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# Functional Form of Nonmanipulable Social Choice Functions with Two Alternatives

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**Abstract:** We propose a new functional form characterization of binary nonmanipulable social choice functions on a universal domain and an arbitrary, possibly infinite, set of agents. In order to achieve this, we considered the more general case of two-valued social choice functions and describe the structure of the family consisting of groups of agents having no power to determine the values of a nonmanipulable social choice function. With the help of such a structure, we introduce a class of functions that we call powerless revealing social choice functions and show that the binary nonmanipulable social choice functions are the powerless revealing ones.

**Keywords:** social choice functions; group strategy-proofness; indifference; universal domain; functional form characterization

**MSC:** 91B14

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## 1. Introduction

A social choice function is a mathematical model for the collective choice of a public project to be implemented by a society. The implicit assumption is that the members of the society obtain different utilities from the implemented project.

Denote the society by $I$. The available public projects to choose from form a set $A$, whose elements are also named alternatives.

The fact that the implementation of the project $a \in A$ may give different advantages to different individuals is modeled by the assumption that every agent $i \in I$ expresses a preference relation $R_i$ over the set $A$. Given a profile $R = (R_i)_{i \in I}$ of declared preferences, the social choice function $\varphi$ selects exactly one alternative $\varphi(R)$. In other words, a social choice function is a function:

$$\varphi : \mathcal{D} \to A,$$

where the domain $\mathcal{D}$ is the set of admissible profiles (i.e., the profiles that can be at the disposal of society $I$).

Several desirable properties may be considered for a social choice function. Probably the most important one is that the social choice function should induce truth-telling as the dominant strategy for all the agents. What we seek is social choice functions that exclude the possibility that a group of agents (coalition) strategically declare some false preferences in order to induce their favorite alternative to be the collective choice. The social choice functions we are talking about are named nonmanipulable or, equivalently, (group) strategy-proof.

A central topic in the political theory of the aggregation of collective preferences [1–3] is the identification of the social choice functions that are nonmanipulable. We aimed to contribute with a complete description of such social choice functions in the case of two alternatives, allowing individuals to declare indifference. Needless to say, problems of collective choice between two alternatives extensively arise in real-life situations. Observe that we did not assume that the society $I$ is a finite set.

There is a huge literature on social choice functions, offering several characterizations of nonmanipulability (see [2] (Vol II), and [4]). Most often, the analysis is restricted to the case in which all profiles in $\mathcal{D}$ are strict, i.e., every agent is forbidden to declare indifference among alternatives. In principle, $\mathcal{D}$ can be any subset of the set of all possible profiles. We confined our attention to the case of the so-called "universal domain hypothesis", which means that $\mathcal{D}$ is either the set of all the profiles of the (weak) preferences or that of all the profiles of the strict preferences.

Some of the characterizations are singled out as "functional form characterizations" (see, for example, [5]). They can be considered as more significant in the sense that they consist of formulas that *describe* the class of social choice functions under investigation. "Describing"means that the assignment rule of the alternative corresponding to every profile is explicitly given.

The best-known functional characterization is the Gibbard–Satterthwaite theorem:

**Theorem 1.** *Let $\mathcal{D} = \mathcal{S}$ be the set of all strict profiles. Suppose $|I| = n$, $\varphi : \mathcal{S} \to A$ is nonmanipulable, $|\varphi(\mathcal{S})| \geq 3$. Then, the social choice function $\varphi$ necessarily is one of the $n$ "dictatorial functions"$\varphi_1, \varphi_2, \ldots, \varphi_n$, which are defined as follows. Let $R$ be a profile; the social choice $\varphi_j(R)$ is the alternative in the range of $\varphi$, which is the best according to $R_j$.*

The above celebrated theorem dates back to the 1970s. In 2006, Larsson and Svensson [6] added a functional characterization in case the set of alternatives is of cardinality two and the social choice function is onto.

Before stating the result (Theorem 2 of [6]), we recall that a *committee* is a nonempty upward inclusive family $\mathcal{C}$ of coalitions of agents, where $\mathcal{C}$ being *upward inclusive* means that if $C \in \mathcal{C}$, then $C'$ also belongs to $\mathcal{C}$, whenever $C' \supseteq C$. In the study of strategic voting, committees have been considered for a long time, as one sees in [7].

**Theorem 2.** *Let $\mathcal{D} = \mathcal{S}$ be the set of all strict profiles. Suppose $|I| = n$, $\varphi : \mathcal{S} \to A = \{a, b\}$ is nonmanipulable, $|\varphi(\mathcal{S})| = 2$. Then, the social choice function $\varphi$ necessarily is a "voting by committee function"$\varphi_{\mathcal{C}}$, which is defined as follows. Let $\mathcal{C}$ be a committee and let $R$ be a profile. The social choice $\varphi_{\mathcal{C}}(R)$ is the alternative $a$ if and only if the coalition of agents that prefer the alternative $a$ under the profile $R$ belongs to $\mathcal{C}$.*

It is worth mentioning that, of course, dictatorial functions, as well as voting by committee functions are nonmanipulable; hence, these results actually provide functional form characterizations. Moreover, the extreme case of a constant $\varphi : \mathcal{S} \to \{a, b\}$ can be treated formally as in Theorem 2 by assuming that also the empty set is a coalition, although this assumption is somewhat unusual.

Under the group nonmanipulability assumption, Theorem 2 was generalized in [8] (Proposition 4.1) to the case in which the sets $I$ and $A$ are arbitrary.

Recently, functional form characterizations of nonmanipulable social choice functions in the case that indifference was admitted and collective choice involving only two alternatives was discovered [9–12]. As far as we know, at a similar level of generality, there are not yet functional descriptions for the case of nonmanipulable social choice functions whose range is of cardinality at least three and where indifference is allowed.

The purpose of this paper is to contribute to the literature by presenting a new functional form characterization of nonmanipulable social choice functions under the assumption that indifference is allowed and the set of alternatives has cardinality two. Our characterization result (Theorem 4) can be compared with:

- Theorem 1 of [12], which is however limited to the case that $I$ is finite;
- Theorem 4.2 of [9], which involves a nontrivial transfinite argument to cover the infinite society case.

In particular, our paper is closer in spirit to [12]. While [12] emphasized the role of coalitions of indifferent agents, we point our attention here to powerless groups of agents. This is not merely a different point of view. It has the remarkable consequence that, with respect to [12], our functional form is much easier to describe and explicitly establishes a one-to-one correspondence with the parameters (uniqueness of the representation).

Our approach differs more from [9]. The latter indeed recovers all group nonmanipulable functions first by lifting the indifference relation between the two alternatives to a larger class of "committees" and, then, sequentially moving along a well-ordered set of committees, till one of them is not indifferent and so determines the social choice.

Organization of the paper: In Section 2, we present the details of the model. Section 3 contains our main results: we used suitable sets of agents (see Definition 6 of ineffective families) to derive the functional form characterization of binary nonmanipulable social choice functions as powerless revealing functions, a class of social choice function we introduce. Examples are provided both to make clear the concepts and to allow the comparison with the literature. In Section 4, we analyze the particular case of strict preferences. In Section 5, we draw our final conclusions.

## 2. The Model

Denote the *society* by $I$. Each one of its members is called an *agent*. Any nonempty group (subset) of agents is called a *coalition*. A *preference* over the set $A$ of alternatives is a complete and transitive binary relation on $A$. We denote by $\mathcal{R}$ the set of all preferences over $A$. A preference is said to be *strict* if it is antisymmetric (different alternatives cannot be indifferent to each other). The set of all strict preferences on $A$ is denoted by $\mathcal{S}$. A *profile* is a specification of a preference for each agent. We denote by $\mathcal{R}^I$ the set of all (weak) profiles and by $\mathcal{S}^I$ the set of all strict profiles.

A social choice function selects an alternative on the basis of each admissible profile. The term *admissible* is reserved for the profiles forming the set $\mathcal{D}$ of all possible starting points of the collective choice.

**Definition 1.** *Let $\mathcal{D} \subseteq \mathcal{R}^I$. A* **social choice function** *is a function:*

$$\varphi : \mathcal{D} \to A.$$

We restrict the analysis to the collective choice between two alternatives, which we denote by $a$ and $b$, distinguishing the *binary* case $A = \{a, b\}$ from the more general, *two-valued* case $A \supseteq \{a, b\}$. Moreover, we limit our attention to the universal domain hypothesis, which means to the case of social choice functions (sometimes, we simply write function to mean social choice function) either defined on the set of all profiles or on the set of all strict profiles. We start analyzing the case $\mathcal{D} = \mathcal{R}^I$, so, until differently specified, a social choice function $\varphi$ is:

$$\varphi : \mathcal{R}^I \to \{a, b\}.$$

We shall switch to the case $\mathcal{D} = \mathcal{S}^I$ in Section 4.

**Notation 1.** *For a given profile $R = (R_i)_{i \in I}$, we denote by $(R, a)$ (resp. $(R, b)$) the set of agents who strictly prefer a over b (resp. b over a) in the profile R. Since the preferences are complete, we can write:*

$$(R, a) = \{i \in I : (b, a) \notin R_i\},$$
$$(R, b) = \{i \in I : (a, b) \notin R_i\}.$$

For a given profile $R$, we denote by $(R, \sim)$ the set of agents that consider the two alternatives $a$ and $b$ indifferent in that profile:

$$(R, \sim) = \{i \in I : (a, b) \in R_i \text{ and } (b, a) \in R_i\}.$$

The set $\{(R,a),(R,b),(R,\sim)\}$ is a partition of the set $I$ of agents.

**Notation 2.** *If $R = (R_i)_{i\in I}$ is a profile and $D$ ($\neq I$) is a coalition, we sometimes write $R = (R_D; R_{-D})$, where $R_D = (R_i)_{i\in D}$, and $-D$ stands for the complement $D^c = I \setminus D$. Hence, if two profiles $Q$ and $R$ coincide on a coalition $D$, we can write:*

$$Q = (Q_D; Q_{-D}) = (R_D; Q_{-D}),$$

*and, analogously,*

$$R = (R_D; R_{-D}) = (Q_D; R_{-D}).$$

One of the most important properties a social choice function should satisfy is to discourage lies: A "good"function should induce all the agents to reveal their true preferences in any situation. In other words, an aggregation mechanism is often required to work in such a way that agents never have a reason to manipulate it by misrepresenting their preferences.

A coalition $D$ can manipulate a social choice function $\varphi$ in the profile $R$ if there exists a profile $Q$ coinciding with $R$ on the complementary coalition $D^c = I \setminus D$ (hence, $R_j = Q_j \quad \forall j \notin D$) such that, according to the preferences of profile $R$, all the agents in the coalition $D$ strictly prefer the alternative $\varphi(Q)$ over $\varphi(R)$. Formally:

**Definition 2.** *A coalition $D$ can* **manipulate** *the social choice function $\varphi$ in the profile $R$ if there exists a profile $Q$ such that:*

$$\begin{aligned} &R_j = Q_j && \forall j \in D^c \text{ and} \\ &(\varphi(R), \varphi(Q)) \notin R_j && \forall j \in D. \end{aligned}$$

**Definition 3.** *A social choice function $\varphi$ such that no coalition can manipulate it in any profile is said to be* **nonmanipulable** *or* **group strategy-proof** *(GSP).*

In [5], various concepts of strategy-proofness were introduced and compared. It is not difficult to check that our definition of GSP corresponds to the *weakly group strategy-proof* in [5] (see also the footnote on p. 795).

Limiting the manipulation to coalitions that are singletons, the previous definition gives back the notion of individual strategy-proofness. The two concepts, group and individual strategy-proofness, are not equivalent in our setting, since we allow for a society consisting of possibly infinitely many agents. It is well known that if $I$ is finite, the two notions are equivalent.

Our aim was to describe a functional form of social choice functions that are nonmanipulable. We shall do this by using suitable collections of sets. In particular, we concentrated our attention on sets of agents with insufficient power to get the society to select the alternative they prefer. We speak of sets rather than coalitions since, being interested in the lack of power, we have to involve the empty set as well.

Let us explain and formalize the meaning of "power" we have in mind. Fix a profile $R$. With reference to a group $M$ of agents, consider a situation in which $M$ does not contain agents having the opposite opinion in $R$ regarding alternatives $a$ and $b$. Assume for example that in the profile $R$, no agent in $M$ strictly prefers $a$ over $b$. A group of agents in $M$ strictly preferring $b$ over $a$ may, or may not, obtain as the outcome of the choice the (preferred) alternative $b$. In the first case, which means $\varphi(R) = b$, it is strong enough to obtain $b$; in the opposite case, $\varphi(R) = a$, it is not. It is reasonable to call powerful in $M$ the subsets of $M$ having this force at every profile, and the others powerless. In other words, the powerless subsets of $M$ are those that fail to be decisive in at least one case (one profile) and for at least one of the two alternatives.

We focus our attention on powerless groups in the next definition.

**Definition 4.** *Let $\varphi$ be a social choice function; let $R$ be a profile; let $X \subseteq M \subseteq I$.*

*We say that $X$* **lacks power** *to determine the alternative $b$ under $\varphi$ in the set $M$ at the profile $R$ if the following three conditions hold:*

$$(*) \qquad X = (R, b), \qquad (R, a) \subseteq M^c, \qquad \varphi(R) = a.$$

We avoid explicitly referring to the function $\varphi$ when it is clear from the context. In this case, instead of saying that the set $X$ lacks power to determine the alternative $b$ under $\varphi$ in the set $M$ at the profile $R$, we only say that $X$ lacks power for $b$ in $M$ at $R$.

If the role of $a$ and $b$ in $(*)$ is reversed, we say that $X$ lacks power for $a$ in $M$ at $R$.

Without loss of generality, we considered groups lacking power for $b$.

With the aim of selecting powerless for $b$ groups of agents, we adopted the term Y**-family** (on $I$) to indicate a map that associates with each subset $M$ of $I$ a part of its power set $\mathcal{P}(M)$. In other words, a Y-family $\mathcal{F}$ assigns an element $\mathcal{F}_M \subseteq \mathcal{P}(M)$ to every $M \subseteq I$. We use the notation:

$$\mathcal{F} = (F_M)_{M \in I}$$

to indicate such a family.

Given a social choice function $\varphi$, we constructed a Y-family, $\mathcal{F}^{(\varphi)}$, collecting, $M$ by $M$, powerless (for $b$) groups of agents in $M$:

**Definition 5.** *Let $\varphi$ be a social choice function. The* Y*-family $\mathcal{F}^{(\varphi)}$ defined by:*

$$X \in \mathcal{F}_M^{(\varphi)} \Leftrightarrow \exists R \in \varphi^{-1}(a) \,:\, (R, a) \subseteq M^c \text{ and } X = (R, b)$$

*is said to be* **generated by** *$\varphi$.*

We present some examples of Y-families generated by social choice functions.

**Example 1** (*dictatorial function*)**.** *We remind that a function $\varphi$ is said to be dictatorial if there exists an agent $j$ such that:*

$$j \in (R, a) \Rightarrow \varphi(R) = a,$$
$$j \in (R, b) \Rightarrow \varphi(R) = b.$$

Of course, there are many social choice functions having a fixed agent $j$ as a dictator. Since our setting involves indifference, not all of them are group strategy-proof. Let us refer to the two "extreme" cases, in which the indifference of the dictator is solved a priori, respectively, in favor of $a$ or of $b$. The two functions we hence introduce are denoted respectively by $\varphi_{ja}$ and $\varphi_{jb}$. They are both examples of GSP functions.

The function $\varphi_{ja}$ (by default, the indifference of the dictator determines $a$) is defined as follows:

$$\varphi_{ja}(R) = b \Leftrightarrow j \in (R, b).$$

Similarly, the function $\varphi_{jb}$ (default $b$) is defined by:

$$\varphi_{jb}(R) = a \Leftrightarrow j \in (R, a).$$

The corresponding Y-families $\mathcal{F}^{ja}$ and $\mathcal{F}^{jb}$ generated by these functions are, respectively:

$$\mathcal{F}_M^{ja} = \mathcal{P}(M \setminus \{j\}) \qquad \forall M \subseteq I, \text{ and}$$

$$\mathcal{F}_M^{jb} = \begin{cases} \emptyset & \text{if } j \in M, \\ \mathcal{P}(M) & \text{if } j \notin M. \end{cases}$$

**Example 2** (*Measure-type function; see [9], Example 2.3*)**.** *Denote by $\mu$ a real-valued monotone (increasing) set function defined on the power set of $I$. The social choice functions $\varphi_{\mu a}$, $\varphi_{\mu b}$ defined by means of:*

$$\varphi_{\mu a}(R) = a \Leftrightarrow \mu((R,a)) \geq \mu((R,b))$$

*and:*

$$\varphi_{\mu b}(R) = a \Leftrightarrow \mu((R,a)) > \mu((R,b)),$$

*respectively, generate the* Y*-families*

$\mathcal{F}^{\mu a}$*:*

$$X \in \mathcal{F}_M^{\mu a} \Leftrightarrow X \subseteq M \text{ and } \mu(X) \leq \mu(M^c)$$

*and:*

$\mathcal{F}^{\mu b}$*:*

$$X \in \mathcal{F}_M^{\mu b} \Leftrightarrow X \subseteq M \text{ and } \mu(X) < \mu(M^c).$$

**Example 3.** *Let $\varphi$ be the social choice function defined by:*

$$\varphi(R) = b \Leftrightarrow [(R,a) = \emptyset \quad \text{and} \quad R_i = R_j \quad \forall i,j \in (R,\sim)].$$

*In case at least one agent strictly prefers a over b, the alternative a is selected. If $(R,a)$ is empty, the alternative b is selected only if all agents in $(R,\sim)$ have the same preference; otherwise, the selected alternative is a. The corresponding* Y*-family the function $\varphi$ generates is:*

$$\mathcal{F}_M = \begin{cases} \mathcal{P}(M) & \text{if } M \neq I, \\ \{X \subseteq I : |I \setminus X| \geq 2\} & \text{if } M = I. \end{cases}$$

In the next definition, we isolate a property that a Y-family may have. We name this property *ineffectiveness*, and it is intuitive, considering the interpretation that a Y-family has when it is generated by a social choice function.

**Definition 6.** *A* Y*-family is said to be* **ineffective** *(and each element of $\mathcal{F}_M$ is said to be ineffective at M) if:*

*(i)* *It is downward inclusive: $X \in \mathcal{F}_M$, $Y \subseteq X \Rightarrow Y \in \mathcal{F}_M$;*
*(ii)* *Ineffective sets have ineffective traces: $X \in \mathcal{F}_M$, $N \subseteq M \Rightarrow X \cap N \in \mathcal{F}_N$.*

Observe that given $(i)$, the implication in $(ii)$ above can be obviously replaced by:

$$X \in \mathcal{F}_M,\ N \subseteq M,\ X \subseteq N \Rightarrow X \in \mathcal{F}_N.$$

The idea behind Definition 6 is the following. When $\mathcal{F}$ is generated by a function, with reference to profiles where a partition $\{M, M^c\}$ *separates* agents strictly preferring one of the two possible outcomes over the other (agents strictly preferring $b$ over $a$ are all in $M$; agents strictly preferring $a$ over $b$ all belong to the complement $M^c$), the family $\mathcal{F}_M$ selects, among the agents in $M$, the (powerless) groups of agents that in one of these profiles strictly prefer the nonselected alternative, $b$, over the other.

It is then intuitive (and such intuition is supported by Proposition 1) that, when the generating function is GSP, if the power of a group is not big enough to determine the collective choice, the same is true for any of its subgroups (Condition (*i*)). It is equally natural to realize that the same group of agents is also powerless in case the set of "disagreeing" agents is even bigger (Condition (*ii*)).

It is obvious that both:

- $\mathcal{F}_M = \emptyset$, for every $M$;

and:

- $\mathcal{F}_M = \mathcal{P}(M)$, for every $M$

define ineffective families. Examples 1, 2, and 3 above are less trivial cases of ineffective Y-families.

## 3. Results

With the aim of obtaining a functional characterization of nonmanipulable social choice functions in the binary case, we linked ineffective families introduced in Definition 6 with group strategy-proof social choice functions. We did this by building a one-to-one correspondence between the two sets: GSP functions on one side, ineffective families on the other.

In our first result, we saw that every two-valued group strategy-proof social choice function generates an ineffective family.

**Proposition 1.** *The* Y*-family* $\mathcal{F}^{(\varphi)}$ *generated by a GSP social choice function* $\varphi$ *is ineffective.*

**Proof.** Let us prove Condition (*i*) of Definition 6:
For a fixed $M \subseteq I$, suppose:

$$Y \subseteq X \in \mathcal{F}_M^{(\varphi)}.$$

Denote by $\succeq$ any element of $\mathcal{R}$ (any preference on $A$) such that both the pairs $(a, b)$ and $(b, a)$ belong to $\succeq$. In other words, according to $\succeq$, the two alternatives $a$ and $b$ are indifferent. Let $R$ be one of the profiles for which it results:

$$\varphi(R) = a \qquad (R, a) \subseteq M^c \qquad (R, b) = X$$

(it exists since $X \in \mathcal{F}_M^{(\varphi)}$). Define the profile $Q$ by:

$$Q_i = \begin{cases} R_i & \text{if} \quad i \notin X \setminus Y, \\ \succeq & \text{if} \quad i \in X \setminus Y. \end{cases}$$

It is easy to check that $(Q, a) = (R, a)$; hence, $(Q, a) \subseteq M^c$. Moreover, $(Q, b) = Y$. The relation $Y \in \mathcal{F}_M^{(\varphi)}$ is proven once we show that $\varphi(Q) = a$.

The latter equality follows from the strategy-proofness of $\varphi$: assume by contradiction that $\varphi(Q) = b$. Observe that $X \setminus Y$ is a coalition (it is not empty otherwise $Q = R$) that can manipulate $\varphi$ in the profile $R$; in fact, agents in $X \setminus Y$ reporting preference $\succeq$ instead of $R_i$ change the profile $R$ into $Q$ and obtain as a result the alternative $b$, which each of them strictly prefers in the profile $R$ over $\varphi(R)$. This is impossible. From the contradiction, we have the thesis.

Let us now prove (*ii*):
Assume:

$$N \subseteq M \qquad X \in \mathcal{F}_M^{(\varphi)}.$$

Use, again, $R$ and $\succeq$ as before, and define the profile $T$ by putting:

$$T_i = \begin{cases} R_i & \text{if} \quad i \notin X \setminus N, \\ \succeq & \text{if} \quad i \in X \setminus N. \end{cases}$$

As before,
$(T, a) = (R, a)$; hence, $(T, a) \in M^c \subseteq N^c$,
$(T, b) = X \cap N$, and
$\varphi(T) = a$.
This is enough to say that $X \cap N \in \mathcal{F}_N^{(\varphi)}$, as wanted. □

**Remark 1.** *Notethatsinceforaa GSPfunction* $\varphi$, *thefamily* $\mathcal{F}_M^{(\varphi)}$ *is proven to be downward inclusive, the definition could be equivalently written in the form*:

$$\mathcal{F}_M^{(\varphi)} = \{X \subseteq M : \exists R \in \varphi^{-1}(a) \, : \, (R,a) \subseteq M^c \text{ and } X \subseteq (R,b) \subseteq M\}.$$

*In both cases (in the formula above and in Definition 5), given the strategy-proofness of* $\varphi$, *the inclusion* $(R,a) \subseteq M^c$ *can be substituted by the equality* $(R,a) = M^c$.

We can reverse the link between social choice functions and Y-families by associating a function with every given Y-family.

**Definition 7.** *Given a* Y*-family* $\mathcal{F}$, *the social choice function* $\varphi^{(\mathcal{F})}$ *defined by:*

$$\varphi^{(\mathcal{F})}(R) = a \Leftrightarrow (R,b) \in \mathcal{F}_{(R,a)^c}.$$

*is said to be* **generated by** $\mathcal{F}$.

Let us describe, for some Y-families, the corresponding generated functions:

**Example 4.** *The* Y*-family, obviously ineffective,*

$$\mathcal{F}_M = \begin{cases} \emptyset & \text{if } M = I, \\ \mathcal{P}(M) & \text{if } M \neq I \end{cases}$$

*generates the so-called consensus rule with disagreement default a and indifference default b (see [13]):*

$$\varphi(R) = \begin{cases} a & \text{if } (R,a) \neq \emptyset, \\ b & \text{if } (R,a) = \emptyset. \end{cases}$$

**Example 5** (*Example 3, continued*)**.** *The family:*

$$\mathcal{F}_M = \begin{cases} \mathcal{P}(M) & \text{if } M \neq I, \\ \{X \subseteq I : |I \setminus X| \geq 2\} & \text{if } M = I \end{cases}$$

*generates the function* $\varphi'$ *defined below:*

$$\varphi'(R) = b \Leftrightarrow [(R,a) = \emptyset \quad \text{and} \quad |(R,\sim)| \leq 1].$$

Observe that $\varphi'$ above differs from the function $\varphi$ of Example 3. However, both the functions $\varphi$ and $\varphi'$ generate the same family $\mathcal{F}$ we are considering. This does not occur in Examples 1 and 2: the families $\mathcal{F}^{ja}$, $\mathcal{F}^{jb}$, $\mathcal{F}^{\mu a}$, and $\mathcal{F}^{\mu b}$ generate the original functions $\varphi_{ja}$, $\varphi_{jb}$, $\varphi_{\mu a}$, and $\varphi_{\mu b}$ from which they come. The key to understand this phenomenon is the independence of irrelevant alternatives, as we discuss later (see the comments following Theorem 3).

Let us now introduce a special class of social choice functions. We name them powerless revealing since they reveal which groups of agents are not powerful.

**Definition 8.** *A social choice function* $\varphi$ *is said to be* **powerless revealing** *if* $\varphi = \varphi^{(\mathcal{F})}$ *for some ineffective family* $\mathcal{F}$.

The powerless revealing functions cannot be manipulated, as the next proposition shows.

**Proposition 2.** *Every powerless revealing function is group strategy-proof.*

**Proof.** Let $\varphi$ be a function and $\mathcal{F}$ an ineffective family such that $\varphi = \varphi^{(\mathcal{F})}$.

Assume $\varphi^{(\mathcal{F})}(R) = a$, and denote by $Q$ any profile coinciding with $R$ on $(R, b)^c$. None of the agents strictly preferring $a$ over $b$ in the profile $R$ change their preference in the passage from $R$ to $Q$. Hence, $(R, a) \subseteq (Q, a)$, which implies the inclusion:

$$(Q, a)^c \subseteq (R, a)^c.$$

On the other side, some of the agents strictly preferring $b$ over $a$ in the profile $R$ may change preference; hence, $(Q, b) \subseteq (R, b)$. The definition of $\varphi^{(\mathcal{F})}$ gives:

$$\varphi^{(\mathcal{F})}(R) = a \Leftrightarrow (R, b) \in \mathcal{F}_{(R,a)^c}.$$

Using Property *(i)*, we obtain:

$$(Q, b) \subseteq (R, b) \in \mathcal{F}_{(R,a)^c} \Rightarrow (Q, b) \in \mathcal{F}_{(R,a)^c}.$$

We now use Property *(ii)*, with $N = (Q, a)^c \subseteq (R, a)^c$ obtaining:

$$(Q, b) \cap N \in \mathcal{F}_N = \mathcal{F}_{(Q,a)^c}.$$

Finally, we observe that the sets $(Q, a)$ and $(Q, b)$ are (obviously) disjoint, then:

$$(Q, b) \cap N = (Q, b) \cap (Q, a)^c = (Q, b).$$

The relation $(Q, b) \in \mathcal{F}_{(Q,a)^c}$ implies that $\varphi^{(\mathcal{F})}(Q) = a$.

Analogously, assume now $\varphi^{(\mathcal{F})}(R) = b$, and denote by $Q$ any profile coinciding with $R$ on $(R, a)^c$. In the passage from $R$ to $Q$, none of the agents strictly preferring $b$ over $a$ change their preference; hence, $(R, b) \subseteq (Q, b)$ and:

$$(Q, a) \subseteq (R, a) \Rightarrow N := (R, a)^c \subseteq (Q, a)^c.$$

The following implications obviously hold:

$$\varphi^{(\mathcal{F})}(Q) = a \Rightarrow (Q, b) \in \mathcal{F}_{(Q,a)^c} \Rightarrow (R, b) \in \mathcal{F}_{(Q,a)^c}.$$

Moreover:

$$(R, b) \cap N = (R, b) \cap (R, a)^c = (R, b),$$

hence

$$(R, b) \cap N = (R, b) \in \mathcal{F}_N = \mathcal{F}_{(R,a)^c},$$

which means $\varphi^{(\mathcal{F})}(R) = a$. □

We are now ready to complete the connection between ineffective families and GSP functions. In particular, we obtained the aforementioned result: in the binary case ($A = \{a, b\}$), there is a one-to-one correspondence between the set of GSP social choice functions and the collection of all ineffective families. In other words, the powerless revealing functions are all and only the GSP functions.

**Theorem 3.** *For every ineffective family $\mathcal{F}$, the equality below holds true:*

$$\mathcal{F}^{(\varphi^{(\mathcal{F})})} = \mathcal{F}.$$

*If, moreover, $A = \{a, b\}$, then it is also true that:*

$$\varphi^{(\mathcal{F}^{(\varphi)})} = \varphi$$

*whenever $\varphi$ is a group strategy-proof social choice function.*

**Proof.** Let us start by proving the equality:

$$\mathcal{F}^{(\varphi^{(\mathcal{F})})} = \mathcal{F}.$$

To simplify the exposition, denote for the moment by $\varrho$ the function $\varphi^{(\mathcal{F})}$. Fix any subset $M$ of $I$; our aim is to prove that:

$$X \in \mathcal{F}_M^{(\varrho)} \Leftrightarrow X \in \mathcal{F}_M.$$

If $X$ is an element of $\mathcal{F}_M^{(\varrho)}$, then $X \subseteq M$, and there exists a profile $R$ such that:
$(*)$ $(R, a) \subseteq M^c$, which gives $M \subseteq (R, a)^c$;
$(**)$ $X = (R, b)$;
$(***)$ $\varrho(R) = a$, which means $\varphi^{(\mathcal{F})}(R) = a$, in other words $(R, b) \in \mathcal{F}_{(R,a)^c}$.
From $(**)$ and $(*)$, it follows that:

$$X \cap M = X \in \mathcal{F}_M.$$

Conversely, assume $X \in \mathcal{F}_M$. Denote by $R$ any profile such that $(R, b) = X$ and $(R, a) = M^c$ (see Remark 1). By definition, it is $\varphi^{(\mathcal{F})}(R) = a$, which gives the thesis.

Assume now that the set $A$ of alternatives does not contain any element but $a$ and $b$. In this case, we want to show that:

$$\varphi(Q) = a \Leftrightarrow \varphi^{(\mathcal{F}^{(\varphi)})}(Q) = a.$$

Assume $\varphi(Q) = a$, and denote by $M$ the set $(Q, a)^c$. Since $(Q, a) \subseteq M^c$ (the two sets coincide) and $(Q, b) \subseteq M$, it is, by the definition of $\mathcal{F}^{(\varphi)}$,

$$(Q, b) \in \mathcal{F}_M^{(\varphi)} = \mathcal{F}_{(Q,a)^c}^{(\varphi)},$$

which means $\varphi^{(\mathcal{F}^{(\varphi)})}(Q) = a$.

Conversely, assume that $\varphi^{(\mathcal{F}^{(\varphi)})}(Q) = a$, in other words that $(Q, b) \in \mathcal{F}_{(Q,a)^c}^{(\varphi)}$. Looking at the definition of the family $\mathcal{F}^{(\varphi)}$ (Definition 5), there exists a profile $R$ such that
$\varphi(R) = a$,
$(R, a) \subseteq (Q, a)$, and
$(R, b) = (Q, b)$.

The relation $\varphi(Q) = b$ would give to coalition $(Q, a) \setminus (R, a)$ in the profile $Q$ the possibility of manipulating $\varphi$. Hence, we have the thesis. □

The reason why we obtained the formula $\varphi^{(\mathcal{F}^{(\varphi)})} = \varphi$ (and, hence, a functional characterization) only in the binary case is that, in the two-valued setting, group strategy-proofness does not imply the independence of the collective choice from irrelevant alternatives. Specifically, different social choice functions may originate the same ineffective family (see Example 3, continued). In other words, the mapping $\varphi \mapsto \mathcal{F}^{(\varphi)}$ is not injective. This is a consequence of the fact that the construction only depends on how agents rank the alternatives $a$ and $b$ in the different profiles; hence, it does not distinguish among functions assuming different values on profiles having the same restrictions to the set $\{a, b\}$. We can equivalently say that the mapping $\mathcal{F} \mapsto \varphi^{(\mathcal{F})}$ is not surjective.

We conclude this section by explicitly presenting the functional form characterization of group strategy-proof social choice functions in the binary setting:

**Theorem 4.** *A social choice function $\varphi : \mathcal{R}^I \to \{a, b\} = A$ is group strategy-proof if and only if it is powerless revealing. Moreover, it has a unique generating family.*

The functional characterization above extends to the two-valued setting if the independence of irrelevant alternatives is assumed. A functional form characterization of two-valued group strategy-proof functions without the independence assumption was obtained by Basile et al. in [10] as an extension of the same authors' previous result for the binary case in [9].

**Remark 2.** *If the set of agents is finite, we can compare our functional form characterization of Theorem 4 with the generalized voting by committee characterization provided by Lahiri and Pramanik in [12], Theorem 1.*

*A generalized voting by committee is a binary social choice function from $\mathcal{R}^I$ to $\{a,b\} = A$ denoted and defined as follows:*

$$\varphi^{\mathcal{I}^d}_{\mathcal{F}_{\mathcal{I}^d}}(R) = \begin{cases} d & \text{if } (R,\sim) \in \mathcal{I}^d, \\ a & \text{if } (R,\sim) \notin \mathcal{I}^d \text{ and } (R,a) \in \mathcal{F}_{(R,\sim)^c,\mathcal{I}^d}, \\ b & \text{otherwise.} \end{cases}$$

*The necessary specification of the "parameters"$\mathcal{I}^d$ and $\mathcal{F}_{\mathcal{I}^d}$ is needed. As for $\mathcal{I}^d$, where d is either a or b, it is a committee whose purpose is to assign the default d to all profiles R for which it is $(R,\sim) \in \mathcal{I}^d$. Concerning $\mathcal{F}_{\mathcal{I}^d}$, it is a family $\{\mathcal{F}_{M,\mathcal{I}^d} : M \subseteq I\}$ where each $\mathcal{F}_{M,\mathcal{I}^d}$ is a committee within the set M. Moreover, the pair of parameters is required to satisfy certain conditions (crucial for the strategy-proofness), which we do not report here for the sake of brevity.*

*In our opinion, the simplicity and intuitiveness of Definition 6 make the characterization based on the concept of powerless revealing functions considerably simpler than the one by means of generalized voting by committee.*

## 4. The Case of Strict Preferences

In this section, we consider social choice functions defined on the set of all profiles consisting of strict preferences:

$$\varphi : \mathcal{S}^I \to \{a,b\} \subseteq A.$$

When indifference between alternatives is not allowed, the reasoning to obtain a functional form characterization for GSP functions is simpler. We describe it and compare the result with the literature.

Each profile $R$ produces now a partition of the set $I$ of agents into two subsets $(R,a)$ and $(R,b)$, the set $(R,\sim)$ being obviously empty in this case.

The first important observation is that limiting the domain of $\varphi$ to the strict profiles, strategy-proofness implies that the value of $\varphi$ on a profile only depends on the considered partition, the other alternatives having no influence (this is not the case if indifference is allowed):

**Observation:** *Let the function $\varphi : \mathcal{S}^I \to \{a,b\} \subseteq A$ be group strategy-proof. Then:*

$$(R,b) = (Q,b) \Rightarrow \varphi(R) = \varphi(Q).$$

For the sake of completeness, we give a proof of the observation.

**Proof.** Assume by contradiction that the values of $\varphi$ are different on two profiles $R$ and $Q$ with $(R,b) = (Q,b)$, say, for example, $\varphi(R) = a$ and $\varphi(Q) = b$. The sets $(R,a) = (Q,a)$ and $(R,b) = (Q,b)$ are both coalitions, since otherwise, the values $\varphi(a)$ and $\varphi(b)$ coincide because of unanimity.

Denote by $D$ the coalition $(R,a)$, and define the profile $T$ coinciding with $R$ on the coalition $D$ and with $Q$ on the complementary coalition $D^c$:

$$T = (R_D; Q_{D^c}).$$

If $\varphi(T) = a$, coalition $D$ can manipulate $\varphi$ in the profile $Q$, if $\varphi(T) = b$ coalition $D^c$ can manipulate $\varphi$ in the profile $R$. In both cases, there is a contradiction. □

The previous observation allows defining the sets:

$$\mathcal{G}_a^{(\varphi)} = \{M \subseteq I : \varphi(R) = a \text{ for some } R \text{ with } M = (R,b)\}$$

and:

$$\mathcal{G}_b^{(\varphi)} = \{M \subseteq I : \varphi(R) = b \text{ for some } R \text{ with } M = (R,b)\}.$$

They form a partition of the power set $\mathcal{P}(I)$ of $I$. The first set, $\mathcal{G}_a^{(\varphi)}$, is downward inclusive. The second, $\mathcal{G}_b^{(\varphi)}$, is upward inclusive.

Let us now recall from Remark 1 that the definition of the ineffective family $\mathcal{F}^{(\varphi)}$ generated by a GSP function $\varphi : \mathcal{S}^I \to \{a,b\}$ is:

$$\mathcal{F}_M^{(\varphi)} = \{X \subseteq M : \exists R \in \varphi^{-1}(a) \, : \, (R,a) \subseteq M^c \text{ and } X \subseteq (R,b) \subseteq M\}.$$

When the two relations $(R,a) \subseteq M^c$ and $(R,b) \subseteq M$ are simultaneously true, it is necessarily $(R,a) = M^c$ and $(R,b) = M$. This means that we can equivalently write:

$$\mathcal{F}_M^{(\varphi)} = \{X \subseteq M : \exists R \in \varphi^{-1}(a) \, : X \subseteq (R,b) = M\}.$$

Moreover, if for a certain $M$, the set $\mathcal{F}_M^{(\varphi)}$ is not empty, which means that a profile $R$ with $(R,b) = M$ and $\varphi(R) = a$ exists, the relation $\varphi(Q) = a$ holds for all profiles $Q$ with $(Q,b) = M$. Hence, it is also:

$$\mathcal{F}_M^{(\varphi)} = \{X \subseteq M : \varphi(R) = a \quad \forall R \text{ with } (R,b) = M\}.$$

The consequence is that $\mathcal{F}_M^{(\varphi)}$ is either empty or it consists of the whole power set $\mathcal{P}(M)$ of $M$. Precisely:

$$\mathcal{F}_M^{(\varphi)} = \begin{cases} \mathcal{P}(M) & \text{if } M \in \mathcal{G}_a^{(\varphi)}, \\ \emptyset & \text{if } M \in \mathcal{G}_b^{(\varphi)}. \end{cases}$$

It is not difficult to check that also in this (strict) case, the family $\mathcal{F}^{(\varphi)}$ is ineffective.

The relation:

$$\varphi^{(\mathcal{F}^{(\varphi)})} = \varphi$$

in the statement of Theorem 3 that in the general case (weak profiles) only holds for $A = \{a,b\}$ here is true for any set $A$, in fact:

$$\varphi^{(\mathcal{F}^{(\varphi)})}(Q) = a \Leftrightarrow (Q,b) \in \mathcal{F}_{(Q,a)^c}^{(\varphi)} \Leftrightarrow \mathcal{F}_{(Q,a)^c}^{(\varphi)} \neq \emptyset \Leftrightarrow$$

$$\Leftrightarrow \mathcal{F}_{(Q,a)^c}^{(\varphi)} = \mathcal{P}\left(\mathcal{F}_{(Q,a)^c}^{(\varphi)}\right) \Leftrightarrow (Q,b) = (Q,a)^c \in \mathcal{G}_a^{(\varphi)} \Leftrightarrow \varphi(Q) = a.$$

Let us now look at the other way round: the definition of a GSP function from an ineffective family.

We limit our considerations to special ineffective families:

Take a downward inclusive subset $\mathcal{H}$ of $\mathcal{P}(I)$, and define the ineffective family $\mathcal{F}^{(\mathcal{H})}$ by:

$$\mathcal{F}_M^{(\mathcal{H})} = \begin{cases} \mathcal{P}(M) & \text{if } M \in \mathcal{H}, \\ \emptyset & \text{if } M \notin \mathcal{H}. \end{cases}$$

The social choice function generated by it (we denote it by $\varphi^{(\mathcal{H})}$ for simplicity) is given by:

$$\varphi^{(\mathcal{H})}(R) = \begin{cases} a & if(R,b) \in \mathcal{H}, \\ b & if(R,b) \notin \mathcal{H}. \end{cases}$$

Summing up: if $\mathcal{H} \subseteq \mathcal{P}(I)$ is downward inclusive, then the function $\varphi^{(\mathcal{H})}$ defined by:

$$\varphi^{(\mathcal{H})}(R) = a \Leftrightarrow (R, b) \in \mathcal{H}$$

is GSP.

On the other side, if the function $\varphi : \mathcal{S}^I \to \{a, b\} \subseteq A$ is GSP, the subset $\mathcal{G}_a^{(\varphi)}$ of the power set $\mathcal{P}(I)$ of $I$ is downward inclusive. Moreover,

$$\varphi^{\left(\mathcal{G}_a^{(\varphi)}\right)} = \varphi \qquad \text{and} \qquad \mathcal{G}_a^{(\varphi^{(\mathcal{H})})} = \mathcal{H}.$$

It is at this point clear that in this case (strict profiles), there is no need to consider a whole (ineffective) family, in the sense that there is no need to specify, for each subset of $M$, a subset of $\mathcal{P}(M)$. It is now more natural to refer to a unique family of subsets of $I$, the only requirement for such a family being the downward inclusion.

We can then conclude the section by stating the version of the functional representation theorem for the case of strict profiles. It is the specialization to this setting of Theorem 4:

**Theorem 5.** *A social choice function $\varphi : \mathcal{S}^I \to \{a, b\} \subseteq A$ is group strategy-proof if and only if it is of the form:*

$$\varphi(R) = a \Leftrightarrow (R, b) \in \mathcal{H}$$

*for a (unique) downward inclusive subset $\mathcal{H}$ of $\mathcal{P}(I)$.*

We observe that Theorem 5 coincides with Proposition 4.1, of [8], of which Theorem 2 of [6] is a corollary.

## 5. Conclusions and Future Directions

We studied two-valued nonmanipulable social choice functions. For such functions, we established the properties of the coalitions failing to be decisive to determine the collective choice. These properties led us to introduce a class of social choice functions that we called powerless revealing functions. We showed that the powerless revealing functions are all and only the nonmanipulable binary social choice functions. Our study concerns the general framework that admits the possibility that agents declare indifference. If this is not the case, i.e., if indifference is not admitted, our approach simplifies and gives that powerless revealing functions are the voting by committee functions.

There are some natural future developments of this work. One is to investigate if the ineffective families approach can be successfully adapted to the case of two-valued functions without assuming the independence of irrelevant alternatives. This would offer an alternative functional form characterization to that obtained in [10].

Whereas in the binary case, the universal domain hypothesis can be considered not too demanding, it can be interesting to relax such a hypothesis in the two-valued case, for example, either keeping the independence of agents (each endowed with an admissible set of available preferences, $\mathcal{D}$ resulting in a Cartesian product) or assuming that $\mathcal{D}$ is not necessarily a Cartesian product, i.e., allowing that individual preferences are not independent.

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