmetic function $z : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$, which is defined as $z(n) := \min\{k \geq 1 : n \mid F_k\}$ (the called order of appearance in the Fibonacci sequence). Recently, Trojovský [13] showed that the natural density of $\{n \geq 1 : 2 \mid z(n)\}$ is equal to 1. Furthermore, he conjectured that the same holds for the set of positive integers $n$ for which $4 \mid z(n)$. In this work, we confirm the expectation: for any $k \geq 1$, the natural density of the set $\{n \geq 1 : 2^k \mid z(n)\}$ is equal to 1. Moreover, we provide a nontrivial lower bound for $\#\{n \leq x : m \mid z(n)\}$ depending on $x$ and $m$ (which is related to the Question 2 of [13]). The proof combines arithmetical and analytical tools in number theory.

**Author Contributions:** E.T. conceived the presented idea on the conceptualization, methodology, and investigation. Writing—review and editing and preparation of program procedures in Mathematica were done by K.V. All authors have read and agreed to the published version of the manuscript.

**Funding:** The authors was supported by the Project of Specific Research PrF UHK no. 2101/2021, University of Hradec Králové, Czech Republic.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Informed consent was obtained from all subjects involved in the study.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References


1. Lucas, E. Théorie des fonctions numériques simplement périodiques. *Am. J. Math.* **1878**, *1*, 289–321. [CrossRef]
2. Sallé, H.J.A. Maximum value for the rank of apparition of integers in recursive sequences. *Fibonacci Quart.* **1975**, *13*, 159–161.
3. Marques, D. Fixed points of the order of appearance in the Fibonacci sequence. *Fibonacci Quart.* **2012**, *50*, 346–352.
4. Somer, L.; Křížek, M. Fixed points and upper bounds for the rank of appearance in Lucas sequences. *Fibonacci Quart.* **2013**, *51*, 291–306.
5. Sun, Z.H.; Sun, Z.W. Fibonacci numbers and Fermat's last theorem. *Acta Arith.* **1992**, *60*, 371–388. [CrossRef]
6. Luca, F.; Tron, E. The distribution of self-Fibonacci divisors, Advances in the theory of numbers. *Fields Inst. Commun.* **2015**, *77*, 149–158.
7. Trojovský, P. On Diophantine equations related to order of appearance in Fibonacci sequence. *Mathematics* **2019**, *7*, 1073. [CrossRef]
8. Trojovská, E. On the Diophantine Equation $z(n) = (2 - 1/k)n$ Involving the Order of Appearance in the Fibonacci Sequence. *Mathematics* **2020**, *8*, 124. [CrossRef]
9. Trojovský, P. On the Natural Density of Sets Related to Generalized Fibonacci Numbers of Order $r$. *Axioms* **2021**, *10*, 144. [CrossRef]
10. Sloane, N.J.A. The On-Line Encyclopedia of Integer Sequences. Published Electronically. Available online: http://www.oeis.org/ (accessed on 20 August 2021)
11. Trojovský, P. Some problems related to the growth of $z(n)$. *Adv. Difference Equ.* **2020**, *2020*, 270. [CrossRef]
12. Marques, D. Sharper upper bounds for the order of appearance in the Fibonacci sequence. *Fibonacci Quart.* **2013**, *51*, 233–238.
13. Trojovský, P. On the Parity of the Order of Appearance in the Fibonacci Sequence. *Mathematics* **2021**, *9*, 1928. [CrossRef]
14. Fulton, J.; Morris, W. On arithmetical functions related to the Fibonacci numbers. *Acta Arith.* **1969**, *16*, 105–110. [CrossRef]
15. Vinson, J. The relation of the period modulo to the rank of apparition of $m$ in the Fibonacci sequence. *Fibonacci Quart.* **1963**, *1*, 37–46.
16. Renault, M. Properties of the Fibonacci Sequence Under Various Moduli. Master's Thesis, Wake Forest University, Winston-Salem, NC, USA, 1996. Available online: http://webspace.ship.edu/msrenault/fibonacci/FibThesis.pdf (accessed on 20 August 2021).
17. Chang, B.; Martin, G. The smallest invariant factor of the multiplicative group. *Int. J. Number Theory* **2020**, *16*, 1377–1405. [CrossRef]
18. De Koninck, J.-M.; Luca, F. Analytic Number Theory-Exploring the Anatomy of Integers. In *Analytic Number Theory: Exploring the Anatomy of Integers*; American Mathematical Soc.: Providence, RI, USA, 2012.
19. Bilu, Y.; Hanrot, G.; Voutier, P. Existence of primitive divisors of Lucas and Lehmer numbers (with an appendix by M. Mignotte). *J. Reine Angew. Math.* **2001**, *539*, 75–122.