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Oscillation Criteria for Third-Order Nonlinear Neutral Dynamic Equations with Mixed Deviating Arguments on Time Scales

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Abstract: Under a couple of canonical and mixed canonical-noncanonical conditions, we investigate the oscillation and asymptotic behavior of solutions to a class of third-order nonlinear neutral dynamic equations with mixed deviating arguments on time scales. By means of the double Riccati transformation and the inequality technique, new oscillation criteria are established, which improve and generalize related results in the literature. Several examples are given to illustrate the main results.

Keywords: time scale; oscillation criterion; third-order neutral dynamic equation; mixed deviating argument; mixed canonical-noncanonical condition; double Riccati transformation

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1. Introduction

The theory of time scales provides a powerful tool for unifying and extending the knowledge about continuous and discrete systems, which has attracted the attention of many scholars in recent years, see the monographs [1,2] for the essentials about the subject. In particular, the research on the oscillation and asymptotic behavior of solutions to different types of differential equations and dynamic equations has been a topic of interest in the past two decades, see, for instance, Refs. [3–32] and the references cited therein.

Following this trend, in this paper, we are concerned with the oscillation and the asymptotic behavior of solutions to third-order nonlinear neutral functional dynamic equations with mixed deviating arguments of the form

$$(b(t)((a(t)(z(t))^{\Delta})^{\Delta})^{\alpha})^{\Delta} + f(t, x(\delta(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{1}$$

on a time scale $\mathbb{T}$, where

$$z(t) = x(t) + p(t)x(\tau(t)).$$

Throughout this paper, we assume that the following hypotheses are fulfilled:

(A_1) $a(t), b(t), p(t) \in C_{rd}(\mathbb{T}, \mathbb{R})$, $0 \le p(t) \le p_0 < \infty$;
(A_2) $\alpha > 0$ is a quotient of odd positive integers;
(A_3) $\delta(t), \tau(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ such that $\tau(t) \le t$, $\delta(t) \ge \tau(t)$, $\tau^{\Delta}(t) \ge \tau_0 > 0$, $\delta^{\Delta}(t) > 0$;
(A_4) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ and there exists a function $q(t) \in C^1_{rd}(\mathbb{T}, [0, \infty))$ such that $f(t, u)/u^{\alpha} \ge q(t) \not\equiv 0$ for all $u \ne 0$ and $t \in [t_1, \infty)_{\mathbb{T}}$, $t_1 \ge t_0$.

Since we are interested in the oscillatory and asymptotic behavior of solutions, we assume that the given time scale $\mathbb{T}$ is unbounded from above. By a solution of (1), we understand a nontrivial function $x \in C^1_{rd}([T_a, \infty)_{\mathbb{T}}, \mathbb{R})$ with $T_a \in [t_0, \infty)_{\mathbb{T}}$, which has the property and satisfies (1) on $[T_a, \infty)_{\mathbb{T}}$.

We restrict our attention to only those solutions of (1) which exist on some half-line $[T_b, \infty)_{\mathbb{T}}$ and satisfy the condition $\sup\{|x(t)| :\in [T_c, \infty)_{\mathbb{T}}\} > 0$ for any $T_c \in [T_b, \infty)_{\mathbb{T}}$. We tacitly assume that (1) admits such a solution. A solution of Equation (1) is said to be oscillatory, if it is neither eventually positive nor eventually negative. Otherwise, it is called non-oscillatory. Equation (1) is said to be oscillatory, if all its solutions are oscillatory. Otherwise, it is called non-oscillatory.

Equation (1) can be seen as a natural generalization of the half-linear differential equation, whose applications range over many science and technology areas (such as in the investigations of non-Newtonian fluid theory and properties of solutions to porous medium problems); see, e.g., the papers [7,33] for more details.

Below, we mention several existing results for particular cases of (1) or its close generalizations investigated in the literature that inspired our research:

(i) $\alpha = 1, b(t) = a(t) = 1, p(t) = 0, \delta(t) = t$: L. Erbe et al. [12];
(ii) $b(t) = a(t) = 1, p(t) = 0, \delta(t) \leq t$: Z. Han et al. [14];
(iii) $p(t) = 0, \delta(t) \leq t$: R.P. Agarwal et al. [3,4];
(iv) $\alpha = 1, p(t) = 0, \delta(t) \leq t$ and $f(t, x(\delta(t))) = q(t)x(\delta(t))$: R.P. Agarwal et al. [5];
(v) $\mathbb{T} = \mathbb{R}$, $b(t) = a(t) = 1$, $\alpha = 1$, $\delta(t) \leq t$ and $f(t, x(\delta(t))) \geq q(t)x^{\lambda}(g(t))$, where $\lambda > 0$ is the ratio of odd positive integers, $p(t) \geq 1$, $p(t) \not\equiv 1$ is unbounded: G. E. Chatzarakis et al. [10];
(vi) $\mathbb{T} = \mathbb{R}$, $a(t) = 1$, $f(t, x(\delta(t))) = q(t)x^{\alpha}(\delta(t))$, $\tau(t) = t - \tau$, $\tau > 0$ or $\tau < 0$, $p(t) = p_0 \geq 0$: T. Li and Yu. V. Rogovchenko [17];
(vii) $\mathbb{T} = \mathbb{R}, a(t) = 1, f(t, x(\delta(t))) = q(t)x^{\alpha}(\delta(t)), \tau(t) \leq \delta(t), \tau \circ \delta = \delta \circ \tau$: E. Thandapani and T. Li [21];
(viii) $\mathbb{T} = \mathbb{R}, \alpha \geq 1, a(t) = 1, \tau(t) \leq t, \delta(t) \leq t, \tau \circ \delta = \delta \circ \tau$: Y. Jiang et al. [15];
(ix) $0 \leq p(t) \leq p_0 < 1$; $\tau(t) < t$, $\delta(t) > t$ or $\delta(t) < t$: S. H. Saker [22,23], S. H. Saker and J. R. Graef [24].

In addition, B. Baculikova and J. Dzurina [6], T. Li et al. [18,19], T. Candan [8,9], Y. Wang and Z. Xu [25], Y. Wang et al. [26] T. Li et al. [27,28], Z. Zhang et al. [13,29–32], and other scholars have done a lot of results for the oscillatory behavior of various classes of third order functional dynamic equations and differential equations.

One of the most used classification rules for these results depends on which of the following conditions:

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \infty, \quad \int_{t_0}^{\infty} \left(\frac{1}{b(t)}\right)^{\frac{1}{\alpha}} \Delta t = \infty, \tag{2}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \infty, \quad \int_{t_0}^{\infty} \left(\frac{1}{b(t)}\right)^{\frac{1}{\alpha}} \Delta t < \infty, \tag{3}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t < \infty, \quad \int_{t_0}^{\infty} \left(\frac{1}{b(t)}\right)^{\frac{1}{\alpha}} \Delta t = \infty, \tag{4}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t < \infty, \quad \int_{t_0}^{\infty} \left(\frac{1}{b(t)}\right)^{\frac{1}{\alpha}} \Delta t < \infty, \tag{5}$$

is assumed to be satisfied. Conditions (2) are termed double canonical, conditions (3) and (4) are named mixed canonical-noncanonical and mixed noncanonical-canonical, respectively, and conditions (5) are termed double noncanonical. In fact, a majority of the related research (e.g., the results from references stated in (i)–(viii)) were given under the double canonical condition. In (ix), the authors investigated the equation under either the double canonical or the double noncanonical conditions.

To the best of our knowledge, nothing is known regarding the oscillation of (1) when $0 \leq p(t) \leq p_0 < \infty$, and mixed canonical-noncanonical conditions hold. The natural question now arises regarding whether it is possible to find new oscillation conditions which improve the results established in the above-mentioned papers and can be applied in the above case. One of our aims in this paper is to give an affirmative answer to this question and establish some sufficient conditions for oscillation by employing the Riccati substitution and the appropriate inequality technique.

The main results in this paper are organized into two parts in accordance with different assumptions on the coefficients $a(t)$ and $b(t)$. In Section 2.1, under the double canonical conditions (2), oscillation results for Equation (1) are established for $\delta(t) \geq \tau(t)$, $\delta(t) \geq t$ and $\delta(t) \geq \sigma(t)$, respectively. In Section 2.2, under the mixed canonical-noncanonical conditions (3), oscillation results for Equation (1) are given also for $\delta(t) \geq \tau(t)$, $\delta(t) \geq t$ and $\delta(t) \geq \sigma(t)$, respectively. In Section 3, we give three examples to illustrate the results reported in Sections 2.1 and 2.2.

2. Main Results

As usual, we assume that all functional inequalities hold for all t large enough. In the sequel, we adopt the following notation for a compact presentation of our results:

$$d_+(t) = \max\{0, d(t)\},\ \ Q(t) = \min\{q(t), q(\tau(t))\},\ \ \mu(t) = \sigma(t) - t,\ \ A(t_0, t) = \int_{t_0}^{t} \frac{1}{a(t)} \Delta t,$$

$$B(t_0, t) = \int_{t_0}^{t} \left(\frac{1}{b(t)}\right)^{\frac{1}{\alpha}} \Delta t,\ B(t) = \int_{t}^{\infty} \left(\frac{1}{b(t)}\right)^{\frac{1}{\alpha}} \Delta t,\ G(t_1, t) = \frac{b^{\frac{1}{\alpha}}(t) B(t_1, t) A(t, \delta(t))}{b^{\frac{1}{\alpha}}(t) B(t_1, t) + \mu(t)}.$$

We start by recalling an auxiliary result which will be useful in the proofs of our main results.

Lemma 1 (see [21] (Lemma 1, Lemma 2))**.** *Let $X_1 > 0$ and $X_2 > 0$. Then,*

$$X_1^{\alpha} + X_2^{\alpha} \geq \begin{cases} 2^{1-\alpha}(X_1 + X_2)^{\alpha}, & \alpha \geq 1, \\ (X_1 + X_2)^{\alpha}, & 0 < \alpha < 1. \end{cases} \tag{6}$$

2.1. Oscillation Results for (1) When (2) Holds

In this section, we investigate (1) under the double canonical conditions (2).

Theorem 1. *Assume that $\alpha \geq 1$, $\delta(t) \geq t$, (2) and*

$$\int_{t_0}^{\infty} \frac{1}{a(\tau(s))} \int_{s}^{\infty} \left(\frac{1}{b(\tau(v))} \int_{v}^{\infty} Q(u) \Delta u\right)^{\frac{1}{\alpha}} \Delta v \Delta s = \infty \tag{7}$$

hold. If a positive function $\phi(t) \in C_{rd}^{1}(\mathbb{T}, \mathbb{R})$ and $t_1 \geq t_0$ exist, such that

$$\limsup_{t \to \infty} \int_{t_1}^{t} \left[2^{1-\alpha} \phi(s) Q(s) G^{\alpha}(t_1, s) - \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right) \frac{b(s)(\phi^{\Delta}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\phi(s))^{\alpha}}\right] \Delta s = \infty, \tag{8}$$

then every non-oscillatory solution $x(t)$ of Equation (1) tends to zero as $t \to \infty$.

Proof. Assume that $x(t)$ is a non-oscillatory solution of Equation (1). Without loss of generality, one can pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0, x(\delta(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Then, the corresponding function $z(t)$ satisfies one of two possible cases:

case (I): $z(t) > 0, z^{\Delta}(t) > 0, (a(t) z^{\Delta}(t))^{\Delta} > 0, (b(t)((a(t) z^{\Delta}(t))^{\Delta})^{\gamma})^{\Delta} < 0;$

case (II): $z(t) > 0, z^{\Delta}(t) < 0, (a(t) z^{\Delta}(t))^{\Delta} > 0, (b(t)((a(t) z^{\Delta}(t))^{\Delta})^{\gamma})^{\Delta} < 0.$

Assume first that case (I) holds. By virtue of Equation (1) and condition (A_4), we get

$$(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}+q(t)x^{\alpha}(\delta(t))\leq 0. \tag{9}$$

Since $(b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha})^{\Delta}=(b((az^{\Delta})^{\Delta})^{\alpha})^{\Delta}(\tau)\tau^{\Delta}(t)$, there exists a $t_2\geq t_1$, such that

$$p_0^{\alpha}\frac{(b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha})^{\Delta}}{\tau^{\Delta}(t)}+p_0^{\alpha}q(\tau(t))x^{\alpha}(\delta(\tau(t)))\leq 0,\ \text{ for } t\geq t_2.$$

In addition, from the condition $\tau^{\Delta}(t)\geq\tau_0\geq 0$, the above inequality can be reduced to

$$p_0^{\alpha}\frac{(b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha})^{\Delta}}{\tau_0}+p_0^{\alpha}q(\tau(t))x^{\alpha}(\delta(\tau(t)))\leq 0,\ \text{ for } t\geq t_2. \tag{10}$$

From (9) and (10), we have

$$\begin{aligned}(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}&+p_0^{\alpha}\frac{(b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha})^{\Delta}}{\tau_0}\\ &\leq -q(t)x^{\alpha}(\delta(t))-p_0^{\alpha}q(\tau(t))x^{\alpha}(\delta(\tau(t)))\\ &\leq -\min\{q(t),q(\tau(t))\}(x^{\alpha}(\delta(t))+p_0^{\alpha}x^{\alpha}(\delta(\tau(t))))\\ &= -Q(t)(x^{\alpha}(\delta(t))+p_0^{\alpha}x^{\alpha}(\tau(\delta(t)))),\ \text{ for } t\geq t_2.\end{aligned} \tag{11}$$

Using the first inequality in (6) and the definition of $z(t)$, we get

$$x^{\alpha}(\delta(t))+p_0^{\alpha}x^{\alpha}(\tau(\delta(t)))\geq 2^{1-\alpha}(x(\delta(t))+p_0x(\tau(\delta(t))))^{\alpha}\geq 2^{1-\alpha}z^{\alpha}(\delta(t)). \tag{12}$$

Substituting the above inequality into (11), we obtain

$$(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}+\frac{p_0^{\alpha}}{\tau_0}(b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha})^{\Delta}\leq -2^{1-\alpha}Q(t)z^{\alpha}(\delta(t)),\ \text{ for } t\geq t_2. \tag{13}$$

Now, we define the function $\omega(t)$ by the generalized Riccati substitution

$$\omega(t)=\frac{b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}}{(a(t)z^{\Delta}(t))^{\alpha}},\ \text{ for } t\geq t_2. \tag{14}$$

Obviously, we see that $\omega(t)>0$ and, by the product rule and the quotient rule on time scales,

$$\omega^{\Delta}(t)=\frac{(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}}-\frac{b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}((a(t)z^{\Delta}(t))^{\alpha})^{\Delta}}{(a(t)z^{\Delta}(t))^{\alpha}(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}}. \tag{15}$$

By the Pötzsche chain rule (see [1] (Theorem 1.90)), when $\alpha\geq 1$, we have

$$((a(t)z^{\Delta}(t))^{\alpha})^{\Delta}\geq\alpha(a(t)z^{\Delta}(t))^{\alpha-1}(a(t)z^{\Delta}(t))^{\Delta}. \tag{16}$$

Substituting the above inequality into (15), we obtain

$$\omega^{\Delta}(t)\leq\frac{(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}}-\alpha\frac{b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}(a(t)z^{\Delta}(t))^{\Delta}}{a(t)z^{\Delta}(t)(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}}. \tag{17}$$

Since $(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}<0$ and $(a(t)z^{\Delta}(t))^{\Delta}>0$, then

$$\begin{aligned}b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}&\geq b(\sigma(t))((a(\sigma(t))z^{\Delta}(\sigma(t)))^{\Delta})^{\alpha},\\ a(t)z^{\Delta}(t)&\leq a(\sigma(t))z^{\Delta}(\sigma(t)),\ \text{for } t\geq t_2,\ \text{ for } t\geq t_2.\end{aligned} \tag{18}$$

In addition, by (17), (18) and the definition (14) of $\omega(t)$, we can obtain

$$\omega^{\Delta}(t) \leq \frac{(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} - \alpha b^{-\frac{1}{\alpha}}(t)(\omega(\sigma(t)))^{1+\frac{1}{\alpha}}. \tag{19}$$

Furthermore, we define another function $v(t)$, by the generalized Riccati substitution

$$v(t) = \frac{b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha}}{(a(t)z^{\Delta}(t))^{\alpha}}, \ \text{for } t \geq t_2. \tag{20}$$

It is clear that $v(t) > 0$ and $\tau(t) < \sigma(t)$. Similarly to the process of proving (19), we can easily get

$$v^{\Delta}(t) \leq \frac{(b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} - \alpha b^{-\frac{1}{\alpha}}(t)(v(\sigma(t)))^{1+\frac{1}{\alpha}}. \tag{21}$$

Combining (13), (19) and (21), we have

$$\begin{aligned} \omega^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}v^{\Delta}(t) \leq & -2^{1-\alpha}Q(t)\frac{z^{\alpha}(\delta(t))}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} - \alpha b^{-\frac{1}{\alpha}}(t)(\omega(\sigma(t)))^{1+\frac{1}{\alpha}} \\ & - \frac{\alpha p_0^{\alpha}}{\tau_0}b^{-\frac{1}{\alpha}}(t)(v(\sigma(t)))^{1+\frac{1}{\alpha}}, \ \text{for } t \geq t_2. \end{aligned} \tag{22}$$

Since $(a(t)z^{\Delta}(t))^{\Delta} > 0$ and $\delta(t) \geq t$, then

$$z(\delta(t)) - z(t) = \int_t^{\delta(t)} \frac{a(s)z^{\Delta}(s)}{a(s)}\Delta s \geq a(t)z^{\Delta}(t)\int_t^{\delta(t)} \frac{1}{a(s)}\Delta s,$$

that is,

$$z(\delta(t)) \geq a(t)z^{\Delta}(t)A(t,\delta(t)), \ \text{for } t \geq t_2. \tag{23}$$

Hence,

$$\frac{z(\delta(t))}{a(\sigma(t))z^{\Delta}(\sigma(t))} \geq \frac{a(t)z^{\Delta}(t)}{a(\sigma(t))z^{\Delta}(\sigma(t))} \cdot A(t,\delta(t)), \ \text{for } t \geq t_2. \tag{24}$$

Using that $(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta} < 0$, we get

$$a(t)z^{\Delta}(t) - a(t_1)z^{\Delta}(t_1) = \int_{t_1}^{t} \frac{b^{\frac{1}{\alpha}}(s)(a(s)z^{\Delta}(s))^{\Delta}}{b^{\frac{1}{\alpha}}(s)}\Delta s \geq b^{\frac{1}{\alpha}}(t)(a(t)z^{\Delta}(t))^{\Delta}\int_{t_1}^{t} \frac{1}{b^{\frac{1}{\alpha}}(s)}\Delta s,$$

which implies that

$$(a(t)z^{\Delta}(t))^{\Delta} \leq \frac{a(t)z^{\Delta}(t)}{b^{\frac{1}{\alpha}}(t)B(t_1,t)}, \ \text{for } t \geq t_2. \tag{25}$$

In addition, since $a(\sigma(t))z^{\Delta}(\sigma(t)) = a(t)z^{\Delta}(t) + \mu(t)(a(t)z^{\Delta}(t))^{\Delta}$ and (25), we have

$$\frac{a(\sigma(t))z^{\Delta}(\sigma(t))}{a(t)z^{\Delta}(t)} = 1 + \mu(t)\frac{(a(t)z^{\Delta}(t))^{\Delta}}{a(t)z^{\Delta}(t)} \leq 1 + \frac{\mu(t)}{b^{\frac{1}{\alpha}}(t)B(t_1,t)}, \tag{26}$$

which also implies that

$$\frac{a(t)z^{\Delta}(t)}{a(\sigma(t))z^{\Delta}(\sigma(t))} \geq \frac{b^{\frac{1}{\alpha}}(t)B(t_1,t)}{b^{\frac{1}{\alpha}}(t)B(t_1,t) + \mu(t)}, \ \text{for } t \geq t_2. \tag{27}$$

Thus, combining (24) and (27) yields

$$\frac{z(\delta(t))}{(a(\sigma(t))z^{\Delta}(\sigma(t)))} \geq \frac{b^{\frac{1}{\alpha}}(t)B(t_1,t)A(t,\delta(t))}{b^{\frac{1}{\alpha}}(t)B(t_1,t)+\mu(t)} =: G(t_1,t). \tag{28}$$

Substituting (28) into (22), we conclude that

$$\begin{aligned} \omega^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}v^{\Delta}(t) \leq &-2^{1-\alpha}Q(t)G^{\alpha}(t_1,t) - \alpha b^{-\frac{1}{\alpha}}(t)(\omega(\sigma(t)))^{1+\frac{1}{\alpha}} \\ &- \frac{\alpha p_0^{\alpha}}{\tau_0}b^{-\frac{1}{\alpha}}(t)(v(\sigma(t)))^{1+\frac{1}{\alpha}}, \quad \text{for } t \geq t_2. \end{aligned} \tag{29}$$

Multiplying both sides of (29) by $\phi(t)$ and replacing t with s, integrating with respect to s from t_2 to $t > t_2$, we have

$$\begin{aligned} \int_{t_2}^{t} 2^{1-\alpha}Q(s)\phi(s)G^{\alpha}(t_1,s)\Delta s \leq &\int_{t_2}^{t} -\left[\omega^{\Delta}(s)\phi(s) + \alpha b^{-\frac{1}{\alpha}}(s)\phi(s)(\omega(\sigma(s)))^{1+\frac{1}{\alpha}}\right] \\ &-\frac{p_0^{\alpha}}{\tau_0}\left[v^{\Delta}(s)\phi(s) + \alpha b^{-\frac{1}{\alpha}}(s)\phi(s)(v(\sigma(s)))^{1+\frac{1}{\alpha}}\right]\Delta s. \end{aligned} \tag{30}$$

Using integration by parts, (30) yields

$$\begin{aligned} \int_{t_2}^{t} 2^{1-\alpha}Q(s)\phi(s)G^{\alpha}(t_1,s)\Delta s \leq &\phi(t_2)\omega(t_2) - \phi(t)\omega(t) + \frac{p_0^{\alpha}}{\tau_0}(\phi(t_2)v(t_2) - \phi(t)v(t)) \\ &+ \int_{t_2}^{t}\left[(\phi^{\Delta}(s))_{+}\omega(\sigma(s)) - \alpha b^{-\frac{1}{\alpha}}(s)\phi(s)(\omega(\sigma(s)))^{1+\frac{1}{\alpha}}\right]\Delta s \\ &+ \frac{p_0^{\alpha}}{\tau_0}\int_{t_2}^{t}\left[(\phi^{\Delta}(s))_{+}v(\sigma(s)) - \alpha b^{-\frac{1}{\alpha}}(s)\phi(s)(v(\sigma(s)))^{\frac{\alpha+1}{\alpha}}\right]\Delta s. \end{aligned} \tag{31}$$

Let

$$g(\nu) = (\phi^{\Delta}(s))_{+}\nu - \alpha b^{-\frac{1}{\alpha}}(s)\phi(s)\nu^{\frac{\alpha+1}{\alpha}}. \tag{32}$$

It is easy to verify that $\Phi(\nu)$ reaches its maximum value on $[0,\infty)$ at

$$\nu_0 = \frac{b(s)}{(\alpha+1)^{\alpha}}\left(\frac{(\phi^{\Delta}(s))_{+}}{\phi(s)}\right)^{\alpha},$$

see [32] (Lemma 2.4), and so

$$\begin{aligned} g(\nu) &\leq g(\nu_0) \\ &= (\phi^{\Delta}(s))_{+}\frac{b(s)}{(\alpha+1)^{\alpha}}\left(\frac{(\phi^{\Delta}(s))_{+}}{\phi(s)}\right)^{\alpha} - \alpha b^{-\frac{1}{\alpha}}(s)\phi(s)\left(\frac{b(s)}{(\alpha+1)^{\alpha}}\left(\frac{(\phi^{\Delta}(s))_{+}}{\phi(s)}\right)^{\alpha}\right)^{\frac{\alpha+1}{\alpha}} \\ &= \frac{1}{(\alpha+1)^{\alpha+1}}\frac{b(s)(\phi^{\Delta}(s))^{\alpha+1}}{(\phi(s))^{\alpha}}. \end{aligned}$$

Substituting the above inequality into (31), we have

$$\begin{aligned} &\int_{t_2}^{t} 2^{1-\alpha}Q(s)\phi(s)G^{\alpha}(t_1,s)\Delta s \leq \phi(t_2)\omega(t_2) - \phi(t)\omega(t) \\ &+\frac{p_0^{\alpha}}{\tau_0}\phi(t_2)v(t_2) - \phi(t)v(t) + \left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\int_{t_2}^{t}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{b(s)(\phi^{\Delta}(s))^{\alpha+1}}{(\phi(s))^{\alpha}}\Delta s, \end{aligned} \tag{33}$$

that is,

$$\begin{aligned}
&\int_{t_2}^{t}\left[2^{1-\alpha}Q(s)\phi(s)G^{\alpha}(t_1,s)-\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{b(s)(\phi^{\Delta}(s))^{\alpha+1}}{(\phi(s))^{\alpha}}\right]\Delta s \\
&\le \phi(t_2)\omega(t_2)-\phi(t)\omega(t)+\frac{p_0^{\alpha}}{\tau_0}\phi(t_2)v(t_2)-\phi(t)v(t) \\
&\le \phi(t_2)\omega(t_2)+\frac{p_0^{\alpha}}{\tau_0}\phi(t_2)v(t_2),\ \text{ for } t\ge t_2.
\end{aligned} \tag{34}$$

Taking the limsup on both sides of (34) as $t\to\infty$ contradicts (8).

If case (II) holds, by $z(t)>0$ and $z^{\Delta}(t)<0$, we know that $z(t)$ is decreasing and $\lim_{t\to+\infty}z(t)=L\ge 0$. We assert that $L=0$. If not, then $L>0$. Integrating (13) from $t\ge t_2$ to ∞, we have

$$-b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}-\frac{p_0^{\alpha}}{\tau_0}b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha}\le-\int_t^{\infty}2^{1-\alpha}Q(s)z^{\alpha}(\delta(s))\Delta s, \tag{35}$$

that is,

$$\begin{aligned}
&b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}+\frac{p_0^{\alpha}}{\tau_0}b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha} \\
&\ge\int_t^{\infty}2^{1-\alpha}Q(s)z^{\alpha}(\delta(s))\Delta s,\ \text{ for } t\ge t_2.
\end{aligned} \tag{36}$$

Taking into account that $(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}<0$ and $\tau(t)<t$, (36) yields

$$b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha}\ge\frac{2^{1-\alpha}}{\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)}\int_t^{\infty}Q(s)z^{\alpha}(\delta(s))\Delta s, \tag{37}$$

and, since $\lim_{t\to\infty}z(t)=L>0$, then

$$(a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta}\ge\left(\frac{2^{1-\alpha}}{\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)}\frac{L^{\alpha}}{b(\tau(t))}\int_t^{\infty}Q(s)\Delta s\right)^{\frac{1}{\alpha}},\ \text{ for } t\ge t_2. \tag{38}$$

Now, integrating the above inequality from sufficiently large $t\ge t_2$ to ∞, we have

$$-z^{\Delta}(\tau(t))\ge\frac{\tau_0}{a(\tau(t))}\int_t^{\infty}\left(\frac{2^{1-\alpha}}{\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)}\frac{L^{\alpha}}{b(\tau(s))}\int_s^{\infty}Q(u)\Delta u\right)^{\frac{1}{\alpha}}\Delta s. \tag{39}$$

In addition, integrating (39) from $t\ge t_2$ to ∞, we can obtain

$$z(\tau(t_2))\ge\tau_0\int_{t_2}^{\infty}\frac{\tau_0}{a(\tau(s))}\int_s^{\infty}\left(\frac{2^{1-\alpha}}{\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)}\frac{L^{\alpha}}{b(\tau(v))}\int_v^{\infty}Q(u)\Delta u\right)^{\frac{1}{\alpha}}\Delta v\Delta s, \tag{40}$$

which contradicts (7). Hence, $L=0$, which implies that $\lim_{t\to\infty}z(t)=\lim_{t\to\infty}x(t)=0$. The proof is complete. □

Theorem 2. *Assume that* $0<\alpha<1$, $\delta(t)\ge t$, (2) *and* (7) *hold. If a positive function* $\phi(t)\in C^1_{rd}(\mathbb{T},\mathbb{R})$ *and* $t_1\ge t_0$ *exist, such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\phi(s)Q(s)G^{\alpha}(t_1,s)-\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{b(s)(\phi^{\Delta}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\phi(s))^{\alpha}}\right]\Delta s=\infty, \tag{41}$$

then every non-oscillatory solution $x(t)$ *of Equation* (1) *tends to zero as* $t\to\infty$.

Proof. The proof is similar to that of Theorem 7 except that the second inequality in (6) is used instead of the first one to obtain (12). The details are omitted. □

If we put $\phi(t) = C$ in Theorems 7 and 8, we obtain the following corollary.

Corollary 1. *Assume that* $\alpha > 0$, $\delta(t) \geq t$, (2), (7) *and there exists a* $t_1 \geq t_0$, *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \min\{q(\tau(s)), q(s)\} \left(\frac{b^{\frac{1}{\alpha}}(t)B(t_1, s)A(s, \delta(s))}{b^{\frac{1}{\alpha}}(s)B(t_1, s) + \mu(s)} \right)^{\alpha} \Delta s = \infty, \tag{42}$$

then every non-oscillatory solution $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Remark 1. *Clearly, Corollary 1 removes the restriction* $0 \leq p(t) \leq p_0 < 1$ *as in [8,22–24,32]. In addition, Ref. [22] (Theorem 2.1) is included in Corollary 1.*

When $\delta(t) \geq \tau(t)$, since $(a(t)z^{\Delta}(t))^{\Delta} > 0$, then

$$z(\delta(t)) - z(\tau(t)) = \int_{\tau(t)}^{\delta(t)} \frac{a(s)z^{\Delta}(s)}{a(s)} \Delta s \geq a(\tau(t))z^{\Delta}(\tau(t)) \int_{\tau(t)}^{\delta(t)} \frac{1}{a(s)} \Delta s,$$

that is,

$$z(\delta(t)) \geq a(\tau(t))z^{\Delta}(\tau(t))A(\tau(t), \delta(t)), \quad \text{for } t \geq t_2,$$

which is similar to (23). Hence, it is not difficult to get

$$\frac{z(\delta(t))}{a(\tau(\sigma(t)))z^{\Delta}(\tau(\sigma(t)))} \geq \frac{b^{\frac{1}{\alpha}}(\tau(t))B(\tau(t_1), \tau(t))A(\tau(t), \delta(t))}{b^{\frac{1}{\alpha}}(\tau(t))B((t_1), \tau(t)) + \mu(t)} =: G(\tau(t_1), \tau(t)),$$

which is similar to (28). Thus, it is easier to prove the following theorems, which include Theorems 5 and 6 in [21], respectively.

Theorem 3. *Assume that* $\alpha \geq 1$, $\delta(t) \geq \tau(t)$, (2), *and* (7) *hold. If a positive function* $\phi(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ *and* $t_1 \geq t_0$ *exist, such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[2^{1-\alpha}\phi(s)Q(s)G^{\alpha}(\tau(t_1), \tau(s)) - \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right) \frac{b(s)(\phi^{\Delta}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\phi(s))^{\alpha}} \right] \Delta s = \infty \tag{43}$$

holds, then every non-oscillatory solution $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Theorem 4. *Assume that* $0 < \alpha < 1$, $\delta(t) \geq \tau(t)$, (2) *and* (7) *hold. If a positive function* $\phi(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ *and* $t_1 \geq t_0$ *exist, such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\phi(s)Q(s)G^{\alpha}(\tau(t_1), \tau(s)) - \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right) \frac{b(s)(\phi^{\Delta}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\phi(s))^{\alpha}} \right] \Delta s = \infty \tag{44}$$

holds, then every non-oscillatory solution $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Corollary 2. *Assume that* $\alpha > 0$, $\delta(t) \geq \tau(t)$, (2) *and* (7) *hold. If* $t_1 \geq t_0$ *exists, such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \min\{q(\tau(s)), q(s)\} \left(\frac{b^{\frac{1}{\alpha}}(\tau(s))B(\tau(t_1), \tau(s))A(\tau(s), \delta(s))}{b^{\frac{1}{\alpha}}(\tau(s))B(\tau(t_1), \tau(s)) + \mu(s)} \right)^{\alpha} \Delta s = \infty \tag{45}$$

holds, then every non-oscillatory solution $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

In addition, the case when $\delta(t) \geq \sigma(t)$ has not been considered in the literature (see, e.g., [23]), hence we will obtain new theorems in this case.

Theorem 5. *Assume that $\alpha \geq 1$, $\delta(t) \geq \sigma(t)$, (2) and (7) hold. If a positive function $\phi(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ and $t_1 \geq t_0$ exist, such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[2^{1-\alpha} Q(s)\phi(s) - \left(1 + \frac{p_0^\alpha}{\tau_0}\right) \frac{a^\alpha(\delta(s))(\phi^\Delta(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\phi(s)\delta^\Delta(s)B(s,\sigma(s)))^\alpha} \right] \Delta s = \infty \tag{46}$$

holds, then every non-oscillatory solution $x(t)$ of Equation (1) tends to zero as $t \to \infty$.

Proof. Assume $x(t)$ is a non-oscillatory solution of Equation (1). Without loss of generality, one can pick $t_1 \in [t_0, \infty)_\mathbb{T}$ such that $x(t) > 0, x(\tau(t)) > 0, x(\delta(t)) > 0$ for all $t \in [t_1, \infty)_\mathbb{T}$. Then, the corresponding $z(t)$ satisfies case (I) or case (II) in the proof of Theorem 7.

If case (I) holds, first we define a function $\omega(t)$, by the generalized Riccati substitution as follows:

$$\omega(t) = \phi(t) \frac{b(t)((a(t)z^\Delta(t))^\Delta)^\alpha}{z^\alpha(\delta(t))}, \quad \text{for } t \geq t_1. \tag{47}$$

Noting that $\omega(t) > 0$, by the product rule and the quotient rule on time scales and (47), we have

$$\begin{aligned} \omega^\Delta(t) =& \left(\frac{\phi(t)}{z^\alpha(\delta(t))}\right)^\Delta b(\sigma(t))((a(\sigma(t))z^\Delta(\sigma(t)))^\Delta)^\alpha + \frac{\phi(t)}{z^\alpha(\delta(t))}(b(t)((a(t)z^\Delta(t))^\Delta)^\alpha)^\Delta \\ =& \frac{\phi^\Delta(t)}{\phi(\sigma(t))}\omega(\sigma(t)) - \frac{\phi(t)(z^\alpha(\delta(t)))^\Delta}{z^\alpha(\delta(t))z^\alpha(\delta(\sigma(t)))} b(\sigma(t))((a(\sigma(t))z^\Delta(\sigma(t)))^\Delta)^\alpha \\ &+ \frac{\phi(t)}{z^\alpha(\delta(t))}(b(t)((a(t)z^\Delta(t))^\Delta)^\alpha)^\Delta, \quad \text{for } t \geq t_1. \end{aligned} \tag{48}$$

Since $(b(t)((a(t)z^\Delta(t))^\Delta)^\alpha)^\Delta < 0$, for all $t \geq t_1$, then

$$\begin{aligned} a(\sigma(t))z^\Delta(\sigma(t)) - a(t)z^\Delta(t) &= \int_t^{\sigma(t)} \frac{b^{\frac{1}{\alpha}}(s)(a(s)z^\Delta(s))^\Delta}{b^{\frac{1}{\alpha}}(s)} \Delta s \\ &\geq b^{\frac{1}{\alpha}}(\sigma(t))(a(\sigma(t))z^\Delta(\sigma(t)))^\Delta \int_t^{\sigma(t)} \frac{1}{b^{\frac{1}{\alpha}}(s)} \Delta s, \end{aligned} \tag{49}$$

which yields

$$a(\sigma(t))z^\Delta(\sigma(t)) \geq b^{\frac{1}{\alpha}}(\sigma(t))(a(\sigma(t))z^\Delta(\sigma(t)))^\Delta B(t, \sigma(t)). \tag{50}$$

In addition, from $(a(t)z^\Delta(t))^\Delta > 0$ and $\delta(t) \geq \sigma(t)$, we have

$$a(\delta(t))z^\Delta(\delta(t)) \geq b^{\frac{1}{\alpha}}(\sigma(t))(a(\sigma(t))z^\Delta(\sigma(t)))^\Delta B(t, \sigma(t)), \quad \text{for } t \geq t_1. \tag{51}$$

Moreover, since $z^\Delta(t) > 0$ and $\delta^\Delta(t) > 0$, it is obvious that

$$z(\delta(t)) < z(\delta(\sigma(t))). \tag{52}$$

In combination with (48), (51) and (52), it is obtained that

$$\begin{aligned} \omega^\Delta(t) \leq& (\phi^\Delta(t))_+ \left(\frac{\omega(\sigma(t))}{\phi(\sigma(t))}\right) - \frac{\alpha\phi(t)\delta^\Delta(t)B(t,\sigma(t))}{a(\delta(t))} \cdot \left(\frac{\omega(\sigma(t))}{\phi(\sigma(t))}\right)^{1+\frac{1}{\alpha}} \\ &+ \frac{\phi(t)}{z^\alpha(\delta(t))}(b(t)((a(t)z^\Delta(t))^\Delta)^\alpha)^\Delta, \quad \text{for } t \geq t_1. \end{aligned} \tag{53}$$

Similarly, another function $v(t)$ is defined by generalized Riccati substitution

$$v(t) = \phi(t)\frac{b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha}}{z^{\alpha}(\delta(t))}, \text{ for } t \geq t_1. \tag{54}$$

It is clear that $v(t) > 0$. In addition, analogously to the process of proving (53), we get

$$\begin{aligned} v^{\Delta}(t) \leq &(\phi^{\Delta}(t))_+\left(\frac{v(\sigma(t))}{\phi(\sigma(t))}\right) - \frac{\alpha\phi(t)\delta^{\Delta}(t)B(t,\sigma(t))}{a(\delta(t))}\left(\frac{v(\sigma(t))}{\phi(\sigma(t))}\right)^{1+\frac{1}{\alpha}} \\ &+ \frac{\phi(t)}{z^{\alpha}(\delta(t))}(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}, \text{ for } t \geq t_1. \end{aligned} \tag{55}$$

By adding (53) to (55), we have

$$\begin{aligned} \omega^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}v^{\Delta}(t) \leq &\frac{\phi(t)}{z^{\alpha}(\delta(t))}(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta} + \frac{p_0^{\alpha}}{\tau_0}\frac{\phi(t)}{z^{\alpha}(\delta(t))}(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta} \\ &+ (\phi^{\Delta}(t))_+\left(\frac{\omega(\sigma(t))}{\phi(\sigma(t))}\right) - \frac{\alpha\phi(t)\delta^{\Delta}(t)B(t,\sigma(t))}{a(\delta(t))}\left(\frac{\omega(\sigma(t))}{\phi(\sigma(t))}\right)^{\frac{\alpha+1}{\alpha}} \\ &+ \frac{p_0^{\alpha}}{\tau_0}\left[(\phi^{\Delta}(t))_+\left(\frac{v(\sigma(t))}{\phi(\sigma(t))}\right) - \frac{\alpha\phi(t)\delta^{\Delta}(t)B(t,\sigma(t))}{a(\delta(t))}\left(\frac{v(\sigma(t))}{\phi(\sigma(t))}\right)^{\frac{\alpha+1}{\alpha}}\right], \end{aligned} \tag{56}$$

for all $t \geq t_1$. Similarly, for (32), set

$$g(\nu) = \phi^{\Delta}(t))_+\nu - \frac{\alpha\phi(t)\delta^{\Delta}(t)B(t,\sigma(t))}{a(\delta(t))}\nu^{\frac{\alpha+1}{\alpha}}. \tag{57}$$

It is easy to see that, for $\nu = \nu_0 = \frac{((\phi^{\Delta}(t))_+a(\delta(t)))^{\alpha}}{((\alpha+1)\phi(t)\delta^{\Delta}(t)B(t,\sigma(t)))^{\alpha}}$, $g(\nu)$ gets its maximum value on $[0,\infty)$, which implies that

$$g(\nu) \leq g(\nu_0) = \frac{1}{(1+\alpha)^{1+\alpha}}\frac{(\phi^{\Delta}(t))^{\alpha+1}a^{\alpha}(\delta(t))}{(\phi(t)\delta^{\Delta}(t)B(t,\sigma(t)))^{\alpha}}. \tag{58}$$

By substituting (13) and (58) into (56), we get

$$\omega^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}v^{\Delta}(t) \leq -2^{1-\alpha}Q(t)\phi(t) + \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right)\frac{(\phi^{\Delta}(t))^{\alpha+1}a^{\alpha}(\delta(t))}{(1+\alpha)^{1+\alpha}(\phi(t)\delta^{\Delta}(t)B(t,\sigma(t)))^{\alpha}},$$

for all $t \geq t_2 \geq t_1$. By integrating the above inequality from t_2 to $t > t_2$, we have

$$\begin{aligned} &\int_{t_2}^{t}\left[2^{1-\alpha}Q(s)\phi(s) - \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right)\frac{(\phi^{\Delta}(s))^{\alpha+1}a^{\alpha}(\delta(s))}{(1+\alpha)^{1+\alpha}(\phi(s)\delta^{\Delta}(s)B(s,\sigma(s)))^{\alpha}}\right]\Delta s \\ \leq &\omega(t_2) - \omega(t) + \frac{p_0^{\alpha}}{\tau_0}v(t_2) - v(t) \leq \omega(t_2) + \frac{p_0^{\alpha}}{\tau_0}v(t_2), \end{aligned} \tag{59}$$

which contradicts (46).

If case (II) holds, we proceed the same as in Theorem 7 to arrive at the desired conclusion. The proof is complete. □

Theorem 6. *Assume that $0 < \alpha < 1$, $\delta(t) \geq \sigma(t)$, (2) and (7) hold. If a positive function $\phi(t) \in C^1_{rd}(\mathbb{T},\mathbb{R})$ and $t_1 \geq t_0$ exist, such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Q(s)\phi(s) - \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right)\frac{a^{\alpha}(\delta(s))(\phi^{\Delta}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\phi(s)\delta^{\Delta}(s)B(s,\sigma(s)))^{\alpha}}\right]\Delta s = \infty \tag{60}$$

holds, then every non-oscillatory solution $x(t)$ of Equation (1) tends to zero as $t \to \infty$.

Corollary 3. *Assume that* $\alpha > 0$, $\delta(t) \geq \sigma(t)$, (2), (7) *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \min\{q(\tau(s)), q(s)\}\Delta s = \infty \tag{61}$$

hold, then every non-oscillatory solution $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

2.2. Oscillation Results for (1) When (3) Holds

In this section, we will consider (1) under the mixed canonical-noncanonical conditions (3).

Theorem 7. *Assume that* $\alpha \geq 1$, $\delta(t) \geq t$, (3), (7) *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[2^{1-\alpha} Q(s) B^{\alpha}(\sigma(s)) G^{\alpha}(t_1, s) - K \frac{A^{\alpha}(s, \delta(s))}{b^{\frac{1}{\alpha}}(s) G^{\alpha}(t_1, s)} \frac{B^{\alpha^2-1}(s)}{B^{\alpha^2}(\sigma(s))} \right] \Delta s = \infty \tag{62}$$

hold, where

$$K = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right).$$

If a positive function $\phi(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ *and* $t_1 \geq t_0$ *exist, such that* (8) *holds, then every non-oscillatory solution* $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Proof. Assume $x(t)$ is a non-oscillatory solution of Equation (1). Without loss of generality, one can pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0, x(\delta(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Based on condition (3), the corresponding $z(t)$ satisfies one of three possible cases: (I), (II) (as those in the Proof of Theorem 7), and

$$\text{case (III)}: \; z(t) > 0, z^{\Delta}(t) > 0, (a(t)z^{\Delta}(t))^{\Delta} < 0, (b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta} < 0.$$

If case (I) or case (II) holds, we turn back to the proof of Theorem 7. If case (III) holds, by virtue of the $(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta} < 0$, we have

$$a(l)z^{\Delta}(l) - a(t)z^{\Delta}(t) = \int_t^l \frac{b^{\frac{1}{\alpha}}(s)(a(s)z^{\Delta}(s))^{\Delta}}{b^{\frac{1}{\alpha}}(s)} \Delta s \leq b^{\frac{1}{\alpha}}(t)(a(t)z^{\Delta}(t))^{\Delta} \int_t^l \frac{1}{b^{\frac{1}{\alpha}}(s)} \Delta s,$$

for all $t \geq t_1$. Letting $l \to \infty$, the above inequality implies that

$$-a(t)z^{\Delta}(t) \leq \int_t^{\infty} \frac{b^{\frac{1}{\alpha}}(s)(a(s)z^{\Delta}(s))^{\Delta}}{b^{\frac{1}{\alpha}}(s)} \Delta s \leq b^{\frac{1}{\alpha}}(t)(a(t)z^{\Delta}(t))^{\Delta} B(t),$$

that is,

$$\frac{b^{\frac{1}{\alpha}}(t)(a(t)z^{\Delta}(t))^{\Delta}}{a(t)z^{\Delta}(t)} B(t) \geq -1, \tag{63}$$

and thus,

$$\frac{b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}}{(a(t)z^{\Delta}(t))^{\alpha}} B^{\alpha}(t) \geq -1, \quad \text{for } t \geq t_1. \tag{64}$$

Define the function $\eta(t)$ by

$$\eta(t) = \frac{b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}}{(a(t)z^{\Delta}(t))^{\alpha}}, \quad \text{for } t \geq t_1. \tag{65}$$

Note that $\eta(t) < 0$ and

$$\eta(t)B^{\alpha}(t) \geq -1. \tag{66}$$

By the product quotient rules on time scales, we have

$$\eta^{\Delta}(t) \leq \frac{(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} - \frac{b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha}((a(t)z^{\Delta}(t))^{\alpha})^{\Delta}}{(a(t)z^{\Delta}(t))^{\alpha}(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}}. \tag{67}$$

By (16) and $(a(t)z^{\Delta}(t))^{\Delta} < 0$, we also have

$$\frac{((a(t)z^{\Delta}(t))^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} \geq \frac{\alpha(a(t)z^{\Delta}(t))^{\alpha-1}(a(t)z^{\Delta}(t))^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} = \frac{\alpha(a(t)z^{\Delta}(t))^{\alpha}(a(t)z^{\Delta}(t))^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}(a(t)z^{\Delta}(t))}. \tag{68}$$

Combining (67) and (68), we obtain

$$\eta^{\Delta}(t) \leq \frac{(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} - \alpha\frac{(a(t)z^{\Delta}(t))^{\alpha}}{b^{\frac{1}{\alpha}}(t)(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}}(\eta(t))^{\frac{\alpha+1}{\alpha}}. \tag{69}$$

As in Theorem 7, we can easily get the inequality (27):

$$\frac{a(t)z^{\Delta}(t)}{a(\sigma(t))z^{\Delta}(\sigma(t))} \geq \frac{b^{\frac{1}{\alpha}}(t)B(t_1,t)}{b^{\frac{1}{\alpha}}(t)B(t_1,t)+\mu(t)} := G_0(t_1,t). \tag{70}$$

Therefore, combining (69) and (70) yields

$$\eta^{\Delta}(t) \leq \frac{(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} - \frac{\alpha G_0^{\alpha}(t_1,t)}{b^{\frac{1}{\alpha}}(t)}(\eta(t))^{\frac{\alpha+1}{\alpha}}. \tag{71}$$

In addition, from $(b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha})^{\Delta} < 0$ and $\tau(t) < t$, we obtain

$$b(t)((a(t)z^{\Delta}(t))^{\Delta})^{\alpha} < b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha}. \tag{72}$$

Next, define the another function $\vartheta(t)$ by

$$\vartheta(t) = \frac{b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha}}{(a(t)z^{\Delta}(t))^{\alpha}}, \quad \text{for } t \geq t_1. \tag{73}$$

From (65), (72) and (73), we know $\eta(t) < \vartheta(t) < 0$. By virtue of (66), we have

$$\vartheta(t)B^{\alpha}(t) \geq -1. \tag{74}$$

Similarly to the process of proving (71), we get

$$\vartheta^{\Delta}(t) \leq \frac{(b(\tau(t))((a(\tau(t))z^{\Delta}(\tau(t)))^{\Delta})^{\alpha})^{\Delta}}{(a(\sigma(t))z^{\Delta}(\sigma(t)))^{\alpha}} - \frac{\alpha G_0^{\alpha}(t_1,t)}{b^{\frac{1}{\alpha}}(t)}(\vartheta(t))^{\frac{\alpha+1}{\alpha}}. \tag{75}$$

In addition, from (28), (71) and (75), we have

$$\eta^{\Delta}(t) + \frac{p_0^{\alpha}}{\tau_0}\vartheta^{\Delta}(t) \leq$$
$$\leq -2^{1-\alpha}Q(t)G^{\alpha}(t_1,t) - \frac{\alpha G_0^{\alpha}(t_1,t)}{b^{\frac{1}{\alpha}}(t)}(\eta(t))^{\frac{\alpha+1}{\alpha}} - \frac{p_0^{\alpha}}{\tau_0}\frac{\alpha G_0^{\alpha}(t_1,t)}{b^{\frac{1}{\alpha}}(t)}(\vartheta(t))^{\frac{\alpha+1}{\alpha}}, \quad \text{for } t \geq t_1.$$

Multiplying the above inequality by $B^{\alpha}(\sigma(t))$, with t replaced by s, and integrating with respect to s from t_1 to $t \geq t_1$, and since $(B^{\alpha}(s))^{\Delta} = -\alpha B^{\alpha-1}(s) b^{-\frac{1}{\alpha}}(s)$, so we have

$$\begin{aligned}
&\int_{t_1}^{t} 2^{1-\alpha} Q(s) B^{\alpha}(\sigma(s)) G^{\alpha}(t_1,s) \Delta s \\
&\leq B^{\alpha}(t_1)\eta(t_1) - B^{\alpha}(t)\eta(t) \\
&+ \int_{t_1}^{t} \left[(B^{\alpha}(s))^{\Delta}\eta(s) - \frac{\alpha G_0^{\alpha}(t_1,s)}{b^{\frac{1}{\alpha}}(s)} B^{\alpha}(\sigma(s))(\eta(s))^{\frac{\alpha+1}{\alpha}} \right] \Delta s \\
&+ \frac{p_0^{\alpha}}{\tau_0}(B^{\alpha}(t_1)\vartheta(t_1) - B^{\alpha}(t)\vartheta(t)) \\
&+ \frac{p_0^{\alpha}}{\tau_0} \int_{t_1}^{t} \left[(B^{\alpha}(s))^{\Delta}\vartheta(s) - \frac{\alpha G_0^{\alpha}(t_1,s)}{b^{\frac{1}{\alpha}}(s)} B^{\alpha}(\sigma(s))(\vartheta(s))^{\frac{\alpha+1}{\alpha}} \right] \Delta s \\
&= B^{\alpha}(t_1)\eta(t_1) - B^{\alpha}(t)\eta(t) \\
&- \alpha \int_{t_1}^{t} \left[\frac{B^{\alpha-1}(s)}{b^{\frac{1}{\alpha}}(s)}\eta(s) + \frac{G_0^{\alpha}(t_1,s)}{b^{\frac{1}{\alpha}}(s)} B^{\alpha}(\sigma(s))(\eta(s))^{\frac{\alpha+1}{\alpha}} \right] \Delta s \\
&+ \frac{p_0^{\alpha}}{\tau_0}(B^{\alpha}(t_1)\vartheta(t_1) - B^{\alpha}(t)\vartheta(t)) \\
&- \frac{\alpha p_0^{\alpha}}{\tau_0} \int_{t_1}^{t} \left[\frac{B^{\alpha-1}(s)}{b^{\frac{1}{\alpha}}(s)}\vartheta(s) + \frac{G_0^{\alpha}(t_1,s)}{b^{\frac{1}{\alpha}}(s)} B^{\alpha}(\sigma(s))(\vartheta(s))^{\frac{\alpha+1}{\alpha}} \right] \Delta s.
\end{aligned} \tag{76}$$

Let

$$F(x) = \frac{B^{\alpha-1}(s)}{b^{\frac{1}{\alpha}}(s)} x + \frac{G_0^{\alpha}(t_1,s)}{b^{\frac{1}{\alpha}}(s)} B^{\alpha}(\sigma(s)) x^{\frac{\alpha+1}{\alpha}}, \text{ for } x \leq 0. \tag{77}$$

By the inequality,

$$A = -x\left[\frac{G_0^{\alpha}(t_1,s)}{b^{\frac{1}{\alpha}}(s)} B^{\alpha}(\sigma(s)) \right]^{\frac{\alpha}{\alpha+1}}, \quad B = \left[\frac{\alpha}{\alpha+1} \frac{B^{\alpha-1}(s)}{b^{\frac{1}{\alpha}}(s)} \left(\frac{G_0^{\alpha}(t_1,s)}{b^{\frac{1}{\alpha}}(s)} B^{\alpha}(\sigma(s)) \right)^{-\frac{\alpha}{\alpha+1}} \right]^{\alpha},$$

and by the inequality (see [16])

$$\frac{\alpha+1}{\alpha} A B^{\frac{1}{\alpha}} - A^{\frac{\alpha+1}{\alpha}} \leq \frac{1}{\alpha} B^{\frac{\alpha+1}{\alpha}}, \text{ for } A \geq 0 \text{ and } B \geq 0, \tag{78}$$

we have

$$F(x) \geq -\frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{b^{\frac{1}{\alpha}}(s) G_0^{\alpha^2}(t_1,s)} \frac{B^{\alpha^2-1}}{B^{\alpha^2}(\sigma(s))}, \text{ for } x \leq 0. \tag{79}$$

Since we have that $G(t_1,t) = G_0(t_1,t)A(t,\delta(t))$, substituting (79) and (77) into (76), it can be concluded that

$$\begin{aligned}
&\int_{t_1}^{t} \left[2^{1-\alpha} Q(s) B^{\alpha}(\sigma(s)) G^{\alpha}(t_1,s) - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(1 + \frac{p_0^{\alpha}}{\tau_0} \right) \frac{A^{\alpha}(s,\delta(s))}{b^{\frac{1}{\alpha}}(s) G^{\alpha^2}(t_1,s)} \frac{B^{\alpha^2-1}(s)}{B^{\alpha^2}(\sigma(s))} \right] \Delta s \\
&\leq B^{\alpha}(t_1)\eta(t_1) - B^{\alpha}(t)\eta(t) + \frac{p_0^{\alpha}}{\tau_0}(B^{\alpha}(t_1)\vartheta(t_1) - B^{\alpha}(t)\vartheta(t)) \\
&= B^{\alpha}(t_1)\eta(t_1) + 1 + \frac{p_0^{\alpha}}{\tau_0}(B^{\alpha}(t_1)\vartheta(t_1) + 1).
\end{aligned}$$

Taking the limsup on both sides of the above inequality as $t \to \infty$, which contradicts (62). The proof is complete. $\square$

Theorem 8. *Assume that* $0 < \alpha < 1$, $\delta(t) \geq t$, (3), (7) *and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[Q(s)B^{\alpha}(\sigma(s))G^{\alpha}(t_1,s) - K\frac{A^{\alpha}(s,\delta(s))}{b^{\frac{1}{\alpha}}(s)G^{\alpha}(t_1,s)} \frac{B^{\alpha^2-1}(s)}{B^{\alpha^2}(\sigma(s))} \right] \Delta s = \infty \tag{80}$$

hold, where

$$K = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right).$$

If a positive function $\phi(t) \in C_{rd}^{1}(\mathbb{T},\mathbb{R})$ *and* $t_1 \geq t_0$ *exist, such that* (41) *holds, then every non-oscillatory solution* $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Theorem 9. *Assume that* $\alpha \geq 1$, $\delta(t) \geq \tau(t)$, (3), (7) *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[Q(s)B^{\alpha}(\sigma(s))G^{\alpha}(\tau(t_1),\tau(s)) - \frac{K_1A^{\alpha}(\tau(s),\delta(s))}{b^{\frac{1}{\alpha}}(s)G^{\alpha}(\tau(t_1),\tau(s))} \frac{B^{\alpha^2-1}(s)}{B^{\alpha^2}(\sigma(s))} \right] \Delta s = \infty \tag{81}$$

hold, where

$$K_1 = 2^{\alpha-1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right).$$

If a positive function $\phi(t) \in C_{rd}^{1}(\mathbb{T},\mathbb{R})$ *and* $t_1 \geq t_0$ *exist, such that* (43) *holds, then every non-oscillatory solution* $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Theorem 10. *Assume that* $0 < \alpha < 1$, $\delta(t) \geq \tau(t)$, (3), (7) *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[Q(s)B^{\alpha}(\sigma(s))G^{\alpha}(\tau(t_1),\tau(s)) - \frac{KA^{\alpha}(\tau(s),\delta(s))}{b^{\frac{1}{\alpha}}(s)G^{\alpha}(\tau(t_1),\tau(s))} \frac{B^{\alpha^2-1}(s)}{B^{\alpha^2}(\sigma(s))} \right] \Delta s = \infty \tag{82}$$

hold, where

$$K = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right).$$

If a positive function $\phi(t) \in C_{rd}^{1}(\mathbb{T},\mathbb{R})$ *and* $t_1 \geq t_0$ *exist, such that* (44) *holds, then every non-oscillatory solution* $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Theorem 11. *Assume that* $\alpha \geq 1$, $\delta(t) \geq \sigma(t)$, (3), (7) *and* (62) *hold. If a positive function* $\phi(t) \in C_{rd}^{1}(\mathbb{T}\mathbb{R})$ *and* $t_1 \geq t_0$ *exist, such that* (46) *holds, then every non-oscillatory solution* $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Theorem 12. *Assume that* $0 < \alpha < 1$, $\delta(t) \geq \sigma(t)$, (3), (7) *and* (80) *hold. If a positive function* $\phi(t) \in C_{rd}^{1}(\mathbb{T},\mathbb{R})$ *and a* $t_1 \geq t_0$ *exist, such that* (60) *holds, then every non-oscillatory solution* $x(t)$ *of Equation* (1) *tends to zero as* $t \to \infty$.

Remark 2. *With an appropriate choice of the function* $\phi(t)$, *one can derive a number of oscillation criteria for Equation* (1) *using Theorems 7–12.*

3. Examples

In this section, we give three examples to illustrate our main results in this paper.

Example 1. *Consider a third-order neutral dynamic equation*

$$\left(t\left(\left(\frac{1}{t}(x(t)+\frac{p_0t}{1+t}x\left(\frac{t}{2}\right))^{\Delta}\right)^{\Delta}\right)^{3}\right)^{\Delta} + \frac{\lambda}{t^7}x^3(\frac{3t}{2}) = 0, \quad t \in [1,\infty)_{\mathbb{T}}, \tag{83}$$

where $\lambda > 0$, $p_0 > 0$ are two constants. Here, $\alpha = 3$, $b(t) = t$, $a(t) = \frac{1}{t}$, $0 < p(t) = \frac{p_0 t}{1+t} < p_0$, $q(t) = \frac{\lambda}{t^7}$, $\delta(t) = \frac{3t}{2}$, $\tau(t) = \frac{t}{2} < t$, $\tau_0 = \frac{1}{2}$.

For instance, while $\mathbb{T} = \overline{2^{\mathbb{Z}}} = 2^{\mathbb{Z}} \cup \{0\}$, then $\sigma(t) = 2t > \delta(t) > t$. It is not difficult to verify the conditions (A_1)–(A_4), (2) and (7) are satisfied. Now, we prove that (42) is true, that is,

$$\limsup_{t\to\infty} \int_{t_0}^{t} Q(s)G^{\alpha}(t_0, s)\Delta s = \infty,$$

where $t_0 = 1$. In fact, it is not hard to see here

$$Q(t) = \min\{\frac{\lambda}{t^7}, \frac{2^7\lambda}{t^7}\} = \frac{\lambda}{t^7}, \quad B(t_0, t) = \int_1^t s^{-\frac{1}{3}}\Delta s = \frac{s^{\frac{2}{3}}|_1^t}{2^{\frac{2}{3}} - 1} > t^{\frac{2}{3}},$$

and

$$A(t, \delta(t)) = \int_t^{\delta(t)} \frac{1}{1/s}\Delta s = \frac{1}{2^2 - 1} s^2|_t^{\frac{3}{2}t} = \frac{5}{12}t^2,$$

hence

$$G(t_0, t) = \frac{b^{\frac{1}{\alpha}}(t)B(1, t)A(t, \delta(t))}{b^{\frac{1}{\alpha}}(t)B(1, t) + \mu(t)} > \frac{t^{\frac{1}{3}} \cdot t^{\frac{2}{3}} \cdot \frac{5}{12}t^2}{t^{\frac{1}{3}}t^{\frac{2}{3}} + \sigma(t) - t} = \frac{5}{24}t^2.$$

Obviously,

$$\limsup_{t\to\infty} \int_{t_0}^{t} Q(s)G^{\alpha}(t_0, s)\Delta s > \limsup_{t\to\infty} \int_1^t \frac{\lambda}{s^7} \cdot 1 \cdot (\frac{5}{24}s^2)^3 \Delta s = \infty.$$

Thus, (42) holds and every non-oscillatory solution $x(t)$ of (83) tends to zero as $t \to \infty$, by Corollary 1.

Example 2. *Consider a third-order neutral dynamic equation*

$$\left(t\left(\left(\frac{1}{t}(x(t) + \frac{p_0 t}{1+t}x(\frac{t}{2}))^{\Delta}\right)^{\Delta}\right)^3\right)^{\Delta} + \frac{\lambda}{t^6}x^3(3t) = 0, \quad t \in [1, \infty)_{\mathbb{T}}, \tag{84}$$

where $\lambda > 0$ and $p_0 > 0$ are constants. Here, $\alpha = 3$, $b(t) = t$, $a(t) = \frac{1}{t}$, $0 < p(t) = \frac{p_0 t}{1+t} < p_0$, $q(t) = \frac{\lambda}{t^6}$, $\delta(t) = 3t$, $\tau(t) = \frac{t}{2} < t$, $\tau_0 = \frac{1}{2}$.

For instance, while $\mathbb{T} = \overline{2^{\mathbb{Z}}} = 2^{\mathbb{Z}} \cup \{0\}$, it is obvious that $\sigma(t) = 2t < \delta(t) = 3t$, conditions (A_1)–(A_4), (2) and (7) hold, and also

$$Q(t) = \min\{q(t), q(\tau(t))\} = \frac{\lambda}{t^6}, \quad B(t_0, t) = \int_1^t s^{-\frac{1}{3}}\Delta s > t^{\frac{2}{3}}.$$

If we set $\phi(t) = t^5$, it is easy to obtain that

$$\begin{aligned}
&\limsup_{t\to\infty} \int_1^t \left[2^{1-\alpha}Q(s)\phi(s) - \left(1 + \frac{p_0^{\alpha}}{\tau_0}\right)\frac{((\phi^{\Delta}(s))_+)^{\alpha+1}a^{\alpha}(\delta(s))}{(1+\alpha)^{1+\alpha}(\phi(s)\delta^{\Delta}(s)B(s, \sigma(s)))^{\alpha}}\right]\Delta s \\
&\geq \limsup_{t\to\infty} \int_1^t \left[\frac{\lambda}{4s} - \frac{5^4(1 + 2p_0^3)}{4 \cdot 9^3 s^4}\right]\Delta s = \infty.
\end{aligned}$$

Hence, (46) holds and any non-oscillatory solution $x(t)$ of (84) tends to zero as $t \to \infty$, by Theorem 14.

Example 3. *Consider a third-order neutral dynamic equation*

$$\left(t^{\frac{9}{2}}\left(\left(\frac{1}{t}(x(t)+\frac{p_0 t}{1+t}x(\frac{t}{2}))^{\Delta}\right)^{3}\right)\right)^{\Delta}+\frac{\lambda}{t^{\frac{11}{2}}}x^3\left(\frac{3t}{2}\right)=0,\quad t\in[1,\infty)_{\mathbb{T}}, \tag{85}$$

where $\lambda > 0$, $p_0 > 0$ *are two constants. Here,* $\alpha = 3$, $b(t) = t^{\frac{9}{2}}$, $a(t) = \frac{1}{t}$, $0 < p(t) = \frac{p_0 t}{1+t} < p_0$, $q(t) = \frac{\lambda}{t^{\frac{11}{2}}}$, $\delta(t) = \frac{3}{2}t$, $\tau(t) = \frac{t}{2} < t$, $\tau_0 = \frac{1}{2}$.

While $\mathbb{T} = \mathbb{R}$, *then* $\sigma(t) = t < \delta(t)$. *It is easy to see that assumptions* (A_1)*–*(A_4), (3), (7) *and* (8) *hold. Since*

$$\mu(t)=0,\ Q(t)=\min\{q(t),q(\tau(t))\}=\min\left\{\frac{\lambda}{t^{\frac{11}{2}}},\frac{2^{\frac{11}{2}}\lambda}{t^{\frac{11}{2}}}\right\}=\frac{\lambda}{t^{\frac{11}{2}}},$$

$$B(t)=\int_t^{\infty}b^{-\frac{1}{\gamma}}(s)\Delta s=\int_t^{\infty}(s^{\frac{9}{2}})^{-\frac{1}{3}}\Delta s=2t^{-\frac{1}{2}},\ A(t,\delta(t))=\int_t^{\frac{3}{2}t}sds=\frac{5}{8}t^2=G(t_0,t),$$

and $t_0 = 1$, *we obtain*

$$R(t):=Q(t)B^{\alpha}(\sigma(t))G^{\alpha}(t_0,t)=Q(t)B^{\alpha}(t)A^{\alpha}(t,\delta(t))=\frac{\lambda}{t^{\frac{11}{2}}}(2t^{-\frac{1}{2}})^3(\frac{5}{8}t^2)^3=(\frac{5}{4})^3\lambda\frac{1}{t}.$$

In order to prove (62), *we show that*

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_0}^{t}\left[2^{1-\alpha}R(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{1}{b^{\frac{1}{\alpha}}(s)G_0^{\alpha}(t_0,s)}\frac{B^{\alpha^2-1}(s)}{B^{\alpha^2}(\sigma(s))}\right]\Delta s\\
&=\limsup_{t\to\infty}\int_1^t\left[2^{1-\alpha}R(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{1}{b^{\frac{1}{\alpha}}(s)B(s)}\right]ds\\
&=\limsup_{t\to\infty}\int_1^t\left[2^{-2}(\frac{5}{4})^3\lambda\frac{1}{s}-\left(\frac{3}{4}\right)^4\left(1+2p_0^3\right)\frac{1}{2s}\right]ds\\
&=\left[\frac{1}{4}\left(\frac{5}{4}\right)^3\lambda-\left(\frac{3}{4}\right)^4\cdot\frac{1}{2}(1+2p_0^3)\right]\limsup_{t\to\infty}\int_1^t\frac{1}{s}ds=\infty
\end{aligned}$$

if

$$\lambda > \frac{81(1+2p_0^3)}{250}. \tag{86}$$

Thus, every non-oscillatory solution $x(t)$ *of* (85) *tends to zero as* $t \to \infty$, *if* (86) *holds, by Theorem 7.*

Remark 3. *The three examples above demonstrate that many theorems in this papers apply also in the case* $p_0 > 1$, *when all the relevant results obtained in the related literature fail.*

4. Conclusions

Inspired by previous research, under double canonical and mixed canonical-noncanonical conditions, we discussed the oscillation criteria of Equation (1) without the restrictive condition $0 \le p(t) \le p_0 < 1$ in the literature, for instance in [8,22–24,32]. By using the double Riccati transformation and inequality technique, we established new oscillation theorems which include, improve, and generalize some results in the literature, for instance [21–24]. At the same time, the effects of $\tau(t) \le \delta(t)$, $t \le \delta(t) \le \sigma(t)$ and $\sigma(t) < \delta(t)$ on the oscillation results were considered. Finally, three examples were given to illustrate the validity of the theorems in this paper.

Actually, there are many directions for future research: first, it is an interesting task to generalize the obtained results for odd-order dynamic equations with deviating arguments. Second, with regard to the results of this paper, there remains an open problem associated with removing all non-oscillatory solutions of (1). Third, our further plan is to take advantage of a recent approach [34] developed for third-order differential equations in order to state more effective results for dynamic equations on time scales.

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