mathematics

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Characterization of Rectifying Curves by Their Involutes and Evolutes

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Abstract: A rectifying curve is a twisted curve with the property that all of its rectifying planes pass through a fixed point. If this point is the origin of the Cartesian coordinate system, then the position vector of the rectifying curve always lies in the rectifying plane. A remarkable property of these curves is that the ratio between torsion and curvature is a nonconstant linear function of the arc-length parameter. In this paper, we give a new characterization of rectifying curves, namely, we prove that a curve is a rectifying curve if and only if it has a spherical involute. Consequently, rectifying curves can be constructed as evolutes of spherical twisted curves; we present an illustrative example of a rectifying curve obtained as the evolute of a spherical helix. We also express the curvature and the torsion of a rectifying spherical curve and give necessary and sufficient conditions for a curve and its involute to be both rectifying curves.

Keywords: rectifying curves; involutes; evolutes

Citation: Jianu, M.; Achimescu, S.; Dăuş, L.; Mihai, A.; Roman, O.-A.; Tudor, D. Characterization of Rectifying Curves by Their Involutes and Evolutes. *Mathematics* **2021**, *9*, 3077. https://doi.org/10.3390/math9233077

Academic Editor: Juan De Dios Pérez

Received: 7 November 2021
Accepted: 26 November 2021
Published: 29 November 2021

1. Introduction

The term *rectifying curve* was introduced by Chen in [1] to designate a curve whose position vector always lies in its rectifying plane. Kim et al. [2] also studied the properties of the "space curves satisfying $\tau/\kappa = as + b$". They proved that a curve has this property if and only if there exists a point such that all of the rectifying planes to the curve pass through this point. By a translation, any curve of this type turns out to be a *rectifying curve* in the restricted sense of [1]. In this paper, we consider, for a rectifying curve, the definition from [2] because it employs only the intrinsic properties of the curve.

The geometric properties of rectifying curves are presented in detail in [1,3]. It is shown that any rectifying curve has a parametrization of the form $\mathbf{x}(t) = a\sec(t+t_0)\mathbf{y}(t)$, where $\mathbf{y}$ is a unit speed curve on the unit sphere S^2, $a > 0$ and t_0 are some constants. Chen [4] proved that a geodesic on a conical surface is either a rectifying curve or an open portion of a ruling.

The relationship between rectifying curves and the notion of centrodes in mechanics is pointed out in [3] and developed in [5]. The *centrode* of a curve is the curve defined by the Darboux vector $\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}$. Chen and Dillen showed in [3] that the centrode of a unit speed curve with constant (nonzero) curvature and nonconstant torsion is a rectifying curve (and the same result holds for a curve with constant (nonzero) torsion and nonconstant curvature). A generalization of these results is given in [5], where it is proven that the centrode of a twisted nonhelical curve is a rectifying curve if and only if the curvature κ and torsion τ are not both constants and if they satisfy a nonhomogeneous linear equation of the form $a\kappa - b\tau = c$, where $a^2 + b^2 \neq 0$ and $c \neq 0$. In the same paper, Deshmukh et al.

show that, for any twisted nonhelical curve, the dilated centrode $\bar{\mathbf{d}} = \frac{1}{\kappa}\mathbf{d} = \frac{\tau}{\kappa}\mathbf{t} + \mathbf{b}$ is a rectifying curve.

In this paper, we give a new characterization of rectifying curves based on the relationship with their evolutes/involutes. An *involute* of a curve $\mathbf{x}$ is a curve $\mathbf{y}$ that lies on the tangent surface to $\mathbf{x}$ and intersects the tangent lines orthogonally. If the curve $\mathbf{y}$ is an involute of $\mathbf{x}$, then $\mathbf{x}$ is said to be an *evolute* of $\mathbf{y}$. We prove that a curve is a rectifying curve if and only if it has a spherical involute (see Corollary 2). We also express the curvature and the torsion of a spherical rectifying curve (in Theorem 3) and give necessary and sufficient conditions for a curve and its involute to be both rectifying curves (in Theorem 5). A direct consequence of Theorem 6 is that rectifying curves can be constructed as evolutes of spherical non-planar curves.

The outline of the paper is as follows. Section 2 contains some basic notions and formulas needed for our work. In Section 3, we briefly present the main features of rectifying curves and give (in Theorem 3) necessary and sufficient conditions for a rectifying curve to lie on a sphere. Section 4 is devoted to the involute of a rectifying curve and establishes in which conditions an involute of a rectifying curve is also a rectifying curve. In Section 5, we state and prove a new defining feature of rectifying curves: a curve is a rectifying curve if and only if its involutes are spherical curves. This result (Theorem 6) allows us to develop a new method of constructing rectifying curves: as evolutes of spherical twisted curves. The method is applied in Section 6 to construct rectifying curves as evolutes of a spherical helix.

2. Preliminaries

We denote by $\mathbb{E}^3$ the Euclidean 3-space with its inner product $\langle\, , \rangle$.

Let I be a real interval and $\mathbf{x} : I \to \mathbb{E}^3$ be a space curve of class C^∞, parametrized by the arc-length s (a *unit-speed* curve). The *unit* vector field $\mathbf{t} = \mathbf{x}'$ is called the *tangent vector field*. The *curvature* of the curve $\mathbf{x}$ is the positive function $\kappa(s) = \|\mathbf{t}'(s)\|$. We assume that $\kappa(s) > 0$ for any $s \in I$ and define the *principal normal vector field* as $\mathbf{n} = \dfrac{\mathbf{t}'}{\kappa}$. Since $\langle \mathbf{t}(s), \mathbf{t}(s)\rangle = 1$ for all $s \in I$, it follows that $\langle \mathbf{t}', \mathbf{t}\rangle = 0$; hence, $\mathbf{n}$ is orthogonal to $\mathbf{t}$. The *binormal vector field* is defined by the cross product $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. It is a unit vector field orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. The torsion of the curve is the function $\tau(s)$ by the equation $\mathbf{b}' = -\tau\mathbf{n}$. The orthogonal unit vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ form the well-known *Frenet frame* or *Frenet trihedron*.

The vector fields $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ satisfy the following *Frenet–Serret* formulas (see, for instance, [6] or [7]):

$$\begin{array}{rcccc} \mathbf{t}' = & & \kappa\mathbf{n} & & \\ \mathbf{n}' = & -\kappa\mathbf{t} & + & \tau\mathbf{b} & \\ \mathbf{b}' = & & -\tau\mathbf{n} & . & \end{array} \tag{1}$$

At each point of the curve, the planes determined by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$ and $\{\mathbf{n}, \mathbf{b}\}$ are known as the *osculating plane*, the *rectifying plane* and the *normal plane*, respectively. A curve in $\mathbb{E}^3$ is called *twisted* if it has nonzero curvature and torsion.

3. Rectifying Curves

In the following, we consider $\mathbf{x} : I \to \mathbb{E}^3$ to be a unit speed curve of class C^∞ with $\kappa(s) \neq 0$ for all $s \in I$.

Definition 1. *A curve $\mathbf{x}$ is a rectifying curve if there exists a point $\mathbf{x}_0 \in \mathbb{E}^3$ that belongs to all of the rectifying planes to the curve. The point $\mathbf{x}_0$ (uniquely determined) is said to be the reference point of the rectifying curve (see [4]).*

By Definition 1, a rectifying curve can be written as:

$$\mathbf{x}(s) = \mathbf{x}_0 + \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s), \tag{2}$$

for any $s \in I$, where $\lambda, \mu : I \to \mathbb{R}$ are functions of class C^∞. These functions cannot be arbitrary; they have a well-defined form established by the following theorem (see [1,2]):

Theorem 1. *The unit speed curve $\mathbf{x} : I \to \mathbb{E}^3$ is a rectifying curve if and only if there exist $\mathbf{x}_0 \in \mathbb{E}^3$ and the constants $\nu, \mu \in \mathbb{R}$, $\mu \neq 0$, such that*

$$\mathbf{x}(s) = \mathbf{x}_0 + (s+\nu)\mathbf{t}(s) + \mu\mathbf{b}(s),\ s \in I. \tag{3}$$

A rectifying curve can also be defined by the ratio τ/κ. It is known that a space curve is said to be a a generalized helix if and only if the ratio τ/κ is a nonzero constant. For rectifying curves, the ratio τ/κ must be a (nonconstant) linear function of the arc-length s. Thus, by a simple differentiation of (3) and using Frenet–Serret Formula (1), we obtain that

$$\mathbf{x}'(s) = \mathbf{t}(s) = \mathbf{t}(s) + (s+\nu)\kappa(s)\mathbf{n}(s) - \mu\tau(s)\mathbf{n}(s), \tag{4}$$

so

$$\frac{\tau(s)}{\kappa(s)} = \frac{1}{\mu}s + \frac{\nu}{\mu} \tag{5}$$

and the next theorem follows (see also [1,2]):

Theorem 2. *The unit speed curve $\mathbf{x} : I \to \mathbb{E}^3$ is a rectifying curve if and only if there exist the constants $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that*

$$\frac{\tau(s)}{\kappa(s)} = \alpha s + \beta,\ s \in I. \tag{6}$$

The first main result of the paper is the following theorem, which gives necessary and sufficient conditions for a rectifying curve to lie on a sphere.

Theorem 3. *A unit speed rectifying curve lying on a sphere of radius r has the curvature and torsion expressed by the following equations:*

$$\kappa(s) = \frac{1}{\sqrt{r^2 - \left(\frac{1}{2}\alpha s^2 + \beta s + \gamma\right)^2}}, \tag{7}$$

$$\tau(s) = \frac{\alpha s + \beta}{\sqrt{r^2 - \left(\frac{1}{2}\alpha s^2 + \beta s + \gamma\right)^2}}, \tag{8}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha \neq 0$.

Conversely, if the curvature and the torsion of a curve are given by (7) and (8), then it is a spherical rectifying curve.

Proof. Let $\mathbf{x} : I \to \mathbb{E}^3$ be a unit speed rectifying curve that lies on a sphere of radius r, centered at $\mathbf{c}_0$. Thus,

$$\|\mathbf{x}(s) - \mathbf{c}_0\|^2 = \langle \mathbf{x}(s) - \mathbf{c}_0, \mathbf{x}(s) - \mathbf{c}_0 \rangle = r^2,\ \forall\, s \in I. \tag{9}$$

By differentiating the relation above, we obtain that

$$\langle \mathbf{x}(s) - \mathbf{c}_0, \mathbf{t}(s) \rangle = 0,\ \forall\, s \in I, \tag{10}$$

so we can write

$$\mathbf{x} - \mathbf{c}_0 = \delta\mathbf{n} + \eta\mathbf{b}, \tag{11}$$

where $\delta = \delta(s)$ and $\eta = \eta(s)$ are two functions, such that

$$\|\mathbf{x} - \mathbf{c}_0\|^2 = \delta^2 + \eta^2 = r^2. \tag{12}$$

We differentiate (11) and use Frenet–Serret Formula (1):

$$\mathbf{t} = \delta'\mathbf{n} + \delta(-\kappa\mathbf{t} + \tau\mathbf{b}) + \eta'\mathbf{b} - \eta\tau\mathbf{n}, \tag{13}$$

so we obtain:

$$1 + \delta\kappa = 0 \tag{14}$$

$$\delta' - \eta\tau = 0 \tag{15}$$

$$\eta' + \delta\tau = 0. \tag{16}$$

From (14), we have

$$\delta = -\frac{1}{\kappa} \tag{17}$$

and, after replacing in (16), we obtain:

$$\eta' = \frac{\tau}{\kappa}. \tag{18}$$

Since $\mathbf{x}$ is a rectifying curve, by Theorem 2, there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$, such that the ratio τ/κ satisfies Equation (6). Hence, Equation (18) is written $\eta' = \alpha s + \beta$ and we obtain by integration that

$$\eta = \tfrac{1}{2}\alpha s^2 + \beta s + \gamma, \tag{19}$$

where $\gamma \in \mathbb{R}$ is a constant. From Equation (12), we obtain that

$$\frac{1}{\kappa^2} + \left(\tfrac{1}{2}\alpha s^2 + \beta s + \gamma\right)^2 = r^2, \tag{20}$$

so the Formula (7) readily follows. For Equation (8), we simply apply (6).

Conversely, let $\mathbf{x}$ be a curve with the curvature κ and the torsion τ be given by the Formulas (7) and (8). Then $\tau/\kappa = \alpha s + \beta$, hence $\mathbf{x}$ is a rectifying curve. It can also be proven, by a straightforward calculation, that $\kappa(s)$ and $\tau(s)$ satisfy the equation

$$\frac{1}{\kappa^2} + \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]^2 = r^2, \tag{21}$$

so the curve $\mathbf{x}$ lies on a sphere of radius r ([7], p. 32). $\square$

4. The Involute of a Rectifying Curve

An involute of a curve $\mathbf{x} : I \to \mathbb{E}^3$ is a curve $\mathbf{y}$ lying on the surface generated by the tangent lines to $\mathbf{x}$, and having the property that these tangent lines are orthogonal to the corresponding tangent lines to $\mathbf{y}$ (see [6,7]). It can be proven that $\mathbf{y} : I \to \mathbb{E}^3$ is of the form:

$$\mathbf{y} = \mathbf{x} + (c - s)\mathbf{t}, \tag{22}$$

where $c \in \mathbf{R}$ is a constant.

If $\mathbf{x} : I \to \mathbb{E}^3$ is a rectifying curve defined by (3) and $\mathbf{y} : I \to \mathbb{E}^3$ is one of its involutes, defined by (22), then

$$\mathbf{y}(s) = \mathbf{x}_0 + (c + \nu)\mathbf{t}(s) + \mu\mathbf{b}(s),\ s \in I, \tag{23}$$

so

$$\|\mathbf{y}(s) - \mathbf{x}_0\| = \sqrt{(c + \nu)^2 + \mu^2} = \text{constant},\ \forall s \in I \tag{24}$$

and the following theorem is proven:

Theorem 4. *Any involute of a rectifying curve lies on a sphere centered at the reference point.*

The next theorem establishes in which conditions an involute of a rectifying curve is also a rectifying curve.

Theorem 5. *Let $\mathbf{x} = \mathbf{x}(s)$ be a rectifying curve (satisfying Equation (6)). Let*

$$\mathbf{y} = \mathbf{x}(s) + (c - s)\mathbf{t}(s),\ s \in I \tag{25}$$

be an involute of $\mathbf{x}$ ($c \in \mathbb{R}$ is a constant). Then, $\mathbf{y}$ is also a rectifying curve if and only if the curvature $\kappa(s)$ of the curve $\mathbf{x}$ is expressed by:

$$\kappa(s) = \gamma\Big(1 + (\alpha s + \beta)^2\Big)^{-\frac{5}{4}}\left[1 + (\alpha c + \beta)(\alpha s + \beta) + \delta\sqrt{1 + (\alpha s + \beta)^2}\right]^{-\frac{1}{2}} \tag{26}$$

where δ and γ are real constants, $\gamma \neq 0$.

Proof. Assume that $c - s > 0$ on I.

Let $\tilde{s}$ be the arc-length parameter of $\mathbf{y}$. Then, since $y'(s) = (c - s)\kappa(s)\mathbf{n}(s)$,

$$\tilde{s} = \int_{s_0}^{s} \|y'(\sigma)\| d\sigma = \int_{s_0}^{s} (c - \sigma)\kappa(\sigma) d\sigma, \tag{27}$$

hence

$$\frac{d\tilde{s}}{ds} = \kappa(s)(c - s). \tag{28}$$

We can write the unit vectors of the Frenet trihedron of the curve $\mathbf{y}$ as follows. Obviously, $\tilde{\mathbf{t}} = \mathbf{n}$, and we have:

$$\tilde{\mathbf{n}} = \frac{\frac{d\tilde{\mathbf{t}}}{d\tilde{s}}}{\left\|\frac{d\tilde{\mathbf{t}}}{d\tilde{s}}\right\|} = \frac{\mathbf{n}'}{\|\mathbf{n}'\|} = \frac{-\kappa\mathbf{t} + \tau\mathbf{b}}{\sqrt{\kappa^2 + \tau^2}}, \tag{29}$$

$$\tilde{\mathbf{b}} = \tilde{\mathbf{t}} \times \tilde{\mathbf{n}} = \frac{\kappa\mathbf{b} + \tau\mathbf{t}}{\sqrt{\kappa^2 + \tau^2}}. \tag{30}$$

Thus, the relationship between the Frenet frames $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$ and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ can be written:

$$\begin{aligned}
\tilde{\mathbf{t}} &= \mathbf{n}, \\
\tilde{\mathbf{n}} &= -\frac{1}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}}\mathbf{t} + \frac{\frac{\tau}{\kappa}}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}}\mathbf{b}, \\
\tilde{\mathbf{b}} &= \frac{\frac{\tau}{\kappa}}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}}\mathbf{t} + \frac{1}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}}\mathbf{b},
\end{aligned} \tag{31}$$

and the curvature $\tilde{\kappa}$ and the torsion $\tilde{\tau}$ of $\mathbf{y}$ are given by the formulas (see also [8]):

$$\tilde{\kappa} = \frac{1}{c - s}\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}, \tag{32}$$

$$\tilde{\tau} = \frac{1}{\kappa(c - s)} \frac{\left(\frac{\tau}{\kappa}\right)'}{1 + \left(\frac{\tau}{\kappa}\right)^2}. \tag{33}$$

Hence,

$$\frac{\tilde{\tau}}{\tilde{\kappa}} = \frac{\left(\frac{\tau}{\kappa}\right)'}{\kappa\Big(1 + \left(\frac{\tau}{\kappa}\right)^2\Big)^{\frac{3}{2}}}. \tag{34}$$

However, $\mathbf{x}$ is a rectifying curve, so we can write, by Equation (6),

$$\frac{\tilde{\tau}}{\tilde{\kappa}} = \frac{\alpha}{\kappa(s)[1+(\alpha s+\beta)^2]^{\frac{3}{2}}}. \tag{35}$$

The involute $\mathbf{y}$ is a rectifying curve if and only if

$$\frac{d}{d\tilde{s}}\left(\frac{\tilde{\tau}}{\tilde{\kappa}}\right) = A = \text{constant} \neq 0. \tag{36}$$

It follows that the curvature $\kappa(s)$ must satisfy the following (Bernoulli) differential equation:

$$\kappa' + \frac{3\alpha(\alpha s+\beta)}{1+(\alpha s+\beta)^2}\cdot\kappa + \frac{A}{\alpha}(c-s)\Big(1+(\alpha s+\beta)^2\Big)^{\frac{3}{2}}\cdot\kappa^3 = 0. \tag{37}$$

The general solution of this equation is given by Formula (26), where $\gamma = \sqrt{\frac{\alpha^3}{2A}}$, and so the proof is complete. □

Remark 1. *If we look at Formula (33), we can say that $\tilde{\tau} \neq 0$ (because $\frac{\tau}{\kappa} = \alpha \neq 0$), so we conclude that the involute of a rectifying curve is always a non-planar curve.*

Corollary 1. *If a rectifying curve $\mathbf{x}$ has an involute $\mathbf{y}$ that is a rectifying curve, then any other involute of $\mathbf{x}$ is not a rectifying curve.*

Proof. Suppose that

$$\mathbf{y}_1 = \mathbf{x} + (c_1 - s)\mathbf{t} \tag{38}$$

and

$$\mathbf{y}_2 = \mathbf{x} + (c_2 - s)\mathbf{t} \tag{39}$$

are two involutes of the rectifying curve $\mathbf{x}$, which are also rectifying curves. Then, the differential Equation (37) must be satisfied for $c = c_1$, $A = A_1$ and for $c = c_2$, $A = A_2$, respectively. It follows that

$$A_1(c_1 - s) = A_2(c_2 - s),\ \forall s \in I, \tag{40}$$

hence $A_1 = A_2$ and $c_1 = c_2$ (the two curves coincide). □

Remark 2. *If the curve $\mathbf{x}$ and its involute $\mathbf{y}$ are both rectifying curves, they cannot have the same reference point.*

Indeed, if we suppose that $\mathbf{x}_0$ is the reference point of both curves, they could be written:

$$\mathbf{x} = \mathbf{x}_0 + (s+\nu)\mathbf{t} + \mu\mathbf{b}, \tag{41}$$

$$\mathbf{y} = \mathbf{x}_0 + (\tilde{s}+\tilde{\nu})\tilde{\mathbf{t}} + \tilde{\mu}\tilde{\mathbf{b}} = \mathbf{x} + (c-s)\mathbf{t}, \tag{42}$$

where $\tilde{s}$ is the arc-length parameter of the curve $\mathbf{y}$, and $\tilde{\mu}$, $\tilde{\nu}$ are constants. Thus, we could write:

$$(\tilde{s}+\tilde{\nu})\tilde{\mathbf{t}} + \tilde{\mu}\tilde{\mathbf{b}} = (c+\nu)\mathbf{t} + \mu\mathbf{b}. \tag{43}$$

However, since $\tilde{\mathbf{t}} = \mathbf{n}$, the equation above implies that $\tilde{s} + \tilde{\nu} = 0$, which is not possible.

5. The Evolute of a Rectifying Curve

Recall that an evolute of a curve $\mathbf{y}$ is a curve $\mathbf{x}$ with the property that $\mathbf{y}$ is an involute of $\mathbf{x}$.

Let $\mathbf{y} = \mathbf{y}(s), s \in I$, be a unit speed curve with the Frenet frame $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$, the curvature $\tilde{\kappa}$ and the torsion $\tilde{\tau}$. If $\mathbf{x}$ is an evolute of $\mathbf{y}$, then the vector $\mathbf{x} - \mathbf{y}$ is orthogonal to $\tilde{\mathbf{t}}$, so

$$\mathbf{x} - \mathbf{y} = \lambda \tilde{\mathbf{n}} + \theta \tilde{\mathbf{b}}, \tag{44}$$

where $\lambda = \lambda(s)$ and $\theta = \theta(s)$ are two functions such that, for any $s \in I$, the vector $\mathbf{x}'$ is collinear to $\mathbf{x} - \mathbf{y}$. We differentiate (44) with respect to s, the arc-length parameter of the curve $\mathbf{y}$:

$$\frac{d\mathbf{x}}{ds} - \tilde{\mathbf{t}} = \lambda' \tilde{\mathbf{n}} + \lambda(-\tilde{\kappa}\tilde{\mathbf{t}} + \tilde{\tau}\tilde{\mathbf{b}}) + \theta' \tilde{\mathbf{b}} - \theta\tilde{\tau}\tilde{\mathbf{n}}. \tag{45}$$

Thus, the vector $\frac{d\mathbf{x}}{ds}$ has the expression:

$$\frac{d\mathbf{x}}{ds} = (1 - \lambda\tilde{\kappa})\tilde{\mathbf{t}} + (\lambda' - \theta\tilde{\tau})\tilde{\mathbf{n}} + (\theta' + \lambda\tilde{\tau})\tilde{\mathbf{b}}. \tag{46}$$

Since $\mathbf{x} - \mathbf{y}$ and $\frac{d\mathbf{x}}{ds}$ are collinear vectors, by (44) and (46), we obtain:

$$\lambda = \frac{1}{\tilde{\kappa}} \tag{47}$$

and

$$\frac{\lambda' - \theta\tilde{\tau}}{\lambda} = \frac{\theta' + \lambda\tilde{\tau}}{\theta}. \tag{48}$$

From the last equation we have

$$\theta'\lambda - \theta\lambda' = -(\lambda^2 + \theta^2)\tilde{\tau}, \tag{49}$$

so

$$\frac{\left(\frac{\theta}{\lambda}\right)'}{1 + \left(\frac{\theta}{\lambda}\right)^2} = -\tilde{\tau}. \tag{50}$$

We integrate this equation and, using (47), we find:

$$\arctan \tilde{\kappa}\theta = -T(s), \tag{51}$$

where $T(s)$ is a primitive of the torsion $\tilde{\tau}(s)$. Thus, the expression of $\theta(s)$ is:

$$\theta = -\frac{1}{\tilde{\kappa}} \tan T, \tag{52}$$

so any evolute of the curve $\mathbf{y}$ has the expression:

$$\mathbf{x} = \mathbf{y} + \frac{1}{\tilde{\kappa}}\big(\tilde{\mathbf{n}} - \tan T\tilde{\mathbf{b}}\big). \tag{53}$$

Remark 3. *By Theorem 4 and Remark 1, if the evolute $\mathbf{x}$ of a curve $\mathbf{y}$ is a rectifying curve, then $\mathbf{y}$ must be a non-planar curve that lies on a sphere centered at the reference point of $\mathbf{x}$. We prove in the next theorem that any evolute of any spherical non-planar curve is a rectifying curve.*

Theorem 6. *The evolutes of a spherical non-planar curve are rectifying curves with the reference point at the center of the sphere.*

Proof. Suppose that $\mathbf{y} = \mathbf{y}(s)$ is a unit speed curve lying on a sphere centered at c_0. If $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$ is the Frenet frame, $\tilde{\kappa}$ is the curvature and $\tilde{\tau}$ is the torsion of $\mathbf{y}$, then, by Equations (11), (14) and (15), we can write:

$$\mathbf{y} - c_0 = \delta\tilde{\mathbf{n}} + \eta\tilde{\mathbf{b}}, \tag{54}$$

where $\delta = -\frac{1}{\tilde{\kappa}}$ and $\eta = \frac{\delta'}{\tilde{\tau}} = \frac{\tilde{\kappa}'}{\tilde{\kappa}^2\tilde{\tau}}$. Thus, we have:

$$\mathbf{y} = c_0 - \frac{1}{\tilde{\kappa}}\tilde{\mathbf{n}} + \frac{\tilde{\kappa}'}{\tilde{\kappa}^2\tilde{\tau}}\tilde{\mathbf{b}}. \tag{55}$$

Let $\mathbf{x}$ be an evolute of $\mathbf{y}$. Then, by Equation (53), one can write:

$$\mathbf{x} = c_0 + \frac{1}{\tilde{\kappa}}\left(\frac{\tilde{\kappa}'}{\tilde{\kappa}\tilde{\tau}} - \tan T\right)\tilde{\mathbf{b}}, \tag{56}$$

where $T = T(s)$ is a primitive of the torsion $\tilde{\tau}(s)$.

Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ denote the unit vectors of the Frenet frame and κ, τ denote the curvature and the torsion of the curve $\mathbf{x}$. By Formula (31), since $\mathbf{y}$ is an involute of $\mathbf{x}$, we have:

$$\tilde{\mathbf{b}} = \frac{\frac{\tau}{\kappa}}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}}\mathbf{t} + \frac{1}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}}\mathbf{b}, \tag{57}$$

and so Equation (56) is written:

$$\mathbf{x} - c_0 = \frac{1}{\tilde{\kappa}}\left(\frac{\tilde{\kappa}'}{\tilde{\kappa}\tilde{\tau}} - \tan T\right)\left(\frac{\frac{\tau}{\kappa}}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}}\mathbf{t} + \frac{1}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}}\mathbf{b}\right) = \lambda\mathbf{t} + \theta\mathbf{b}. \tag{58}$$

Hence, $\mathbf{x}$ is a rectifying curve with the reference point c_0. □

Following on from Theorems 4 and 6, the next corollary establishes a new defining feature of rectifying curves:

Corollary 2. *A curve* $\mathbf{x}$ *is a rectifying curve if and only if it has a spherical involute. In this case, the reference point of* $\mathbf{x}$ *is the center of the sphere and all of the involutes of* $\mathbf{x}$ *lie on concentric spheres centered at this point.*

6. Construction of Rectifying Curves

To date, there exist two main sources for the construction of rectifying curves: as dilations of unit speed, spherical curves (see Theorem 7), or as (dilated) centrodes of some type of curves (see Theorems 8 and 9).

Theorem 7 ([3,5])**.** *If* $\mathbf{y}(t)$ *is a unit speed curve that lies on the unit sphere* S^2 *and is not an arc of a great circle, then the curve is defined by*

$$\mathbf{x}(t) = a\sec(t + t_0)\mathbf{y}(t), \tag{59}$$

where $a > 0$*, is a rectifying curve.*

Theorem 8 ([5])**.** *If* $\mathbf{y}(t)$ *is a unit speed curve that is neither a planar curve nor a helix, such that the curvature* κ *and torsion* τ *are not both constants and satisfy a nonhomogeneous linear equation of the form* $a\kappa - b\tau = c$*, where* $a^2 + b^2 \neq 0$ *and* $c \neq 0$*, then the centrode of* $\mathbf{y}$*,*

$$\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}, \tag{60}$$

is a rectifying curve.

Theorem 9 ([5])**.** *If* $\mathbf{y}(t)$ *is a unit speed curve that is neither a planar curve nor a helix, then the dilated centrode*

$$\bar{\mathbf{d}} = \frac{1}{\kappa}\mathbf{d} = \frac{\tau}{\kappa}\mathbf{t} + \mathbf{b} \tag{61}$$

is a rectifying curve.

By Theorem 6, we introduce a new method for constructing rectifying curves: as evolutes of spherical twisted curves. We illustrate this method with the following example:

Example 1. *Consider the spherical helix* $\mathbf{y}$ *that lies on the unit sphere* S^2*:*

$$\mathbf{y} = \left(\cos t \cos at + \frac{1}{a}\sin t \sin at, \cos t \sin at - \frac{1}{a}\sin t \cos at, \frac{\sqrt{a^2-1}}{a}\sin t\right), \tag{62}$$

$t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $a > 1$.

The unit vectors of the Frenet frame of the curve $\mathbf{y}$ *are:*

$$\begin{aligned} \tilde{\mathbf{t}} &= \left(-\frac{\sqrt{a^2-1}}{a}\sin at, \frac{\sqrt{a^2-1}}{a}\cos at, \frac{1}{a}\right), \\ \tilde{\mathbf{n}} &= (-\cos at, -\sin at, 0), \\ \tilde{\mathbf{b}} &= \left(\frac{1}{a}\sin at, -\frac{1}{a}\cos at, \frac{\sqrt{a^2-1}}{a}\right), \end{aligned} \tag{63}$$

and the curvature $\tilde{\kappa}$ *and the torsion* $\tilde{\tau}$ *are given by the formulas:*

$$\tilde{\kappa} = \frac{1}{\cos t}, \quad \tilde{\tau} = \frac{1}{\sqrt{a^2-1}\cos t}. \tag{64}$$

Let s denote the arc-length parameter of $\mathbf{y}$*:*

$$s(t) = \int_{-\frac{\pi}{2}}^{t} \|\mathbf{y}'(\sigma)\| d\sigma = \sqrt{a^2-1}(1+\sin t), \ \frac{ds}{dt} = \sqrt{a^2-1}\cos t. \tag{65}$$

We use the Formula (53) to construct an evolute of $\mathbf{y}$*, which is, by Theorem 6, a rectifying curve. Let* $T(s)$ *be a primitive of the torsion* $\tilde{\tau}(s)$*,* $T(s) = \int \tilde{\tau}(s)ds$*. We have:*

$$T(s(t)) = \int \tilde{\tau}(s(t))\frac{ds}{dt}dt = \int \frac{1}{\sqrt{a^2-1}\cos t}\sqrt{a^2-1}\cos t dt = t + C, \tag{66}$$

where C is a constant. Thus, the following family of rectifying curves is obtained:

$$\mathbf{x}(t) = \left(-\frac{\sin C \sin at}{a\cos(t+C)}, \frac{\sin C \cos at}{a\cos(t+C)}, -\frac{\sin C\sqrt{a^2-1}}{a\cos(t+C)}\right). \tag{67}$$

We remark that any rectifying curve defined by (67) can be written as a dilation of a unit speed circle on the sphere S^2*,*

$$\mathbf{z}(t) = \left(-\frac{1}{a}\sin at, \frac{1}{a}\cos at, -\frac{\sqrt{a^2-1}}{a}\right). \tag{68}$$

Thus, if we denote $\sin C = c$*, we can write:*

$$\mathbf{x}(t) = c\sec(t+C)\mathbf{z}(t). \tag{69}$$

We also note that all of the rectifying curves defined by (67) lie on the cone of equation

$$x^2 + y^2 = \frac{z^2}{a^2-1}. \tag{70}$$

Moreover, as Chen proved in [4], they are geodesics on this cone.

In Figure 1, we represented the spherical helix $\mathbf{y}$ *defined by* (62) *(for* $a = 6$*) and three rectifying curves (evolutes of* $\mathbf{y}$ *given by the Formula* (67)*, obtained for* $C = 2$*,* $C = 2.2$ *and* $C = 2.4$*, respectively).*

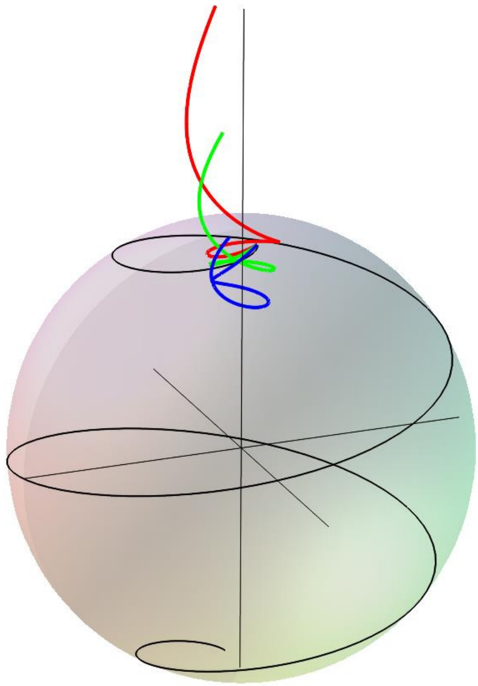

Figure 1. Three rectifying curves constructed from a spherical curve: a spherical helix and three evolutes of it.

7. Conclusions

In this paper, we state and prove some new properties of rectifying curves based on the relationship with their involutes/evolutes. Thus, we express the curvature and the torsion of a rectifying spherical curve and give necessary and sufficient conditions for a curve and its involute to be both rectifying curves.

We also give a new characterization of rectifying curves: we prove that a necessary and sufficient condition for a curve to be a rectifying curve is for it to posses a spherical involute. Consequently, rectifying curves can be constructed as evolutes of spherical twisted curves. We apply this method to construct the rectifying curves obtained as the evolutes of a spherical helix.

Author Contributions: Conceptualization, M.J., S.A. and A.M.; methodology, M.J., S.A., L.D., A.M. and D.T.; validation, L.D., O.-A.R. and D.T.; writing—original draft preparation, M.J.; writing—review and editing, M.J., S.A., L.D., A.M., O.-A.R. and D.T.; supervision, A.M.; project administration, A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the internal competition in the Technical University of Civil Engineering Bucharest, based on special funding for academic research development accorded by the Romanian Ministry of Education (UTCB-CDI-2021-012).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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