mathematics

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On (k, p)-Fibonacci Numbers

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Abstract: In this paper, we introduce and study a new two-parameters generalization of the Fibonacci numbers, which generalizes Fibonacci numbers, Pell numbers, and Narayana numbers, simultaneously. We prove some identities which generalize well-known relations for Fibonacci numbers, Pell numbers and their generalizations. A matrix representation for generalized Fibonacci numbers is given, too.

Keywords: Fibonacci numbers; Pell numbers; Narayana numbers

MSC: 11B39; 11B83; 11C20

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Citation: Bednarz, N. On (k,p)-Fibonacci Numbers. *Mathematics* **2021**, *9*, 727. https://doi.org/10.3390/math9070727

Received: 11 March 2021
Accepted: 26 March 2021
Published: 28 March 2021

1. Introduction

By numbers of the Fibonacci type we mean numbers defined recursively by the r-th order linear recurrence relation of the form

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_r a_{n-r}, \text{ for } n \geqslant r, \tag{1}$$

where $r \geqslant 2$ and $b_i \geqslant 0$, $i = 1, 2, \cdots, r$ are integers.

For special values of r and b_i, $i = 1, 2, \cdots r$, the Equality (1) defines well-known numbers of the Fibonacci type and their generalizations. We list some of them:

1. Fibonacci numbers: $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$, with $F_0 = F_1 = 1$.
2. Lucas numbers: $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$, with $L_0 = 2, L_1 = 1$.
3. Pell numbers: $P_n = 2P_{n-1} + P_{n-2}$ for $n \geqslant 2$, with $P_0 = 0, P_1 = 1$.
4. Pell–Lucas numbers: $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geqslant 2$, with $Q_0 = 1$, $Q_1 = 3$.
5. Jacobsthal numbers: $J_n = J_{n-1} + 2J_{n-2}$ for $n \geqslant 2$, with $J_0 = 0, J_1 = 1$.
6. Jacobsthal–Lucas numbers: $j_n = j_{n-1} + 2j_{n-2}$ for $n \geqslant 2$, with $j_0 = 2, j_1 = 1$.
7. Narayana numbers: $N_n = N_{n-1} + N_{n-3}$ for $n \geqslant 3$, with $N_0 = 0$, $N_1 = N_2 = 1$.

Numbers of the Fibonacci type defined recursively by the second-order linear recurrence relation were introduced by A. F. Horadam in [1] by the following way

$$w_n = pw_{n-1} + qw_{n-2} \text{ for } n \geqslant 2$$

with $w_0 = a, w_1 = b$, where a, b, p, q are arbitrary integers. For their general properties see, for example [1–3]. A special case of the generalized Fibonacci numbers introduced by Horadam are k-Fibonacci numbers and k-Pell numbers presented below

1. k-Fibonacci numbers (Falcón, Á. Plaza [4]):
   $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, for integers $k \geqslant 1, n \geqslant 2$, with $F_{k,0} = 0$, $F_{k,1} = 1$.
2. k-Pell numbers (Catarino [5]):
   $P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}$ for integers $k \geqslant 1, n \geqslant 2$, with $P_{k,0} = 0$, $P_{k,1} = 1$.

There are many other generalizations of the Fibonacci numbers. We recall only some of them:

1. Fibonacci s-numbers (Stakhov [6]):
$F_s(n) = F_s(n-1) + F_s(n-s-1)$ for integer $s \geqslant 1$ and $n > s+1$, with $F_s(0) = \cdots = F_s(s+1) = 1$.
2. Generalized Fibonacci numbers (Kwaśnik, Włoch [7]):
$F(k,n) = F(k,n-1) + F(k,n-k)$ for integers $k \geqslant 1$ and $n \geqslant k+1$, with $F(k,n) = n+1$ for $0 \leqslant n \leqslant k$.
3. Distance Fibonacci numbers (Bednarz, Włoch, Wołowiec-Musiał [8]):
$Fd(k,n) = Fd(k,n-k+1) + Fd(n-k)$ for integers $k \geqslant 2$, $n \geqslant k$, with $Fd(k,n) = 1$ for $0 \leqslant n \leqslant k-1$.
4. generalized Pell (p,i)-numbers (Kiliç [9]):
$P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1)$ for integers $p \geqslant 1$, $0 \leqslant i \leqslant p$, $n > p+1$, with $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$ and $P_p^{(i)}(i+1) = \cdots = P_p^{(i)}(p+1) = 1$.
5. (k,c)-Jacobsthal numbers (Marques, Trojovský [10]):
$J_n^{(k,c)} = J_{n-1}^{(k,c)} + J_{n-2}^{(k,c)} + \cdots + J_{n-k}^{(k,c)}$ for $k \geqslant 2$ and $n \geqslant k$, with $J_0^{(k,c)} = J_1^{(k,c)} = \cdots = J_{k-2}^{(k,c)} = 0$, $J_{k-1}^{(k,c)} = 1$.

For other generalizations of numbers of the Fibonacci type see, for example, in [11].

In this paper, we extend the numbers of the Fibonacci type. In (1), coefficients $b_i \geqslant 0$, $i = 1,2,\cdots,k$ are integers. We consider a special kind of this equation with the assumption that b_i can be rational.

2. Generalization and Identities

In this section we introduce (k,p)-Fibonacci numbers, denoted by $F_{k,p}(n)$. We prove some identities for $F_{k,p}(n)$, which generalize well-known relations for the Fibonacci numbers, Pell numbers, Narayana numbers, k-Fibonacci numbers, Fibonacci s-numbers and generalized Fibonacci numbers, simultaneously.

Let $k \geqslant 2, n \geqslant 0$ be integers and let $p \geqslant 1$ be a rational number.

The (k,p)-Fibonacci numbers, denoted by $F_{k,p}(n)$, are defined recursively in the following way

$$F_{k,p}(n) = pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k) \quad \text{for } n \geqslant k \tag{2}$$

with initial conditions

$$F_{k,p}(0) = 0 \text{ and } F_{k,p}(n) = p^{n-1} \text{ for } 1 \leqslant n \leqslant k-1. \tag{3}$$

For special values k, p we obtain well-known numbers of the Fibonacci type. We list these special cases.

1. If $k=2$, $p=1$, $n \geqslant 0$, then $F_{2,1}(n+1) = F_n$, where F_n is the nth Fibonacci number.
2. If $k \geqslant 2$, $p=1$, $n \geqslant k$, then $F_{k,1}(n) = F(k,n-k)$, where $F(k,n-k)$ is the $(n-k)$th generalized Fibonacci number.
3. If $k \geqslant 2$, $p=1$, $n \geqslant 0$, then $F_{k,1}(n) = F_{k-1}(n)$, where $F_{k-1}(n)$ is the nth Fibonacci $(k-1)$-number.
4. If $k=2$, $p=\frac{3}{2}$, $n \geqslant 0$, then $F_{2,\frac{3}{2}}(n) = P_n$, where P_n is the nth Pell number.
5. If $k=2$, $p=\frac{t}{2}$, $t \in \mathbb{N}$, $t \geqslant 2$ and $n \geqslant 0$, then $F_{2,p}(n) = F_{2p-1,n}$, where $F_{2p-1,n}$ is the nth $(2p-1)$-Fibonacci number.
6. If $k=3$, $p=1$, $n \geqslant 0$, then $F_{3,1}(n) = N_n$, where N_n is the nth Narayana number.

In [12] some interpretations of the (k,p)-Fibonacci numbers were given.

Now we prove some identities for $F_{k,p}(n)$ with rational p, which generalize well-known relations for numbers of the Fibonacci type. For details of identities see, for example, [4,6,13–20].

We give the generating functions for the (k,p)-Fibonacci sequence. Let $k \geqslant 2$ be integers and let $p \geqslant 1$ be a rational number. Let us consider (k,p)-Fibonacci sequence $\{F_{k,p}(n)\}$.

By the definition of an ordinary generating function of some sequence, considering this sequence, the ordinary generating function associated is defined by

$$f_{k,p}(x) = F_{k,p}(0) + F_{k,p}(1)x + \ldots + F_{k,p}(k-1)x^{k-1} + F_{k,p}(n)x^n + \ldots. \quad (4)$$

Using the initial conditions for $F_{k,p}(n)$ and the recurrence (2) we can write (4) as follows

$$f_{k,p}(x) = x + px^2 + \ldots + p^{k-2}x^{k-1} + \sum_{n=k}^{\infty} pF_{k,p}(n)x^n + \sum_{n=k}^{\infty} F_{k,p}(n-k)x^n. \quad (5)$$

Consider the right side of the Equation (5) and doing some simple calculations, we obtain the following theorem.

Theorem 1. *Let $k \geqslant 2$ be integers and let $p \geqslant 1$ be a rational number. The generating function of the sequence $F_{k,p}(n)$ has the following form*

$$f_{k,p}(x) = \frac{x}{1 - px - (p-1)x^{k-1} - x^k}. \quad (6)$$

From Theorem 1, for special values of k and p, we obtain well-known generating functions for Fibonacci numbers, Pell numbers, and and k-Fibonacci numbers.

Corollary 1. *Let $k \geqslant 2$, $n \geqslant 0$, $t \geqslant 2$ be integers and $p \geqslant 1$ be a rational number. If*

1. *$k = 2$, $p = 1$, then $f(x) = \frac{x}{1-x-x^2}$ is the generating function of the Fibonacci numbers. (Hoggatt [21])*
2. *$k = 2$, $p = \frac{3}{2}$, then $f(x) = \frac{x}{1-2x-x^2}$ is the generating function of the Pell numbers. (Horadam [15])*
3. *$k = 2$, $p = \frac{t}{2}$, then $f(x) = \frac{x}{1-(2p-1)x-x^2}$ is the generating function of the $(2p-1)$-Fibonacci numbers. (Bolat, Kőse [13])*
4. *$k \geqslant 2$, $p = 1$, then $f(x) = \frac{x}{1-x-x^k}$ is the generating function of the Fibonacci $(k-1)$ -numbers. (Kiliç [22])*
5. *$k = 3$, $p = 1$, then $f(x) = \frac{x}{1-x-x^3}$ is the generating function of the Narayana numbers (Shannon, Horadam [23])*

Theorem 2. *Let $k \geqslant 2$, $n \geqslant k-2$ be integers and $p \geqslant 1$ be a rational number. Then*

$$\sum_{i=0}^{n} F_{k,p}(i) = \frac{1}{2p-1}\left[F_{k,p}(n+1) + F_{k,p}(n-k+2) - 1 + p\sum_{i=0}^{k-3} F_{k,p}(n-i)\right]. \quad (7)$$

Proof. Let $k \geqslant 2$, $n \geqslant k-2$ be integers and $p \geqslant 1$ be a rational number.

Put $S = \sum_{i=0}^{n} F_{k,p}(i)$. Then

$$\begin{aligned} pS &= pF_{k,p}(0) + pF_{k,p}(1) + \cdots + pF_{k,p}(k) + \cdots + pF_{k,p}(n), \\ (p-1)S &= (p-1)F_{k,p}(0) + (p-1)F_{k,p}(1) + \cdots + (p-1)F_{k,p}(k) \\ &\quad + \cdots + (p-1)F_{k,p}(n-1) + (p-1)F_{k,p}(n). \end{aligned}$$

Let us consider the following cases:

1. $n = k-2$. Then

$$\begin{aligned} pS + (p-1)S = {} & pF_{k,p}(0) + pF_{k,p}(1) + \cdots + pF_{k,p}(k-3) + pF_{k,p}(k-2) \\ & + (p-1)F_{k,p}(0) + (p-1)F_{k,p}(1) + (p-1)F_{k,p}(2) + \cdots + \\ & + (p-1)F_{k,p}(k-3) + (p-1)F_{k,p}(k-2). \end{aligned}$$

Using the initial conditions for (k, p)-Fibonacci numbers, we obtain

$$(2p-1)S = p \cdot 0 + p \cdot 1 + \cdots + p \cdot p^{k-4} + p \cdot p^{k-3} + (p-1) \cdot 0 + (p-1) \cdot 1 \\ + (p-1) \cdot p + \cdots + (p-1) \cdot p^{k-4} + (p-1) \cdot p^{k-3}.$$

By simple calculation

$$(2p-1)S = p + p^2 + \cdots + p^{k-3} + p^{k-2} + p - 1 + p^2 - p + \cdots + p^{k-3} - p^{k-4} \\ + p^{k-2} - p^{k-3} = p^{k-2} - 1 + p(1 + p + \cdots + p^{k-4} + p^{k-3}).$$

Finally

$$S = \sum_{i=0}^{k-2} F_{k,p}(i) = \frac{1}{2p-1}\left[F_{k,p}(k-1) - 1 + p\sum_{i=0}^{k-3} F_{k,p}(k-2-i)\right].$$

2. $n = k-1$. Then

$$pS + (p-1)S = pF_{k,p}(0) + pF_{k,p}(1) + \cdots + pF_{k,p}(k-2) + pF_{k,p}(k-1) \\ + (p-1)F_{k,p}(0) + (p-1)F_{k,p}(1) + (p-1)F_{k,p}(2) + \cdots + \\ + (p-1)F_{k,p}(k-2) + (p-1)F_{k,p}(k-1).$$

Using the initial conditions for $F_{k,p}(n)$ and proving analogously as in case 1, we have

$$(2p-1)S = p \cdot 0 + p \cdot 1 + \cdots + p \cdot p^{k-3} + p \cdot p^{k-2} + (p-1) \cdot 0 + (p-1) \cdot 1 \\ + (p-1) \cdot p + \cdots + (p-1) \cdot p^{k-3} + (p-1) \cdot p^{k-2}.$$

Then $(2p-1)S = p^{k-1} + p - 1 + p\left(p + p^2 + \cdots + p^{k-2}\right)$ and for $n = k-1$

$$S = \sum_{i=0}^{k-1} F_{k,p}(i) = \frac{1}{2p-1}\left[F_{k,p}(k) + p\sum_{i=0}^{k-3} F_{k,p}(k-1-i)\right].$$

3. $n \geqslant k$. Using the recurrence (2), we have

$$pS + (p-1)S + S = pF_{k,p}(0) + pF_{k,p}(1) + \cdots + pF_{k,p}(k-2) + pF_{k,p}(k-1) \\ + pF_{k,p}(k) + pF_{k,p}(k+1) + \cdots + pF_{k,p}(n-1) + pF_{k,p}(n) \\ + (p-1)F_{k,p}(0) + (p-1)F_{k,p}(1) + (p-1)F_{k,p}(2) + \cdots + \\ + (p-1)F_{k,p}(n-1) + (p-1)F_{k,p}(n) + F_{k,p}(0) + F_{k,p}(1) \\ + F_{k,p}(2) + \cdots + F_{k,p}(n-1) + F_{k,p}(n)$$

and consequently

$$S + (2p-1)S = pF_{k,p}(k-1) + (p-1)F_{k,p}(1) + F_{k,p}(0) + pF_{k,p}(k) + (p-1)F_{k,p}(2) \\ + F_{k,p}(1) + \cdots + pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k) \\ + pF_{k,p}(n) + (p-1)F_{k,p}(n-k+2) + F_{k,p}(n-k+1) + pF_{k,p}(0) \\ + pF_{k,p}(1) + pF_{k,p}(2) + \cdots + pF_{k,p}(k-2) + (p-1)F_{k,p}(0) \\ + (p-1)F_{k,p}(n-k+3) + (p-1)F_{k,p}(n-k+4) + \cdots + \\ + (p-1)F_{k,p}(n-1) + (p-1)F_{k,p}(n) + F_{k,p}(n-k+2) \\ + F_{k,p}(n-k+3) + \cdots + F_{k,p}(n-1) + F_{k,p}(n).$$

Using the recurrence $F_{k,p}(n) = pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k)$ and the initial conditions of $F_{k,p}(n)$, we have

$$\begin{aligned} S + (2p-1)S = &F_{k,p}(k) + F_{k,p}(k+1) + \cdots + F_{k,p}(n) + F_{k,p}(n+1) + p \cdot 0 + p \cdot 1 \\ &+ \cdots + p \cdot p^{k-3} + (p-1) \cdot 0 + pF_{k,p}(n-k+3) + pF_{k,p}(n-k+4) \\ &+ \cdots + pF_{k,p}(n-2) + pF_{k,p}(n-1) + pF_{k,p}(n) + F_{k,p}(n-k+2), \end{aligned}$$

so

$$\begin{aligned} S + (2p-1)S = &\sum_{i=k}^{n} F_{k,p}(i) + F_{k,p}(n+1) + p + p^2 + \cdots + p^{k-2} + pF_{k,p}(n-k+3) \\ &+ pF_{k,p}(n-k+4) + \cdots + pF_{k,p}(n-2) + pF_{k,p}(n-1) + pF_{k,p}(n) \\ &+ F_{k,p}(n-k+2). \end{aligned}$$

Consequently

$$\begin{aligned} S + (2p-1)S = &F_{k,p}(n+1) + \sum_{i=k}^{n} F_{k,p}(i) + F_{k,p}(0) + F_{k,p}(1) - 1 + F_{k,p}(2) + \cdots + \\ &+ F_{k,p}(k-1) + F_{k,p}(n-k+2) + p\sum_{i=0}^{k-3} F_{k,p}(n-i) \\ S + (2p-1)S = &F_{k,p}(n+1) + S - 1 + F_{k,p}(n-k+2) + p\sum_{i=0}^{k-3} F_{k,p}(n-i), \end{aligned}$$

so $(2p-1)S = F_{k,p}(n+1) + F_{k,p}(n-k+2) - 1 + p\sum_{i=0}^{k-3} F_{k,p}(n-i)$. Finally

$$S = \sum_{i=0}^{n} F_{k,p}(i) = \frac{1}{2p-1}\left[F_{k,p}(n+1) + F_{k,p}(n-k+2) - 1 + p\sum_{i=0}^{k-3} F_{k,p}(n-i)\right],$$

what completes the proof.

□

For special values k and p, we obtain well-known identities.

Corollary 2. *Let $k \geqslant 2$, $n \geqslant k-2$, $t \geqslant 2$ be integers and $p \geqslant 1$ be a rational number. If*

1. $k = 2$, $p = 1$, *then* $\sum_{i=1}^{n} F_i = F_{n+2} - 1$. *(E. Lucas [19])*
2. $k \geqslant 2$, $p = 1$, $n \geqslant 1$, *then* $\sum_{i=1}^{n} F_{k-1}(i) = F_{k-1}(n+k) - 1$. *(A.P. Stakhov [6])*
3. $k = 2$, $p = \frac{3}{2}$, *then* $\sum_{i=1}^{n} P_i = \frac{1}{2}[P_{n+1} + P_n - 1]$. *(T. Koshy [17])*
4. $k = 2$, $p = \frac{t}{2}$, *then* $\sum_{i=1}^{n} F_{2p-1,i} = \frac{1}{2p-1}\left[F_{2p-1,n+1} + F_{2p-1,n} - 1\right]$. *(S. Falcón et al. [4])*
5. $k = 3$, $p = 1$, *then* $\sum_{i=1}^{n} N_i = N_{n+3} - 1$. *(J.L.Ramirez, V.F.Sirvent [24])*

Theorem 3. *Let $k \geqslant 2$, $m \geqslant 1$, n be integers and $p \geqslant 1$ be a rational number. Let $S_{2n} = \sum_{i=0}^{n} F_{k,p}(2i)$. If $k = 2m$, then for $n \geqslant m-1$*

$$S_{2n} = \frac{1}{2p-1}\left[F_{2m,p}(2n+1) - 1 + \sum_{i=1}^{m-1}\Big((p-1)F_{2m,p}(2n-2i+2) + F_{2m,p}(2n-2i+1)\Big)\right]. \quad (8)$$

If $k = 2m+1$, then for $n \geqslant m$

$$S_{2n} = \frac{1}{3(2p-1)}\Big[3F_{2m+1,p}(2n+1) + (2p-1)F_{2m+1,p}(2n) + (p-2)^2F_{2m+1,p}(2n-2m+1) + (p-2)F_{2m+1,p}(2n-2m) - p - 1 + \sum_{i=1}^{m-1}\Big((p^2-p+1)F_{2m+1,p}(2n-2i+1) - (p^2-4p+1)F_{2m+1,p}(2n-2i)\Big)\Big]. \tag{9}$$

Proof. (by induction on n).

Let $k \geqslant 2$, $m \geqslant 1$, n be integers and $p \geqslant 1$ be a rational number. First, we will show that for $k = 2m$, $n \geqslant m-1$ we have

$$\sum_{i=0}^{n} F_{2m,p}(2i) = \frac{1}{2p-1}\Big[F_{2m,p}(2n+1) - 1 + \sum_{i=1}^{m-1}\Big((p-1)F_{2m,p}(2n-2i+2) + F_{2m,p}(2n-2i+1)\Big)\Big].$$

If $n = m-1$, then using initial conditions of $F_{k,p}(n)$, we have

$$\begin{aligned}
&\frac{1}{2p-1}\Big[F_{2m,p}(2m-1) - 1 + \sum_{i=1}^{m-1}\Big((p-1)F_{2m,p}(2m-2i) + F_{2m,p}(2m-2i-1)\Big)\Big]\\
&= \frac{1}{2p-1}\Big[F_{2m,p}(2m-1) - 1 + (p-1)F_{2m,p}(2m-2) + F_{2m,p}(2m-3)\\
&\quad + (p-1)F_{2m,p}(2m-4) + F_{2m,p}(2m-5) + \cdots + (p-1)F_{2m,p}(2) + F_{2m,p}(1)\Big]\\
&= \frac{1}{2p-1}\Big[p^{2m-2} - 1 + (p-1)F_{2m,p}(2m-2) + p^{2m-4} + (p-1)F_{2m,p}(2m-4)\\
&\quad + \cdots + p^4 + (p-1)F_{2m,p}(4) + p^2 + (p-1)F_{2m,p}(2) + 1\Big]\\
&= \frac{1}{2p-1}\Big[p \cdot p^{2m-3} - 1 + (p-1)F_{2m,p}(2m-2) + p \cdot p^{2m-5} + (p-1)F_{2m,p}(2m-4)\\
&\quad + \cdots + p \cdot p^3 + (p-1)F_{2m,p}(4) + p \cdot p + (p-1)F_{2m,p}(2)\Big]\\
&= \frac{1}{2p-1}\Big[pF_{2m,p}(2m-2) + (p-1)F_{2m,p}(2m-2) + pF_{2m,p}(2m-4)\\
&\quad + (p-1)F_{2m,p}(2m-4) + \cdots + pF_{2m,p}(2) + (p-1)F_{2m,p}(2)\Big]\\
&= \frac{1}{2p-1}(2p-1)\sum_{i=1}^{m-1} F_{2m,p}(2i) = \sum_{i=0}^{m-1} F_{2m,p}(2i),
\end{aligned}$$

so the Equality (8) is true for $n = m-1$. Assume now that for an integer $n \geqslant m-1$ holds

$$\sum_{i=0}^{n} F_{2m,p}(2i) = \frac{1}{2p-1}\Big[F_{2m,p}(2n+1) - 1 + \sum_{i=1}^{m-1}\Big((p-1)F_{2m,p}(2n-2i+2) + F_{2m,p}(2n-2i+1)\Big)\Big].$$

We shall show that

$$\sum_{i=0}^{n+1} F_{2m,p}(2i) = \frac{1}{2p-1}\Big[F_{2m,p}(2n+3) - 1 + \sum_{i=1}^{m-1}\Big((p-1)F_{2m,p}(2n-2i+4) + F_{2m,p}(2n-2i+3)\Big)\Big].$$

Using the induction hypothesis, we obtain

$$\sum_{i=0}^{n+1} F_{2m,p}(2i) = \sum_{i=0}^{n} F_{2m,p}(2i) + F_{2m,p}(2n+2) = \frac{1}{2p-1}\Big[F_{2m,p}(2n+1) - 1$$
$$+ \sum_{i=1}^{m-1}\Big((p-1)F_{2m,p}(2n-2i+2) + F_{2m,p}(2n-2i+1)\Big)\Big] + F_{2m,p}(2n+2)$$
$$= \frac{1}{2p-1}\Big[(2p-1)F_{2m,p}(2n+2) + F_{2m,p}(2n+1) - 1 + (p-1)F_{2m,p}(2n)$$
$$+ F_{2m,p}(2n-1) + \cdots + (p-1)F_{2m,p}(2n-2m+6) + F_{2m,p}(2n-2m+5)$$
$$+ (p-1)F_{2m,p}(2n-2m+4) + F_{2m,p}(2n-2m+3)\Big],$$

which implies

$$\sum_{i=0}^{n+1} F_{2m,p}(2i) = \frac{1}{2p-1}\Big[pF_{2m,p}(2n+2) + (p-1)F_{2m,p}(2n+2) + F_{2m,p}(2n+1) - 1$$
$$+ (p-1)F_{2m,p}(2n) + F_{2m,p}(2n-1) + \cdots + (p-1)F_{2m,p}(2n-2m+6)$$
$$+ F_{2m,p}(2n-2m+5) + (p-1)F_{2m,p}(2n-2m+4) + F_{2m,p}(2n-2m+3)\Big].$$

Due to

$$F_{2m,p}(2n+3) = pF_{2m,p}(2n+2) + (p-1)F_{2m,p}(2n-2m+4) + F_{2m,p}(2n-2m+3),$$

we have

$$\sum_{i=0}^{n+1} F_{2m,p}(2i) = \frac{1}{2p-1}\Big[F_{2m,p}(2n+3) - 1 + \sum_{i=1}^{m-1}\Big((p-1)F_{2m,p}(2n-2i+4)$$
$$+ F_{2m,p}(2n-2i+3)\Big)\Big],$$

so, the Equality (8) is true.

Similarly, we can show that for $k = 2m+1$, $n \geqslant m$:

$$\sum_{i=0}^{n} F_{2m+1,p}(2i) = \frac{1}{3(2p-1)}\Big[3F_{2m+1,p}(2n+1) + (2p-1)F_{2m+1,p}(2n)$$
$$+ (p-2)^2F_{2m+1,p}(2n-2m+1) + (p-2)F_{2m+1,p}(2n-2m) - p - 1$$
$$+ \sum_{i=1}^{m-1}\Big((p^2-p+1)F_{2m+1,p}(2n-2i+1) - (p^2-4p+1)F_{2m+1,p}(2n-2i)\Big)\Big].$$

□

Corollary 3. *Let $k \geqslant 2$, $n \geqslant \frac{k-2}{2}$, $t \geqslant 2$ be integers and $p \geqslant 1$ be a rational number. If*

1. *$k = 2$, $p = 1$ then $\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1$. (E. Lucas [19])*
2. *$k = 2$, $p = \frac{3}{2}$ then $\sum_{i=0}^{n} P_{2i} = \frac{1}{2}[P_{2n+1} - 1]$. (T. Koshy [17])*
3. *$k = 2$, $p = \frac{t}{2}$ then $\sum_{i=0}^{n} F_{2p-1,2i} = \frac{1}{2p-1}\big[F_{2p-1,2n+1} - 1\big]$. (S. Falcón, Á. Plaza [14])*

If we put $p = 1$ in Theorem 3, by simple calculations we obtain a new identity for Fibonacci s-numbers.

Corollary 4. *Let $s \geqslant 1$, $n \geqslant 1$ be integers. Then*

$$\sum_{i=1}^{n} F_s(2i) = \begin{cases} \frac{1}{3}\left[2F_s(2n+s+1) - 2 + \sum_{i=1}^{s}(-1)^{i+1}F_s(2n+i)\right], & s \text{ is even}, \\ \sum_{i=1}^{s}(-1)^{i+1}F_s(2n+i) - 1, & s \text{ is odd}. \end{cases}$$

Using Theorem 4 and the equality $\sum_{i=1}^{n} F_s(i) = F_s(n+s+1) - 1$ we obtain a sum of the first $n-1$ even terms of $F_s(n)$.

Corollary 5. *Let $s \geqslant 1$, $n \geqslant 1$ be integers. Then*

$$\sum_{i=0}^{n-1} F_s(2i+1) = \begin{cases} \frac{1}{3}\left[F_s(2n+s+1) - 1 + \sum_{i=1}^{s}(-1)^{i}F_s(2n+i)\right], & s \text{ is even}, \\ \sum_{i=1}^{s+1}(-1)^{i}F_s(2n+i), & s \text{ is odd}. \end{cases}$$

Using $\sum_{i=0}^{2n} F_{k,p}(i) = \sum_{i=0}^{n} F_{k,p}(2i) + \sum_{i=0}^{n-1} F_{k,p}(2i+1)$ and Theorems 2 and 3 we obtain the next identity for $F_{k,p}(n)$.

Theorem 4. *Let $k \geqslant 2$, $m \geqslant 1$, n be integers and $p \geqslant 1$ be a rational number. Let $S_{2n-1} = \sum_{i=0}^{n-1} F_{k,p}(2i+1)$. If $k = 2m$, then for $n \geqslant m-1$*

$$S_{2n-1} = \frac{1}{2p-1}\left[F_{2m,p}(2n) + \sum_{i=1}^{m-1}\Big((p-1)F_{2m,p}(2n-2i+1) + F_{2m,p}(2n-2i)\Big)\right].$$

If $k = 2m+1$, then for $n \geqslant m$

$$\begin{aligned} S_{2n-1} = \frac{1}{3(2p-1)} \Big[& 3F_{2m+1,p}(2n+2) + (2-4p)F_{2m+1,p}(2n+1) \\ & + (p-2)^2 F_{k,p}(2n-2m+2) + (p-2)F_{2m+1,p}(2n-2m+1) + p - 2 \\ & + \sum_{i=1}^{m-1}\Big((p^2-p+1)F_{2m+1,p}(2n-2i+2) - (p^2-4p+1)F_{2m+1,p}(2n-2i+1)\Big)\Big]. \end{aligned}$$

Corollary 6. *Let $k \geqslant 2$, $n \geqslant \frac{k-2}{2}$, $t \geqslant 2$ be integers and $p \geqslant 1$ be a rational number. If*

1. $k = 2$, $p = 1$ *then* $\sum_{i=0}^{n-1} F_{2i+1} = F_{2n}$. *(E. Lucas [19])*
2. $k = 2$, $p = \frac{3}{2}$ *then* $\sum_{i=0}^{n-1} P_{2i+1} = \frac{1}{2}P_{2n}$. *(T. Koshy [17])*
3. $k = 2$, $p = \frac{t}{2}$ *then* $\sum_{i=0}^{n-1} F_{2p-1,2i+1} = \frac{1}{2p-1}F_{2p-1,2n}$. *(S. Falcón, Á. Plaza [4])*

For more identities of the (k,p)-Fibonacci numbers see [12].

3. Matrix Generator of (*k*, *p*)-Fibonacci Numbers

In the last few decades, miscellaneous affinities between matrices and linear recurrences were studied, see, for instance [21,25]. The main aim is to obtain numbers defined by recurrences of matrices which are called generating matrices.

For the classical Fibonacci numbers, the matrix generator has the following form $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and it is well-known that for $n \geqslant 2$ we have $Q^n = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$, (see, for example, [21]). This generator gives the well known Cassini formula for the Fibonacci numbers, namely

$$\det Q^n = (-1)^n = F_n F_{n-2} - F_{n-1}^2.$$

For Pell numbers, the matrix generator has the form $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and it is easly established that $M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$, (see, for example, [25]).

In [8] the matrix generator for distance Fibonacci numbers was introduced. Using this idea we introduce the matrix generator for (k, p)-Fibonacci numbers, which generalizes the matrix generator for Fibonacci numbers and Pell numbers, simultaneously.

Let $Q_k = [q_{ij}]_{k\times k}$. For a fixed $1 \leqslant i \leqslant k$ an element q_{i1} is equal to the coefficient of $F_{k,p}(n-i)$ in the Equality (2). Moreover for $j \geqslant 2$ we have

$$q_{ij} = \begin{cases} 1, & \text{if } j = i+1 \\ 0, & \text{otherwise} \end{cases}.$$

For $k = 2, 3, 4$ we obtain matrices

$$Q_2 = \begin{bmatrix} 2p-1 & 1 \\ 1 & 0 \end{bmatrix},\ Q_3 = \begin{bmatrix} p & 1 & 0 \\ p-1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\ Q_4 = \begin{bmatrix} p & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p-1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, for $k > 2$ we have

$$Q_k = \begin{bmatrix} p & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

If $k = 2$ and $p = 1$, then Q_2 is the matrix generator for Fibonacci numbers. If $k = 2$ and $p = \frac{3}{2}$, then Q_2 is the matrix generator for Pell numbers. The matrix Q_k will be named as the companion matrix of the (k, p)-Fibonacci numbers or the (k, p)-Fibonacci matrix. Let A_k be the matrix of initial conditions. Then

$$A_k = \begin{bmatrix} F_{k,p}(2k-2) & F_{k,p}(2k-3) & \cdots & F_{k,p}(k-1) \\ F_{k,p}(2k-3) & F_{k,p}(2k-4) & \cdots & F_{k,p}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(k-1) & F_{k,p}(k-2) & \cdots & F_{k,p}(0) \end{bmatrix}_{k\times k}.$$

Theorem 5. *Let $k \geqslant 2$, $n \geqslant 1$ be integers and $p \geqslant 1$ be a rational number. Then*

$$A_k Q_k^n = \begin{bmatrix} F_{k,p}(n+2k-2) & F_{k,p}(n+2k-3) & \cdots & F_{k,p}(n+k-1) \\ F_{k,p}(n+2k-3) & F_{k,p}(n+2k-4) & \cdots & F_{k,p}(n+k-2) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(n+k-1) & F_{k,p}(n+k-2) & \cdots & F_{k,p}(n) \end{bmatrix}.$$

Proof. (by induction on n). Let k, n, p be as in the statement of the theorem. If $n = 1$ then

$$A_kQ_k = \begin{bmatrix} F_{k,p}(2k-2) & F_{k,p}(2k-3) & \cdots & F_{k,p}(k-1) \\ F_{k,p}(2k-3) & F_{k,p}(2k-4) & \cdots & F_{k,p}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(k) & F_{k,p}(k-1) & \cdots & F_{k,p}(1) \\ F_{k,p}(k-1) & F_{k,p}(k-2) & \cdots & F_{k,p}(0) \end{bmatrix} \cdot \begin{bmatrix} p & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} pF_{k,p}(2k-2) + (p-1)F_{k,p}(k) + F_{k,p}(k-1) & F_{k,p}(2k-2) & \cdots & F_{k,p}(k) \\ pF_{k,p}(2k-3) + (p-1)F_{k,p}(k-1) + F_{k,p}(k-2) & F_{k,p}(2k-3) & \cdots & F_{k,p}(k-1) \\ \vdots & \vdots & \ddots & \vdots \\ pF_{k,p}(k) + (p-1)F_{k,p}(2) + F_{k,p}(1) & F_{k,p}(k) & \cdots & F_{k,p}(2) \\ pF_{k,p}(k-1) + (p-1)F_{k,p}(1) + F_{k,p}(0) & F_{k,p}(k-1) & \cdots & F_{k,p}(1) \end{bmatrix},$$

so

$$A_kQ_k = \begin{bmatrix} F_{k,p}(2k-1) & F_{k,p}(2k-2) & \cdots & F_{k,p}(k+1) & F_{k,p}(k) \\ F_{k,p}(2k-2) & F_{k,p}(2k-3) & \cdots & F_{k,p}(k) & F_{k,p}(k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{k,p}(k+1) & F_{k,p}(k) & \cdots & F_{k,p}(3) & F_{k,p}(2) \\ F_{k,p}(k) & F_{k,p}(k-1) & \cdots & F_{k,p}(2) & F_{k,p}(1) \end{bmatrix}.$$

Assume now that the formula is true for all integers $1, 2, \cdots, n$. We shall show that

$$A_kQ_k^{n+1} = \begin{bmatrix} F_{k,p}(n+2k-1) & F_{k,p}(n+2k-2) & \cdots & F_{k,p}(n+k) \\ F_{k,p}(n+2k-2 & F_{k,p}(n+2k-3) & \cdots & F_{k,p}(n+k-1) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(n+k) & F_{k,p}(n+k-1) & \cdots & F_{k,p}(n+1) \end{bmatrix}.$$

Since $A_kQ_k^{n+1} = \left(A_kQ_k^n\right)Q_k$, so by induction hypothesis and from the recurrence Formula (2) we obtain that $A_kQ_k^{n+1}$ is equal to

$$\begin{bmatrix} F_{k,p}(n+2k-2) & F_{k,p}(n+2k-3) & \cdots & F_{k,p}(n+k-1) \\ F_{k,p}(n+2k-3) & F_{k,p}(n+2k-4) & \cdots & F_{k,p}(n+k-2) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(n+k) & F_{k,p}(n+k-1) & \cdots & F_{k,p}(n+1) \\ F_{k,p}(n+k-1) & F_{k,p}(n+k-2) & \cdots & F_{k,p}(n) \end{bmatrix} \begin{bmatrix} p & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 1 \\ 1 & 0 & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} pF_{k,p}(n+2k-2) + (p-1)F_{k,p}(n+k) + F_{k,p}(n+k-1) & \cdots & F_{k,p}(n+k) \\ pF_{k,p}(n+2k-3) + (p-1)F_{k,p}(n+k-1) + F_{k,p}(n+k-2) & \cdots & F_{k,p}(n+k-1) \\ \vdots & \ddots & \vdots \\ pF_{k,p}(n+k) + (p-1)F_{k,p}(n+2) + F_{k,p}(n+1) & \cdots & F_{k,p}(n+2) \\ pF_{k,p}(n+k-1) + (p-1)F_{k,p}(n+1) + F_{k,p}(n) & \cdots & F_{k,p}(n+1) \end{bmatrix}$$

$$= \begin{bmatrix} F_{k,p}(n+2k-1) & F_{k,p}(n+2k-2) & \cdots & F_{k,p}(n+k+1) & F_{k,p}(n+k) \\ F_{k,p}(n+2k-2 & F_{k,p}(n+2k-3) & \cdots & F_{k,p}(n+k) & F_{k,p}(n+k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{k,p}(n+k+1) & F_{k,p}(n+k) & \cdots & F_{k,p}(n+3) & F_{k,p}(n+2) \\ F_{k,p}(n+k) & F_{k,p}(n+k-1) & \cdots & F_{k,p}(n+2) & F_{k,p}(n+1) \end{bmatrix},$$

which ends the proof. □

Theorem 6. *Let $k \geqslant 2$, $n \geqslant 1$ be integers. Then for an arbitrary rational $p \geqslant 1$ holds*

$$\det Q_k = (-1)^{k+1}, \tag{10}$$

$$\det A_k = (-1)^{\frac{(k+1)(k-2)-2}{2}}, \tag{11}$$

$$\det(A_k Q_k^n) = (-1)^{\frac{(k+1)(k+2n-2)-2}{2}}. \tag{12}$$

Proof. Let $k \geqslant 2$ be an integer. We prove only (11). Using the recurrence (2) and the initial conditions for (k,p)-Fibonacci numbers, we obtain

$$\det A_k = \begin{vmatrix} F_{k,p}(2k-2) & F_{k,p}(2k-3) & \dots & F_{k,p}(k-1) \\ F_{k,p}(2k-3) & F_{k,p}(2k-4) & \dots & F_{k,p}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(k-1) & F_{k,p}(k-2) & \dots & F_{k,p}(0) \end{vmatrix}$$

$$= \begin{vmatrix} pF_{k,p}(2k-3)+(p-1)F_{k,p}(k-1)+F_{k,p}(k-2) & F_{k,p}(2k-3) & \dots & F_{k,p}(k-1) \\ pF_{k,p}(2k-4)+(p-1)F_{k,p}(k-2)+F_{k,p}(k-3) & F_{k,p}(2k-4) & \dots & F_{k,p}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ p^{k-2} & p^{k-3} & \dots & F_{k,p}(0) \end{vmatrix}$$

$$= \begin{vmatrix} F_{k,p}(k-2) & pF_{k,p}(2k-4)+(p-1)F_{k,p}(k-2)+F_{k,p}(k-3) & \dots & F_{k,p}(k-1) \\ F_{k,p}(k-3) & pF_{k,p}(2k-5)+(p-1)F_{k,p}(k-3)+F_{k,p}(k-4) & \dots & F_{k,p}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,p}(0) & p^{k-2} & \dots & F_{k,p}(1) \\ 0 & p^{k-3} & \dots & F_{k,p}(0) \end{vmatrix}$$

$$= \begin{vmatrix} p^{k-3} & F_{k,p}(k-3) & \dots & F_{k,p}(k-1) \\ p^{k-4} & F_{k,p}(k-4) & \dots & F_{k,p}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & F_{k,p}(0) & \dots & F_{k,p}(2) \\ 0 & 0 & \dots & F_{k,p}(1) \\ 0 & 0 & \dots & F_{k,p}(0) \end{vmatrix} = \dots = \begin{vmatrix} p^{k-3} & p^{k-4} & \dots & 1 & 0 & p^{k-2} \\ p^{k-4} & p^{k-5} & \dots & 0 & 0 & p^{k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 & p \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{vmatrix}.$$

Applying Laplace expansion on the last row we obtain

$$\det A_k = 1 \cdot (-1)^{k+k-1} \cdot \begin{vmatrix} p^{k-3} & p^{k-4} & \dots & 1 & p^{k-2} \\ p^{k-4} & p^{k-5} & \dots & 0 & p^{k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p & 1 & \dots & 0 & p^2 \\ 1 & 0 & \dots & 0 & p \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

Expanding one more time along the last row, we have

$$
\begin{aligned}
\det A_k &= (-1)^{2k-1} \cdot (-1)^{k-1+k-1} \cdot \begin{vmatrix} p^{k-3} & p^{k-4} & \dots & 1 \\ p^{k-4} & p^{k-5} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{vmatrix} \\
&= (-1)^{4k-3} \cdot \begin{vmatrix} p^{k-3} & p^{k-4} & \dots & 1 \\ p^{k-4} & p^{k-5} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{vmatrix}.
\end{aligned}
$$

Applying Laplace expansion on the last column we obtain

$$
\begin{aligned}
\det A_k &= (-1)^{4k-3} \cdot (-1)^{k-2+1} \cdot (-1)^{k-3+1} \cdot \dots \cdot (-1)^{3+1} \cdot \begin{vmatrix} p & 1 \\ 1 & 0 \end{vmatrix} \\
&= (-1)^{4k-3} \cdot (-1)^{k-1} \cdot (-1)^{k-2} \cdot \dots \cdot (-1)^{4} \cdot (-1) \\
&= (-1)^{4k-2} (-1)^{\frac{(k+3)(k-4)}{2}} = (-1)^{\frac{(k+1)(k-2)-10}{2}} = (-1)^{\frac{(k+1)(k-2)-2}{2}-4} \\
&= (-1)^{\frac{(k+1)(k-2)-2}{2}},
\end{aligned}
$$

which ends the proof. $\square$

4. Conclusions

In this paper we studied (k, p)-Fibonacci numbers which generalize, among others, Fibonacci numbers, Pell numbers and Narayana numbers. We presented properties of this numbers, including their generating function and matrix representation. It is interesting that the results obtained for the (k, p)-Fibonacci numbers generalize, among others, the results presented in Falcón et al. (2007), Koshy (2001) and (2014), Kwaśnik et al. (2000), Ramírez et al. (2015) and Stakhov (1977).

Based on the suggestion of the reviewer, it seems to be interesting to open a new direction of research by the assumption that the parameter p in the Equality (2) is a real number. Then some interesting results related to the characteristic equation of the sequence recurrence relations can be studied and the explicit form of these numbers perhaps will be obtained.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author wishes to thank referees for the constructive and insightful comments, which have helped to substantially improve the manuscript.

Conflicts of Interest: The author declares no conflict of interest.

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