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Sharp Inequalities for the Hardy–Littlewood Maximal Operator on Finite Directed Graphs

Xiao Zhang ¹, Feng Liu ^{1,*} and Huiyun Zhang ²

¹ College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, China; Xzhang@sdust.edu.cn

² College of Electronic and Information Engineering, Shandong University of Science and Technology, Qingdao 266590, China; zhyphys@sdust.edu.cn

* Correspondence: Fliu@sdust.edu.cn

Abstract: In this paper, we introduce and study the Hardy–Littlewood maximal operator $M_{\vec{G}}$ on a finite directed graph $\vec{G}$. We obtain some optimal constants for the ℓ^p norm of $M_{\vec{G}}$ by introducing two classes of directed graphs.

Keywords: finite directed graphs; Hardy–Littlewood maximal operator; sharp constants; Lebesgue estimates; ℓ^p -norm

MSC: 42B25; 05C20; 05C12



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1. Introduction

The best constants for the Hardy–Littlewood maximal inequalities have always been a challenging topic of research. In 1997, Grafakos and Montgomery-Smith [1] first obtained the sharp $L^p(\mathbb{R})$ ($1 < p < \infty$) norm for the one-dimensional uncentered Hardy–Littlewood maximal operator. Since then, the best constants for Hardy–Littlewood maximal operator have been studied extensively. See [2] for the sharp $L^p(\mathbb{R})$ ($1 < p < \infty$) norm of the one-dimensional centered Hardy–Littlewood maximal operator as well as [3–9] for the optimal constants on the weak $(1, 1)$ norm of the centered Hardy–Littlewood maximal operator. Recently, Soria and Tradacete [10] studied the sharp ℓ^p -norm for the Hardy–Littlewood maximal operators on finite connected graphs. It should be pointed out that geometric structure of a graph plays an important role in studying maximal operators on graphs. Given the significance of this operator, it is an interesting and natural question to ask what happens when we consider the directed graphs. It is the purpose of this paper to investigate the optimal constants for the ℓ^p norm of the Hardy–Littlewood maximal operator in directed graph setting.

Let us now recall some known notations, definitions and backgrounds. Let $G = (V, E)$ be an undirected combinatorial graph with the set of vertices V and the set of edges E . Two vertices $x, y \in V$ are called neighbors if they are connected by an edge $x \sim y \in E$. For a $v \in V$, we use the notation $N_G(v)$ to denote the set of neighbors of v . We say that G is finite if $|V| < \infty$. Here the notation $|A|$ represents the cardinality of A for each subset $A \subset V$. The graph G is called connected if for any distinct $x, y \in V$, there is a finite sequence of vertices $\{x_i\}_{i=0}^k$, $k \in \mathbb{N}$, such that $x = x_0 \sim x_1 \sim \cdots \sim x_k = y$. Let d_G be the metric induced by the edges in E , i.e., given $u, v \in V$, the distance $d_G(u, v)$ is the number of edges in the shortest path connecting u and v . Let $B_G(v, r)$ be the ball centered at v , with radius r on the graph G , i.e.,

$$B_G(v, r) = \{u \in V : d_G(u, v) \leq r\}.$$

For example, $B_G(v, r) = \{v\}$ if $0 \leq r < 1$ and $B_G(v, r) = \{v\} \cup N_G(v)$ if $1 \leq r < 2$. For a function $f : V \rightarrow \mathbb{R}$, the Hardy–Littlewood maximal operator M_G on G is defined as

$$M_G f(v) = \sup_{r \geq 0} \frac{1}{|B_G(v, r)|} \sum_{w \in B_G(v, r)} |f(w)|.$$

If G has n ($n \geq 2$) vertices, the maximal operator M_G can be rewritten by

$$M_G f(v) = \max_{k=0, \dots, n-1} \frac{1}{|B_G(v, k)|} \sum_{w \in B_G(v, k)} |f(w)|.$$

Over the last several years the Hardy–Littlewood maximal operator on graphs has been studied by many authors (see [10–16]). The Hardy–Littlewood maximal operator on graphs was first introduced and studied by Korányi and Picardello [15] who used the above operator to explore the boundary behavior of eigenfunctions of the Laplace operator on trees. Subsequently, Cowling, Meda and Setti [12] studied the Hardy–Littlewood maximal operator on homogeneous trees. Later, some weighted norm inequalities for the Hardy–Littlewood maximal operators on infinite graphs were investigated by Badr and Martell [11]. Recently, Soria and Tradacete [10] studied the best constants for the ℓ^p -norm of the Hardy–Littlewood maximal operators on finite connected graphs. Later, Soria and Tradacete [16] investigated some different geometric properties on infinite graphs, related to the weak-type boundedness of the Hardy–Littlewood maximal operator on infinite connected graphs. One can consult [13,14] for the variation properties of the Hardy–Littlewood maximal operator on finite connected graphs.

We now introduce the ℓ^p spaces on graphs.

Definition 1 ($\ell^p(V)$ space). Let $G = (V, E)$ be a graph with the set of vertices V and the set of edges E . For $0 < p \leq \infty$, let $\ell^p(V)$ be the set of all functions $f : V \rightarrow \mathbb{R}$ satisfying $\|f\|_{\ell^p(V)} < \infty$, where $\|f\|_{\ell^p(V)} = (\sum_{v \in V} |f(v)|^p)^{1/p}$ for all $0 < p < \infty$ and $\|f\|_{\ell^\infty(V)} = \sup_{v \in V} |f(v)|$.

By Hölder’s inequality, we have

$$\|f\|_{\ell^q(V)} \leq \|f\|_{\ell^p(V)} \leq |V|^{1/p-1/q} \|f\|_{\ell^q(V)}, \quad \text{for } 0 < p < q \leq \infty. \quad (1)$$

On the other hand, it is easy to see that $|f(v)| \leq M_G f(v) \leq \|f\|_{\ell^\infty(V)}$ for all $v \in V$. This together with (1) yields that

$$\|f\|_{\ell^p(V)} \leq \|M_G f\|_{\ell^p(V)} \leq |V|^{1/p} \|f\|_{\ell^\infty(V)} \leq |V|^{1/p} \|f\|_{\ell^p(V)}, \quad \text{for } 0 < p \leq \infty. \quad (2)$$

Therefore, the ℓ^p -boundedness for M_G is trivial. Moreover, it follows from (2) that

$$1 \leq \sup_{\|f\|_{\ell^p(V)} \neq 0} \frac{\|M_G f\|_{\ell^p(V)}}{\|f\|_{\ell^p(V)}} \leq |V|^{1/p}, \quad \text{for } 0 < p \leq \infty.$$

In [10], among other things, Soria and Tradacete studied the sharp constants

$$\|M_G\|_p := \sup_{\|f\|_{\ell^p(V)} \neq 0} \frac{\|M_G f\|_{\ell^p(V)}}{\|f\|_{\ell^p(V)}}, \quad \text{for } 0 < p \leq \infty.$$

Precisely, they established the following result.

Theorem 1 ([10]). Let $n \geq 2$.

(i) Let $0 < p \leq 1$. Then, for any graph G with n vertices, we have

$$\left(1 + \frac{n-1}{n^p}\right)^{1/p} \leq \|M_G\|_p \leq \left(1 + \frac{n-1}{2^p}\right)^{1/p}.$$

Moreover,

(a) $\|M_G\|_p = \left(1 + \frac{n-1}{n^p}\right)^{1/p}$ if and only if $G = K_n$. Here K_n denotes the complete graph with n vertices, i.e., $|N_{K_n}(v)| = n-1$ for any $v \in V$.

(b) $\|M_G\|_p = \left(1 + \frac{n-1}{2^p}\right)^{1/p}$ if and only if G is isomorphic to S_n . Here S_n denotes the star graph of n vertices, i.e., there exists a unique $v \in V$ such that $|N_{S_n}(v)| = n-1$ and $|N_{S_n}(w)| = 1$ for every $w \in V \setminus \{v\}$.

(ii) Let $1 < p < \infty$, then

$$\left(1 + \frac{n-1}{n^p}\right)^{1/p} \leq \|M_{K_n}\|_p \leq \left(1 + \frac{n-1}{n}\right)^{1/p},$$

and

$$\left(1 + \frac{n-1}{2^p}\right)^{1/p} \leq \|M_{S_n}\|_p \leq \left(\frac{n+5}{2}\right)^{1/p}.$$

The main motivation of this paper is to extend Theorem A to the directed graph setting. Let $\vec{G} = (V, E)$ be a finite graph with the set of vertices V and the set of edges E . Given an edge $u \sim v \in E$, if $u \rightarrow v$, we say that v (resp., u) is a right (resp., left) neighbor of u (resp., v). Then we write $u \sim v = u \rightarrow v$. For $v \in V$, we denote by $N_{\vec{G},+}(v)$ (resp., $N_{\vec{G},-}(v)$) the set of right (resp., left) neighbors of v . We say that the graph $\vec{G}$ is a directed graph if every edge in E has only a unique direction and $N_{\vec{G},+}(v) \cup N_{\vec{G},-}(v) \neq \emptyset$ for all $v \in V$. The directed graph $\vec{G}$ is called connected if for any distinct $x, y \in V$, there is a finite sequence of vertices $\{x_i\}_{i=0}^k$, $k \in \mathbb{N}$, such that $x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_k = y$.

In what follows, we always assume that the graph $\vec{G} = (V, E)$ with the set of vertices V and the set of edges E . Let $B_{\vec{G}}(v, r)$ be the ball centered at v , with radius r on the graph $\vec{G}$, equipped with the metric $d_{\vec{G}}$ induced by the edges in E , i.e., given $u, v \in V$, the distance $d_{\vec{G}}(u, v)$ is the number of edges in a shortest path connecting from u to v , and

$$B_{\vec{G}}(v, r) = \{u \in V : d_{\vec{G}}(v, u) \leq r\}.$$

For example, $B_{\vec{G}}(v, r) = \{v\}$ if $0 \leq r < 1$ and $B_{\vec{G}}(v, r) = \{v\} \cup N_{\vec{G},+}(v)$ if $1 \leq r < 2$. For a function $f : V \rightarrow \mathbb{R}$, we consider the Hardy–Littlewood maximal operator on $\vec{G}$

$$M_{\vec{G}}f(v) = \sup_{r \geq 0} \frac{1}{|B_{\vec{G}}(v, r)|} \sum_{w \in B_{\vec{G}}(v, r)} |f(w)|.$$

Naturally, when $|V| = n$, the maximal operator $M_{\vec{G}}$ can be redefined in the way that

$$M_{\vec{G}}f(v) = \max_{k=0, \dots, n-1} \frac{1}{|B_{\vec{G}}(v, k)|} \sum_{w \in B_{\vec{G}}(v, k)} |f(w)|.$$

There are some remarks as follows:

Remark 1. (i) This type of operator $M_{\vec{G}}$ has its roots in the ergodic theory in infinite directed graph setting. More precisely, let $\vec{G}_1 = (V_1, E_1)$, where $V_1 = \mathbb{Z}$ and $E_1 = \{i \rightarrow i+1 : i \in \mathbb{Z}\}$.

Then $M_{\vec{G}_1}$ is the usual one-dimensional one-sided discrete Hardy–Littlewood maximal operator M_1 , i.e.,

$$M_1 f(n) = \sup_{r \in \{0,1,\dots\}} \frac{1}{r+1} \sum_{k=0}^r |f(n+k)|, \quad n \in \mathbb{Z}.$$

This type of maximal operator M_1 first arose in Dunford and Schwartz's work [17] and was studied by Calderón [18].

- (ii) It was pointed out in [10] that the complete graph K_n whose maximal operator M_{K_n} is the smallest in the pointwise ordering among all graphs with $n \geq 2$, but there is no graph G whose maximal operator M_G is the largest in the pointwise ordering among all graphs with $n \geq 2$. In Section 2, we point out that there is no directed graph $\vec{G}$ whose maximal operator $M_{\vec{G}}$ is the smallest or largest in the pointwise ordering among all graphs with $n \geq 2$ vertices, which is different from M_G .
- (iii) It should be pointed out that as with M_G , the maximal operator $M_{\vec{G}}$ completely determines the graph $\vec{G}$ (see Proposition 2).

It is not difficult to see that

$$|f(v)| \leq M_{\vec{G}} f(v) \leq \|f\|_{\ell^\infty(V)}, \quad \text{for all } v \in V,$$

which together with (1) leads to $\|M_{\vec{G}}\|_\infty = 1$ and

$$1 \leq \|M_{\vec{G}}\|_p \leq |V|^{1/p}, \quad \text{for } 0 < p < \infty. \quad (3)$$

Based on (3) and Theorem A, finding the sharp ℓ^p -norm of $M_{\vec{G}}$ is a certainly interesting issue, which is the main motivation of this work. In Section 3 we shall introduce the outward star graph $\vec{S}_{O,n}$ and the inward star graph $\vec{S}_{I,n}$ and prove that $\vec{S}_{O,n}$ and $\vec{S}_{I,n}$ are the extremal directed graphs attaining, which completely determine the lower and upper estimates of the ℓ^p -norm for $M_{\vec{G}}$ in the case $0 < p \leq 1$, respectively (see Theorem 2). We also claim that the ℓ^p -norm of $M_{\vec{G}}$ cannot determine the graph $\vec{G}$ (see Proposition 3). In Section 4, we consider the ℓ^p -norm for $M_{\vec{G}}$ in the case $1 < p < \infty$. Actually, the case $1 < p < \infty$ is more complicated than the case $0 < p \leq 1$, even in the finite undirected graph setting. However, some positive results are discussed. In particular, some sharp estimates of restricted type are given in Section 4.

2. General Properties for $M_{\vec{G}}$

It was pointed out in [10] that there exists a smallest operator M_{K_n} , in the pointwise ordering, among all M_G , with G a graph of n vertices. However, there is no directed graph $\vec{G}$ whose maximal operator $M_{\vec{G}}$ is the smallest or largest in the pointwise ordering among all graphs with $n \geq 2$ vertices, which can be seen by the following result.

Proposition 1. Let $\vec{G} = (V, E)$ be a directed connected graph with $n \geq 2$ vertices. Then

- (i) There exist $j \in V$, a function $f : V \rightarrow \mathbb{R}$ and another directed graph $\vec{G}_1 = (V, E_1)$ with $E_1 \neq E$ such that $M_{\vec{G}} f(j) > M_{\vec{G}_1} f(j)$;
- (ii) There exist $j \in V$, a function $f : V \rightarrow \mathbb{R}$ and another directed graph $\vec{G}_2 = (V, E_2)$ with $E_2 \neq E$ such that $M_{\vec{G}} f(j) < M_{\vec{G}_2} f(j)$.

Proof. At first, we prove (i). When $n = 2$, let $\vec{G} = (V, E)$ with $V = \{u, v\}$ and $E = \{u \rightarrow v\}$. Let us consider the function $f : V \rightarrow \mathbb{R}$ with $f(u) = 1$ and $f(v) = 3$, and $\vec{G}_1 = (V, E_1)$ with $E_1 = \{v \rightarrow u\}$. It is clear that $M_{\vec{G}} f(u) = 2$, $M_{\vec{G}_1} f(u) = 1$. This gives $M_{\vec{G}} f(j) > M_{\vec{G}_1} f(j)$ by taking $j = u$. When $n \geq 3$. There exists a vertex $u \in V$ such that $N_{\vec{G},+}(u) \neq \emptyset$. Let us consider the function $f : V \rightarrow \mathbb{R}$ with $f(u) = 1$ and $f(v) = 2$ for all $v \in N_{\vec{G},+}(u)$, and $\vec{G}_1 = (V, E_1)$ be a directed graph with $N_{\vec{G}_1,+}(u) = \emptyset$. It is clear that

$M_{\vec{G}}f(u) \geq \frac{2|N_{\vec{G},+}(u)|+1}{|N_{\vec{G},+}(u)|+1} > 1$ and $M_{\vec{G}_1}f(u) = 1$. This gives $M_{\vec{G}}f(j) > M_{\vec{G}_1}f(j)$ by taking $j = u$.

Now we prove (ii). When $n = 2$, let $\vec{G} = (V, E)$ with $V = \{u, v\}$ and $E = \{u \rightarrow v\}$. Let us consider the function $f : V \rightarrow \mathbb{R}$ with $f(v) = 1$ and $f(u) = 3$, and $\vec{G}_2 = (V, E_2)$ with $E_2 = \{v \rightarrow u\}$. It is clear that $M_{\vec{G}}f(v) = 1$, $M_{\vec{G}_2}f(v) = 2$. This gives $M_{\vec{G}}f(j) < M_{\vec{G}_2}f(j)$ by taking $j = v$. When $n \geq 3$. There exists a vertex $u \in V$ such that $|N_{\vec{G},+}(u)| \geq 2$. Let $w \in V \setminus \{u\}$. Let us consider the function $f : V \rightarrow \mathbb{R}$ with $f(w) = 2$ and $f(v) = 1$ for all $v \in V \setminus \{w\}$, and $\vec{G}_2 = (V, E_2)$ be a directed graph with $N_{\vec{G}_2,+}(u) = \{w\}$ and $N_{\vec{G}_2,+}(w) = \emptyset$. It is clear that $M_{\vec{G}}f(u) \leq \frac{|N_{\vec{G},+}(u)|+2}{|N_{\vec{G},+}(u)|+1} < \frac{3}{2}$ and $M_{\vec{G}_2}f(u) = \frac{3}{2}$. This gives the claim (ii) by letting $j = u$. $\square$

Using the arguments similar to those used to derive the proof of Theorem 2.4 in [10], one can get the following properties for $M_{\vec{G}}$, which tells us that the operator $M_{\vec{G}}$ completely determines the graph $\vec{G}$.

Proposition 2. Let $\vec{G}_1 = (V, E_1)$ and $\vec{G}_2 = (V, E_2)$ be two directed graphs with $V = \{1, 2, \dots, n\}$. For $j \in V$, the function δ_j denotes the Kronecker delta function

$$\delta_j(i) = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Then the following are equivalent:

- (i) $E_1 = E_2$;
- (ii) $\vec{G}_1 = \vec{G}_2$;
- (iii) For every $f : \{1, \dots, n\} \rightarrow \mathbb{R}$, it holds that $M_{\vec{G}_1}f = M_{\vec{G}_2}f$;
- (iv) For every $k \in \{1, \dots, n\}$, it holds that $M_{\vec{G}_1}\delta_k = M_{\vec{G}_2}\delta_k$.

3. Optimal Estimates for $\|M_{\vec{G}}\|_p$ with $0 < p \leq 1$

In this section, we shall present some optimal estimates for $\|M_{\vec{G}}\|_p$ with $0 < p \leq 1$. To state the main results, the following lemma is needed.

Lemma 1. ([10]). Let $n \geq 2$ and $V = \{1, \dots, n\}$. Assume that $T : \ell^p(V) \rightarrow \ell^p(V)$ is a sublinear operator, with $0 < p \leq 1$. Then

$$\|T\|_p = \max_{k \in V} \|T\delta_k\|_{\ell^p(V)}.$$

Before stating our main results, let us introduce two classes of directed graphs. Let $\vec{S}_{I,n}$ be the inward star graph with n vertices, i.e., there exists a unique $u \in V$ such that $N_{\vec{S}_{I,n},+}(u) = \emptyset$ and $N_{\vec{S}_{I,n},+}(v) = \{u\}$ for all $v \in V \setminus \{u\}$. Let $\vec{S}_{O,n}$ be the outward star graph with n vertices, i.e., there exists a unique $u \in V$ such that $N_{\vec{S}_{O,n},+}(u) = V \setminus \{u\}$ and $N_{\vec{S}_{O,n},+}(v) = \emptyset$ for all $v \in V \setminus \{u\}$. Recall that two graphs $\vec{G}_1 = (V, E_1)$, $\vec{G}_2 = (V, E_2)$ are said to be isomorphic if there is a permutation of the vertices $\pi : V \rightarrow V$ such that $u \rightarrow v \in E_1$ if and only if $\pi(u) \rightarrow \pi(v) \in E_2$. In this case, we can write $\vec{G}_1 \sim \vec{G}_2$. Noting that if $\vec{G}_1 \sim \vec{G}_2$, then $M_{\vec{G}_1}f(v) = M_{\vec{G}_2}f(\pi(v))$ and $\|M_{\vec{G}_1}\|_p = \|M_{\vec{G}_2}\|_p$ for all $0 < p \leq \infty$. However, the converse is not true (see Proposition 3).

Theorem 2. Let $\vec{G} = (V, E)$ be a directed graph with $n \geq 2$ vertices. Then for $0 < p \leq 1$, the following optimal estimates hold:

$$\left(1 + \frac{1}{n^p}\right)^{1/p} \leq \|M_{\vec{G}}\|_p \leq \left(1 + \frac{n-1}{2^p}\right)^{1/p}. \quad (4)$$

Moreover,

- (i) $\|M_{\vec{G}}\|_p = (1 + \frac{n-1}{2^p})^{1/p}$ if and only if $\vec{G} \sim \vec{S}_{I,n}$;
- (ii) $\|M_{\vec{G}}\|_p = (1 + \frac{1}{n^p})^{1/p}$ if and only if $\vec{G} \sim \vec{S}_{O,n}$.

Proof. At first, we shall prove (4). Without loss of generality, we may assume that $V = \{1, \dots, n\}$. Invoking Lemma 4, one has

$$\|M_{\vec{G}}\|_p^p = \max_{k \in V} \sum_{j=1}^n (M_{\vec{G}} \delta_k(j))^p. \quad (5)$$

Fix $k \in [1, n]$, it is clear that $M_{\vec{G}} \delta_k(k) = 1$ and

$$M_{\vec{G}} \delta_k(j) = \max_{r \in \{1, \dots, n\}} \frac{1}{|B_{\vec{G}}(j, r)|} \sum_{u \in B_{\vec{G}}(j, r)} \delta_k(u) = \frac{\chi_{d_{\vec{G}}(j, k) \neq 0}}{|B_{\vec{G}}(j, d_{\vec{G}}(j, k))|} \quad (6)$$

for all $j \in V \setminus \{k\}$. Since $|B_{\vec{G}}(j, d_{\vec{G}}(j, k))| \geq 2$ when $d_{\vec{G}}(j, k) \geq 1$, then $M_{\vec{G}} \delta_k(j) \leq \frac{1}{2}$ for all $j \in V \setminus \{k\}$. This together with (5) implies that $\|M_{\vec{G}}\|_p \leq (1 + \frac{n-1}{2^p})^{1/p}$. Therefore, to prove (4), it is enough to show that

$$\|M_{\vec{G}}\|_p \geq \left(1 + \frac{1}{n^p}\right)^{1/p}. \quad (7)$$

Fix $k \in V$. If $d_{\vec{G}}(j, k) = 0$ for all $j \in V \setminus \{k\}$, then $M_{\vec{G}} \delta_k(j) = 0$ for all $j \in V \setminus \{k\}$ and $\sum_{j=1}^n (M_{\vec{G}} \delta_k(j))^p = 1$. If there exists $j_0 \in V \setminus \{k\}$ such that $d_{\vec{G}}(j_0, k) \geq 1$, then we get from (6) that $M_{\vec{G}} \delta_k(j_0) \geq \frac{1}{n}$ since $|B_{\vec{G}}(j_0, d_{\vec{G}}(j_0, k))| \leq n$. Then we have $\sum_{j=1}^n (M_{\vec{G}} \delta_k(j))^p \geq 1 + \frac{1}{n^p}$. Therefore, inequality (7) holds.

Next we prove part (i). Without loss of generality, we may assume that $\vec{S}_{I,n} = (V, E)$, where $V = \{1, \dots, n\}$ and $E = \{2 \rightarrow 1, \dots, n \rightarrow 1\}$. It is not difficult to see that $\|\delta_k\|_{\ell^p(V)} = 1$ for all $k \in V$ and $M_{\vec{S}_{I,n}} \delta_1(1) = 1$, $M_{\vec{S}_{I,n}} \delta_1(i) = \frac{1}{2}$ for all $i = 2, \dots, n$. Moreover, $M_{\vec{S}_{I,n}} \delta_k(k) = 1$, $M_{\vec{S}_{I,n}} \delta_k(i) = 0$ for all $i \in V \setminus \{k\}$ and $k \in \{2, \dots, n\}$. Hence, $\|M_{\vec{S}_{I,n}} \delta_1\|_{\ell^p(V)} = (1 + \frac{n-1}{2^p})^{1/p}$ and $\|M_{\vec{S}_{I,n}} \delta_k\|_{\ell^p(V)} = 1$ for all $k \in V \setminus \{1\}$. Invoking Lemma 1, we have

$$\|M_{\vec{S}_{I,n}}\|_p = \left(1 + \frac{n-1}{2^p}\right)^{1/p}, \quad \text{if } 0 < p \leq 1.$$

Assume that $\|M_{\vec{G}}\|_p = (1 + \frac{n-1}{2^p})^{1/p}$. By Lemma 1, we have

$$\max_{k \in V} \sum_{j=1}^n (M_{\vec{G}} \delta_k(j))^p = 1 + \frac{n-1}{2^p}.$$

We may assume without loss of generality that

$$\sum_{j=1}^n (M_{\vec{G}} \delta_1(j))^p = \max_{k \in V} \sum_{j=1}^n (M_{\vec{G}} \delta_k(j))^p.$$

This implies that

$$\sum_{j=1}^n (M_{\vec{G}} \delta_1(j))^p = 1 + \frac{n-1}{2^p}. \quad (8)$$

Noting that $M_{\vec{G}} \delta_1(1) = 1$ and $M_{\vec{G}} \delta_1(j) \leq \frac{1}{2}$ for all $j \in \{2, \dots, n\}$. This together with (8) yields that $M_{\vec{G}} \delta_1(j) = \frac{1}{2}$ for all $j \in \{2, \dots, n\}$, which is equivalent to that $N_{\vec{G},+}(j) = \{1\}$ for all $j \in \{2, \dots, n\}$. This leads to $\vec{G} \sim \vec{S}_{I,n}$ and finishes the proof of part (i).

It remains to prove part (ii). Without loss of generality, we may assume that $\overrightarrow{S_{O,n}} = (V, E)$, where $V = \{1, \dots, n\}$ and $E = \{1 \rightarrow 2, \dots, 1 \rightarrow n\}$. Clearly, $M_{\overrightarrow{S_{O,n}}} \delta_1(1) = 1$ and $M_{\overrightarrow{S_{O,n}}} \delta_1(i) = 0$ for all $i \in \{2, \dots, n\}$. For $k \in \{2, \dots, n\}$, we have that $M_{\overrightarrow{S_{O,n}}} \delta_k(k) = 1$, $M_{\overrightarrow{S_{O,n}}} \delta_k(1) = \frac{1}{n}$ and $M_{\overrightarrow{S_{O,n}}} \delta_k(i) = 0$ for all $i \in V \setminus \{1, k\}$. Clearly, $\|\delta_k\|_{\ell^p(V)} = 1$ for all $k \in V$ and $\|M_{\overrightarrow{S_{O,n}}} \delta_1\|_{\ell^p(V)} = 1$, $\|M_{\overrightarrow{S_{O,n}}} \delta_k\|_{\ell^p(V)} = (1 + n^{-p})^{1/p}$ for all $k \in \{2, \dots, n\}$. Invoking Lemma 1, we have

$$\|M_{\overrightarrow{S_{O,n}}}\|_p = (1 + n^{-p})^{1/p}, \text{ if } 0 < p \leq 1.$$

Assume that $\|M_{\vec{G}}\|_p = (1 + \frac{1}{n^p})^{1/p}$. We get by Lemma 1 that

$$\max_{k \in V} \sum_{j=1}^n (M_{\vec{G}} \delta_k(j))^p = 1 + \frac{1}{n^p}.$$

We may assume without loss of generality that

$$\sum_{j=1}^n (M_{\vec{G}} \delta_1(j))^p = \max_{k \in V} \sum_{j=1}^n (M_{\vec{G}} \delta_k(j))^p.$$

It follows that

$$\sum_{j=1}^n (M_{\vec{G}} \delta_1(j))^p = 1 + \frac{1}{n^p}. \quad (9)$$

Noting that $M_{\vec{G}} \delta_1(1) = 1$. Moreover, if $M_{\vec{G}} \delta_1(j) \neq 0$ for some $j \in \{2, \dots, n\}$, then $M_{\vec{G}} \delta_1(j) \geq \frac{1}{n}$. Therefore, from (9) we see that there exists $j_0 \in \{2, \dots, n\}$ such that $M_{\vec{G}} \delta_1(j_0) = \frac{1}{n}$ and $M_{\vec{G}} \delta_1(j) = 0$ for all $j \in V \setminus \{1, j_0\}$. Assume that there exist $i_1, i_2 \in V \setminus \{j_0\}$ such that $i_1 \rightarrow i_2 \in E$. In this case we have $M_{\vec{G}} \delta_{i_2}(i_2) = 1$, $M_{\vec{G}} \delta_{i_2}(i_1) = \frac{1}{2}$ and $M_{\vec{G}} \delta_{i_2}(j_0) = \frac{1}{n}$. Consequently,

$$\sum_{j=1}^n (M_{\vec{G}} \delta_{i_2}(j))^p \geq 1 + \frac{1}{2^p} + \frac{1}{n^p} > 1 + \frac{1}{n^p},$$

which is a contradiction. Hence, we have $N_{\vec{G},+}(j_0) = V \setminus \{j_0\}$ and $N_{\vec{G},+}(j) = \emptyset$ for $j \neq j_0$. So $\vec{G} \sim \overrightarrow{S_{O,n}}$. This completes the proof of part (ii). $\square$

It should be pointed out that parts (i) and (ii) in Theorem 2 show that the ℓ^p -norm of $M_{\vec{G}}$ can determine the property of graph $\vec{G}$. However, the following proposition tells us that the ℓ^p -norm of $M_{\vec{G}}$ cannot determine the concrete graph $\vec{G}$ generally.

Proposition 3. Let $0 < p \leq 1$. There exist two graphs $\vec{G}_1 = (V, E_1)$ and $\vec{G}_2 = (V, E_2)$ with $E_1 \neq E_2$ such that $\|M_{\vec{G}_1}\|_p = \|M_{\vec{G}_2}\|_p$.

Proof. Let $\vec{G}_1 = \vec{P}_n = (V, E_1)$, where $V = \{1, \dots, n\}$ and $E_1 = \{n \rightarrow n-1, n-1 \rightarrow n-2, \dots, 2 \rightarrow 1\}$. Given $k \in V$, one can easily check that

$$M_{\vec{P}_n} \delta_k(i) = \begin{cases} 0, & i = 1, \dots, k-1; \\ \frac{1}{i-k+1}, & i = k, \dots, n. \end{cases}$$

Then we get by Lemma 1 that

$$\|M_{\vec{P}_n}\|_p = \left(\sum_{i=1}^n i^{-p} \right)^{1/p}, \text{ if } 0 < p \leq 1.$$

Let $\vec{G}_2 = \vec{C}_n = (V, E_2)$, where $V = \{1, \dots, n\}$ and $E_2 = \{1 \rightarrow 2, 2 \rightarrow 3, \dots, n-1 \rightarrow n, n \rightarrow 1\}$. It is clear that $M_{\vec{C}_n} \delta_1(1) = 1$, $M_{\vec{C}_n} \delta_1(i) = \frac{1}{n+2-i}$ for all $i = 2, \dots, n$. Then we have

$$\|M_{\vec{C}_n} \delta_1\|_{\ell^p(V)}^p = 1 + \sum_{i=2}^n \frac{1}{(n+2-i)^p}.$$

Invoking Lemma 1, we have

$$\|M_{\vec{C}_n}\|_p = \left(\sum_{i=1}^n i^{-p} \right)^{1/p}, \quad \text{if } 0 < p \leq 1.$$

Observing that $\|M_{\vec{P}_n}\|_p = \|M_{\vec{C}_n}\|_p = (\sum_{i=1}^n i^{-p})^{1/p}$. This proves Proposition 3. $\square$

4. Optimal Estimates for $\|M_{\vec{G}}\|_p$ with $1 < p < \infty$

This section is devoted to presenting some positive results for the $\|M_{\vec{G}}\|_p$ with $1 < p < \infty$. Before formulating the main results, let us give the following observation, which is useful in our proof.

Lemma 2. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces and let $T : X \rightarrow Y$ be a sublinear operator, with $0 < p < \infty$. Then the following is valid:

$$\|T\|_{X \rightarrow Y} = \sup_{f: \|f\|_X \neq 0} \frac{\|Tf\|_Y}{\|f\|_X} = \sup_{f: \|f\|_X = 1} \|Tf\|_Y.$$

At first, we present the ℓ^p -norm for $M_{\vec{S}_{I,n}}$ with $1 < p < \infty$.

Theorem 3. Let $n \geq 2$.

(i) If $1 < p < \infty$, then

$$\left(1 + \frac{n-1}{2^p}\right)^{1/p} \leq \|M_{\vec{S}_{I,n}}\|_p < \left(1 + \frac{n-1}{2}\right)^{1/p}.$$

(ii) If $p = 2$, then

$$\|M_{\vec{S}_{I,n}}\|_2 = \max_{k \in [2, n]} \left(-\frac{(3x_0^2 - 2x_0 - 1)(k-1)}{4(n-k+1 + (k-1)x_0^2)} + 1 \right)^{1/2},$$

where

$$x_0 := \frac{2(k-1) - 3n + \sqrt{9n^2 - 8n(k-1)}}{2(k-1)}.$$

Proof. At first, we shall prove part (i). Without loss of generality, we may assume that $\vec{S}_{I,n} = (V, E)$, where $V = \{1, \dots, n\}$ and $E = \{2 \rightarrow 1, \dots, n \rightarrow 1\}$. Clearly, $\|\delta_1\|_{\ell^p(V)} = 1$ and $M_{\vec{S}_{I,n}} \delta_1(1) = 1$, $M_{\vec{S}_{I,n}} \delta_1(i) = \frac{1}{2}$ for all $i = 2, \dots, n$. It follows that $\|M_{\vec{S}_{I,n}} \delta_1\|_{\ell^p(V)} = (1 + \frac{n-1}{2^p})^{1/p}$, which gives

$$\|M_{\vec{S}_{I,n}}\|_p \geq \left(1 + \frac{n-1}{2^p}\right)^{1/p}, \quad \text{for } 1 < p < \infty.$$

We now prove

$$\|M_{\vec{S}_{I,n}}\|_p < \left(1 + \frac{n-1}{2}\right)^{1/p}, \quad \text{for } 1 < p < \infty. \quad (10)$$

Given a function $f = \sum_{i=1}^n a_i \delta_i$, we write

$$M_{S_{I,n}}^{\rightarrow} f(i) = \begin{cases} |a_1|, & \text{for } i = 1; \\ \max \left\{ |a_i|, \frac{1}{2}(|a_i| + |a_1|) \right\}, & \text{for } i = 2, \dots, n. \end{cases}$$

Invoking Lemma 2, one has

$$\begin{aligned} \|M_{S_{I,n}}^{\rightarrow}\|_p^p &= \max \left\{ |a_1|^p + \sum_{i=2}^n \max \left\{ |a_i|^p, \left(\frac{1}{2}(|a_i| + |a_1|) \right)^p \right\} : \sum_{i=1}^n |a_i|^p = 1 \right\} \\ &= \max \left\{ a_1^p + \sum_{i=2}^n \max \left\{ a_i^p, \left(\frac{1}{2}(a_i + a_1) \right)^p \right\} : \right. \\ &\quad \left. \sum_{i=1}^n a_i^p = 1, \quad a_i \geq 0, \quad i = 1, \dots, n \right\}. \end{aligned}$$

For a given sequence $\{a_i\}_{i=1}^n$ with $\sum_{i=1}^n a_i^p = 1$ and all $a_i \geq 0$ ($i = 1, \dots, n$). We set

$$N_1 := \{j \in \{2, \dots, n\} : a_j \geq a_1\}, \quad N_2 := \{j \in \{2, \dots, n\} : a_j < a_1\}.$$

By the Jensen's inequality we have $\left(\frac{a_i + a_1}{2}\right)^p < \frac{1}{2}(a_i^p + a_1^p)$ for all $i \in N_2$ since $1 < p < \infty$. Therefore, we have

$$\begin{aligned} &a_1^p + \sum_{i=2}^n \max \left\{ a_i^p, \left(\frac{1}{2}(a_i + a_1) \right)^p \right\} \\ &= a_1^p + \sum_{i \in N_1} a_i^p + \sum_{i \in N_2} \left(\frac{a_i + a_1}{2} \right)^p \\ &= 1 + \sum_{i \in N_2} \left(\left(\frac{a_i + a_1}{2} \right)^p - a_i^p \right) \\ &< 1 + \frac{1}{2} \sum_{i \in N_2} (a_1^p - a_i^p) < 1 + \frac{n-1}{2}. \end{aligned}$$

This proves (10).

Next, we prove part (ii). Let $f = \sum_{j=1}^n a_j \delta_j$ with each $a_j \geq 0$. If $a_j \geq a_1$ for all $j = 2, \dots, n$, then $M_{S_{I,n}}^{\rightarrow} f(i) = f(i)$ for all $i = 1, \dots, n$. It follows that $\|M_{S_{I,n}}^{\rightarrow} f\|_{\ell^2(V)} \leq \|f\|_{\ell^2(V)}$. Otherwise, there exists $j_0 \in \{2, \dots, n\}$ such that $a_{j_0} < a_1$. Without loss of generality we may assume that

$$a_2 \leq \dots \leq a_k \leq a_1 \leq a_{k+1} \leq \dots \leq a_n.$$

Then we have

$$\begin{aligned} \|M_{S_{I,n}}^{\rightarrow} f\|_{\ell^2(V)}^2 &= a_1^2 + \sum_{i=2}^n \max \left\{ a_i^2, \left(\frac{1}{2}(a_i + a_1) \right)^2 \right\} \\ &= a_1^2 + \sum_{j=2}^k \left(\frac{1}{2}(a_j + a_1) \right)^2 + \sum_{j=k+1}^n a_j^2 \\ &= \left(\frac{k-1}{4} + 1 \right) a_1^2 + \frac{1}{4} \sum_{j=2}^k a_j^2 + \frac{1}{2} \sum_{j=2}^k a_1 a_j + \sum_{j=k+1}^n a_j^2. \end{aligned} \tag{11}$$

Applying the AM-GM inequality, one finds

$$2 \sum_{j=2}^k a_1 a_j \leq \sum_{j=2}^k \left(x a_1^2 + \frac{1}{x} a_j^2 \right),$$

for all $x \in (0, \infty)$, where the above equality is attained if and only if $a_j = xa_1$ for all $j = 2, \dots, k$. This together with (11) implies that

$$\begin{aligned} & \|M_{S_{I,n}}^{-\rightarrow} f\|_{\ell^2(V)}^2 \\ & \leq \left(\frac{k-1}{4} + 1\right)a_1^2 + \frac{1}{4} \sum_{j=2}^k a_j^2 + \frac{1}{4} \sum_{j=2}^k \left(xa_1^2 + \frac{1}{x}a_j^2\right) + \sum_{j=k+1}^n a_j^2 \\ & = \left(\frac{k-1}{4} + 1 + \frac{(k-1)x}{4}\right)a_1^2 + \left(\frac{1}{4} + \frac{1}{4x}\right) \sum_{j=2}^k a_j^2 + \sum_{j=k+1}^n a_j^2 \\ & = \left(\frac{k-1}{4}x^2 + \frac{k-1}{2}x + \frac{k+3}{4}\right)a_1^2 + \sum_{j=k+1}^n a_j^2 \end{aligned} \quad (12)$$

for all $x \in (0, \infty)$, where the first equality in (12) is attained if and only if $a_j = xa_1$ for all $j = 2, \dots, k$. Assume that $a_j = xa_1$ for all $j = 2, \dots, k$ and some $x \in (0, \infty)$, then

$$\sum_{j=k+1}^n a_j^2 = \|f\|_{\ell^2(V)}^2 - ((k-1)x^2 + 1)a_1^2. \quad (13)$$

Combining (13) and (12) implies that

$$\begin{aligned} \|M_{S_{I,n}}^{-\rightarrow} f\|_{\ell^2(V)}^2 & = \left(-\frac{3(k-1)}{4}x^2 + \frac{k-1}{2}x + \frac{k-1}{4}\right)a_1^2 + \|f\|_{\ell^2(V)}^2 \\ & = -\frac{(k-1)}{4}(3x^2 - 2x - 1)a_1^2 + \|f\|_{\ell^2(V)}^2 \end{aligned} \quad (14)$$

for all $x \in (0, \infty)$. Here $a_j = xa_1$ for all $j = 2, \dots, k$ and some $x \in (0, \infty)$. Let us consider two cases:

(i) If $x \geq 1$, then $3x^2 - 2x - 1 \geq 0$. We get from (14) that

$$\|M_{S_{I,n}}^{-\rightarrow} f\|_{\ell^2(V)} \leq \|f\|_{\ell^2(V)}.$$

(ii) If $0 < x < 1$, then $3x^2 - 2x - 1 < 0$. Observing that

$$\|f\|_{\ell^2(V)}^2 = a_1^2 + \sum_{j=2}^k a_j^2 + \sum_{j=k+1}^n a_j^2 \geq \left(n - k + 1 + (k-1)x^2\right)a_1^2,$$

is equivalent to

$$a_1^2 \leq \frac{1}{n - k + 1 + (k-1)x^2} \|f\|_{\ell^2(V)}^2. \quad (15)$$

Here the equality in (15) is attained if and only if $a_j = xa_1$ for all $j = 2, \dots, k$ and $a_j = a_1$ for all $j = k+1, \dots, n$. In light of (14) and (15) we would have

$$\|M_{S_{I,n}}^{-\rightarrow} f\|_{\ell^2(V)}^2 \leq \left(-\frac{(k-1)}{4} \frac{3x^2 - 2x - 1}{n - k + 1 + (k-1)x^2} + 1\right) \|f\|_{\ell^2(V)}^2. \quad (16)$$

Here the equality in (16) is attained if and only if $a_j = xa_1$ for all $j = 2, \dots, k$ and $a_j = a_1$ for all $j = k+1, \dots, n$. Let $g(x) = \frac{3x^2 - 2x - 1}{n - k + 1 + (k-1)x^2}$, $x \in (0, 1)$. One can easily check that there exists a unique $x_0 \in (0, \infty)$ such that $g(x_0) = \min_{x \in (0, +\infty)} g(x)$, where

$$x_0 := x_0(n, k) = \frac{2(k-1) - 3n + \sqrt{9n^2 - 8n(k-1)}}{2(k-1)} \in (0, 1).$$

For convenience, let

$$h(x, n, k) = -\frac{(k-1)}{4} \frac{3x^2 - 2x - 1}{n - k + 1 + (k-1)x^2} + 1.$$

Then we have

$$\|M_{S_{I,n}}^{-\rightarrow} f\|_{\ell^2(V)}^2 \leq \max_{k \in [2,n]} h(x_0, n, k) \|f\|_{\ell^2(V)}^2. \quad (17)$$

In particular, for fixed $k \in [2, n]$, let $f = \sum_{j=1}^n a_j \delta_j$, where $a_j = x_0$ for all $j \in \{2, \dots, k\}$ and $a_j = 1$ for all $j \in \{k+1, \dots, n\}$. One can easily check that

$$\frac{\|M_{S_{I,n}}^{-\rightarrow} f\|_{\ell^2(V)}^2}{\|f\|_{\ell^2(V)}^2} = h(x_0, n, k).$$

This together with (17) yields the conclusion of part (ii). $\square$

As applications of Theorem 3, we get

Corollary 1. (i) If $n = 2$, then $\|M_{S_{I,2}}^{-\rightarrow}\|_2 = (\frac{\sqrt{5}+3}{2})^{1/2}$. Moreover,

$$\|M_{S_{I,2}}^{-\rightarrow} f\|_{\ell^2(V)} = \left(\frac{\sqrt{5}+3}{2}\right)^{1/2} \|f\|_{\ell^2(V)}$$

if and only if $f(2) = (\sqrt{5}-2)f(1) \neq 0$.

(ii) If $n = 3$, then $\|M_{S_{I,3}}^{-\rightarrow}\|_2 = (\frac{\sqrt{33}+7}{8})^{1/2}$. Moreover,

$$\|M_{S_{I,3}}^{-\rightarrow} f\|_{\ell^2(V)} = \left(\frac{\sqrt{33}+7}{8}\right)^{1/2} \|f\|_{\ell^2(V)}$$

if and only if $f(2) = f(3) = \frac{\sqrt{33}-5}{4}f(1) \neq 0$.

(iii) If $n = 4$, then $\|M_{S_{I,4}}^{-\rightarrow}\|_2 = (\frac{2+\sqrt{3}}{2})^{1/2}$. Moreover,

$$\|M_{S_{I,4}}^{-\rightarrow} f\|_{\ell^2(V)} = \left(\frac{2+\sqrt{3}}{2}\right)^{1/2} \|f\|_{\ell^2(V)}$$

if and only if $f(2) = f(3) = f(4) = (\frac{2}{3}\sqrt{3}-1)f(1) \neq 0$.

Proof. Let x_0 and $h(x, n, k)$ be given as in the proof of Theorem 3. When $n = 2$ and $k = 2$, we have $x_0 = \sqrt{5}-2$ and $h(x_0, 2, 2) = (\frac{\sqrt{5}+3}{2})^{1/2}$.

When $n = 3$, if $k = 2$, then $x_0 = \frac{\sqrt{57}-7}{2}$ and $h(x_0, 3, 2) = \frac{\sqrt{57}+11}{16}$. If $k = 3$, then $x_0 = \frac{\sqrt{33}-5}{4}$ and $h(x_0, 3, 2) = \frac{\sqrt{33}+7}{8}$. It is clear that $\frac{\sqrt{33}+7}{8} > \frac{\sqrt{57}+11}{16}$. Therefore, applying Theorem 3 we get

$$\|M_{S_{I,3}}^{-\rightarrow} f\|_{\ell^2(V)} \leq \left(\frac{\sqrt{33}+7}{8}\right)^{1/2} \|f\|_{\ell^2(V)}. \quad (18)$$

We note that the equality in (18) is attained if and only if $f(2) = f(3) = \frac{\sqrt{33}-5}{4}f(1) \neq 0$. Actually, let $\alpha \neq 0$ and $f: V \rightarrow \mathbb{R}$ be defined by $f(1) = \alpha$ and $f(2) = f(3) = \frac{\sqrt{33}-5}{4}\alpha$. One can easily check that $\|M_{S_{I,3}}^{-\rightarrow} f\|_{\ell^2(V)}^2 = \frac{33-\sqrt{33}}{16}$ and $\|f\|_{\ell^2(V)}^2 = \frac{33-5\sqrt{33}}{4}$. Therefore,

$$\frac{\|M_{S_{I,3}}^{-\rightarrow} f\|_{\ell^2(V)}^2}{\|f\|_{\ell^2(V)}^2} = \frac{\sqrt{33}+7}{8},$$

which together with (18) yields that

$$\|M_{S_{I,3}}^{-\rightarrow}\|_2 = \left(\frac{\sqrt{33}+7}{8}\right)^{1/2}.$$

When $n = 4$. If $k = 2$, then $x_0 = 2\sqrt{7} - 5$ and $h(x_0, 4, 2) = \frac{4+\sqrt{7}}{6}$. If $k = 3$, then $x_0 = \sqrt{5} - 2$ and $h(x_0, 4, 3) = \frac{\sqrt{5}+3}{4}$. If $k = 4$, then $x_0 = \frac{2}{3}\sqrt{3} - 1$ and $h(x_0, 4, 4) = \frac{\sqrt{3}+2}{2}$. Therefore, we get by Theorem 3 that

$$\|M_{S_{I,4}}^{\rightarrow} f\|_{\ell^2(V)} \leq \left(\frac{\sqrt{3}+2}{2}\right)^{1/2} \|f\|_{\ell^2(V)}. \quad (19)$$

It should be pointed out that the equality in (19) is attained if and only if $f(2) = f(3) = f(4) = (\frac{2}{3}\sqrt{3} - 1)f(1) \neq 0$. This proves part (ii) and completes the proof. $\square$

The following result presents the estimates for $\|M_{S_{O,n}}^{\rightarrow}\|_p$ with $1 < p < \infty$.

Theorem 4. Let $n \geq 3$.

(i) If $1 < p < \infty$, then

$$(1 + n^{-p})^{1/p} \leq \|M_{S_{O,n}}^{\rightarrow}\|_p < (1 + n^{-1})^{1/p}.$$

(ii) If $p = 2$, then

$$\|M_{S_{O,n}}^{\rightarrow}\|_2 = \left(1 + \frac{1}{2n} + \max_{k \in [1, n-1]} \left(-\frac{1}{2k} + \frac{\sqrt{(n+3k)(n-k)}}{2nk}\right)\right)^{1/2}.$$

Proof. At first, we shall prove part (i). We may assume without loss of generality that $\overrightarrow{S_{O,n}} = (V, E)$, where $V = \{1, \dots, n\}$ and $E = \{1 \rightarrow 2, \dots, 1 \rightarrow n\}$. It is obvious that $M_{S_{O,n}}^{\rightarrow} \delta_2(2) = 1$, $M_{S_{O,n}}^{\rightarrow} \delta_2(1) = \frac{1}{n}$ and $M_{S_{O,n}}^{\rightarrow} \delta_2(i) = 0$ for all $i \in V \setminus \{1, 2\}$. Moreover, $\|\delta_2\|_{\ell^p(V)} = 1$, $\|M_{S_{O,n}}^{\rightarrow} \delta_2\|_{\ell^p(V)} = (1 + n^{-p})^{1/p}$. Hence,

$$\|M_{S_{O,n}}^{\rightarrow}\|_p \geq (1 + n^{-p})^{1/p}, \text{ for } 1 < p < \infty.$$

We now prove

$$\|M_{S_{O,n}}^{\rightarrow}\|_p < (1 + n^{-1})^{1/p}, \text{ for } 1 < p < \infty. \quad (20)$$

Fix $f = \sum_{i=1}^n a_i \delta_i$ with $\sum_{i=1}^n |a_i|^p = 1$, we can write

$$M_{S_{O,n}}^{\rightarrow} f(i) = \begin{cases} \max \left\{ |a_1|, \frac{1}{n} \sum_{j=1}^n |a_j| \right\}, & i = 1; \\ |a_i|, & i = 2, \dots, n. \end{cases}$$

Then we have

$$\|M_{S_{O,n}}^{\rightarrow} f\|_{\ell^p(V)}^p = \sum_{i=2}^n |a_i|^p + \max \left\{ |a_1|^p, \left(\frac{1}{n} \sum_{j=1}^n |a_j| \right)^p \right\}.$$

Therefore, to prove (20), it suffices to show that

$$\sum_{i=2}^n |a_i|^p + \max \left\{ |a_1|^p, \left(\frac{1}{n} \sum_{j=1}^n |a_j| \right)^p \right\} < 1 + \frac{1}{n}. \quad (21)$$

for any sequence $\{a_i\}_{i=1}^n$ with $\sum_{i=1}^n |a_i|^p = 1$.

Given a sequence $\{a_i\}_{i=1}^n$ with $\sum_{i=1}^n |a_i|^p = 1$, we consider two cases:

(a) If $|a_1| \geq \frac{1}{n} \sum_{j=1}^n |a_j|$. Then

$$\sum_{i=2}^n |a_i|^p + \max \left\{ |a_1|^p, \left(\frac{1}{n} \sum_{j=1}^n |a_j| \right)^p \right\} = 1.$$

This proves (21) in this case.

(b) If $|a_1| < \frac{1}{n} \sum_{j=1}^n |a_j|$. By the Jensen's inequality, we have

$$\begin{aligned} & \sum_{i=2}^n |a_i|^p + \max \left\{ |a_1|^p, \left(\frac{1}{n} \sum_{j=1}^n |a_j| \right)^p \right\} \\ &= \sum_{i=2}^n |a_i|^p + \left(\frac{1}{n} \sum_{j=1}^n |a_j| \right)^p < \sum_{i=2}^n |a_i|^p + \frac{1}{n} \sum_{j=1}^n |a_j|^p < 1 + \frac{1}{n}. \end{aligned}$$

This proves (21) in this case.

Next, we prove part (ii). Let $f = \sum_{i=1}^n a_i \delta_i$ with each $a_i \geq 0$ ($i = 1, \dots, n$) and $\sum_{j=1}^n a_j^2 = 1$. Without loss of generality we may assume that

$$a_2 \leq \dots \leq a_k \leq \frac{1}{n} \sum_{j=1}^n a_j \leq a_{k+1} \leq \dots \leq a_n.$$

Assume that $a_1 < \frac{1}{n} \sum_{j=1}^n a_j$. Then we have

$$\begin{aligned} \|M_{S_{O,n}} f\|_{\ell^2(V)}^2 &= \sum_{i=2}^n a_i^2 + \left(\frac{1}{n} \sum_{j=1}^n a_j \right)^2 \\ &= \sum_{i=2}^n a_i^2 + \frac{1}{n^2} \sum_{j=1}^n a_j^2 + \frac{2}{n^2} \left(\sum_{1 \leq i < j \leq k} a_i a_j \right. \\ &\quad \left. + \sum_{k+1 \leq i < j \leq n} a_i a_j + \sum_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq n}} a_i a_j \right) \\ &\leq 1 + \frac{1}{n^2} - a_1^2 + \frac{k-1}{n^2} \sum_{i=1}^k a_i^2 \\ &\quad + \frac{n-k-1}{n^2} \sum_{i=k+1}^n a_i^2 + \frac{2}{n^2} \sum_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq n}} a_i a_j \\ &= 1 + \frac{1}{n^2} + \left(\frac{k(k-1)}{n^2} - 1 \right) \alpha^2 + \frac{(n-k)(n-k-1)}{n^2} \beta^2 \\ &\quad + \frac{2(n-k)k}{n^2} \alpha \beta. \end{aligned} \tag{22}$$

Here the equality in (22) is attained if and only if $a_i = \alpha$ for all $1 \leq i \leq k$ and $a_j = \beta$ for all $k+1 \leq j \leq n$. Moreover, α, β satisfy $k\alpha^2 + (n-k)\beta^2 = 1$ and $\alpha < \beta$. Please note that the AM-GM inequality holds:

$$2\alpha\beta \leq x\alpha^2 + \frac{1}{x}\beta^2,$$

for all $x \in (0, \infty)$, where the above equality holds if and only if $\beta = x\alpha$. This combines with (22) leads to

$$\begin{aligned} & \left(\frac{k(k-1)}{n^2} - 1 \right) \alpha^2 + \frac{(n-k)(n-k-1)}{n^2} \beta^2 + \frac{2(n-k)k}{n^2} \alpha \beta \\ &\leq \left(\frac{k(k-1)}{n^2} - 1 + \frac{k(n-k)x}{n^2} \right) \alpha^2 \\ &\quad + \left(\frac{(n-k)(n-k-1)}{n^2} + \frac{k(n-k)}{n^2 x} \right) \beta^2 \\ &= \left(\frac{k-1}{n^2} - \frac{1}{k} + \frac{(n-k)x}{n^2} \right) k\alpha^2 + \left(\frac{n-k-1}{n^2} + \frac{k}{n^2 x} \right) (n-k)\beta^2 \end{aligned} \tag{23}$$

for all $x \in (0, \infty)$. Here the equality in (23) is attained if and only if $\beta = x\alpha$. There exists a unique $x_0 \in (0, \infty)$ such that

$$\frac{k-1}{n^2} - \frac{1}{k} + \frac{(n-k)x_0}{n^2} = \frac{n-k-1}{n^2} + \frac{k}{n^2 x_0}, \quad (24)$$

where

$$x_0 := \frac{(n+2k)(n-k) + n\sqrt{(n+3k)(n-k)}}{2k(n-k)}.$$

It follows from (22)–(24) that

$$\begin{aligned} & \|M_{S_{O,n}}^{\rightarrow} f\|_{\ell^2(V)}^2 \\ & \leq \max_{k \in [1, n-1]} \frac{k}{n^2} + 1 - \frac{1}{k} + \frac{(n+2k)(n-k) + n\sqrt{(n+3k)(n-k)}}{2n^2 k} \\ & = 1 + \frac{1}{2n} + \max_{k \in [1, n-1]} \left(-\frac{1}{2k} + \frac{\sqrt{(n+3k)(n-k)}}{2nk} \right). \end{aligned} \quad (25)$$

Here the equality in (25) is attained if and only if $a_i = \alpha > 0$ for all $1 \leq i \leq k$ and $a_j = \beta > 0$ for all $k+1 \leq j \leq n$. Moreover, $\beta = x_0\alpha$ and $\alpha < \beta$. This proves part (ii). $\square$

As applications of Theorem 4, we get

Corollary 2. (i) If $n = 3$, then $\|M_{S_{O,3}}^{\rightarrow}\|_2 = \left(\frac{2+\sqrt{3}}{3}\right)^{1/2}$. Moreover,

$$\|M_{S_{O,3}}^{\rightarrow} f\|_{\ell^2(V)} = \left(\frac{2+\sqrt{3}}{3}\right)^{1/2} \|f\|_{\ell^2(V)}$$

if and only if $f(2) = f(3) = \frac{5+3\sqrt{3}}{2} f(1) \neq 0$.

(ii) If $n = 4$, then $\|M_{S_{O,4}}^{\rightarrow}\|_2 = \left(\frac{5+\sqrt{21}}{8}\right)^{1/2}$. Moreover,

$$\|M_{S_{O,4}}^{\rightarrow} f\|_{\ell^2(V)} = \left(\frac{5+\sqrt{21}}{8}\right)^{1/2} \|f\|_{\ell^2(V)}$$

if and only if $f(2) = f(3) = f(4) = (3 + \frac{2\sqrt{21}}{3}) f(1) \neq 0$.

Proof. At first, we prove part (i). For convenience, we set

$$h(n, k) = 1 + \frac{1}{2n} - \frac{1}{2k} + \frac{\sqrt{(n+3k)(n-k)}}{2nk}.$$

Let x_0 be given as in (24). When $n = 3$. If $k = 1$, then $x_0 = \frac{5+3\sqrt{3}}{2}$ and $h(3, 1) = \frac{2+\sqrt{3}}{3}$. If $k = 2$, then $x_0 = 4$ and $h(3, 2) = \frac{7}{6}$. Hence, we get by Theorem 4 that

$$\|M_{S_{O,3}}^{\rightarrow} f\|_{\ell^2(V)} \leq \left(\frac{2+\sqrt{3}}{3}\right)^{1/2} \|f\|_{\ell^2(V)}, \quad (26)$$

where the equality in (26) is attained if and only if $f(2) = f(3) = \frac{5+3\sqrt{3}}{2} f(1) \neq 0$.

We now prove part (ii). When $n = 4$. If $k = 1$, then $x_0 = 3 + \frac{2\sqrt{21}}{3}$ and $h(4, 1) = \frac{5+\sqrt{21}}{8}$. If $k = 2$, then $x_0 = 2 + \sqrt{5}$ and $h(4, 2) = \frac{7+\sqrt{5}}{8}$. If $k = 3$, then $x_0 = \frac{5+2\sqrt{13}}{3}$ and $h(4, 3) = \frac{23+\sqrt{13}}{24}$. Invoking Theorem 4, we get

$$\|M_{S_{O,4}}^{\rightarrow} f\|_{\ell^2(V)} \leq \left(\frac{5+\sqrt{21}}{8}\right)^{1/2} \|f\|_{\ell^2(V)}, \quad (27)$$

where the equality in (27) is attained if and only if $f(j) = (3 + \frac{2\sqrt{21}}{3})f(1) \neq 0$ for all $j = 2, 3, 4$. This proves part (ii). $\square$

Remark 2. Some challenging questions are to find the sharp constants for $\|M_{\vec{G}}\|_p$ with $p \in (1, \infty)$ and $p \neq 2$.

To obtain some sharp constants for $\|M_{\vec{G}}\|_p$ in the range $1 < p < \infty$, we consider the following restricted-type estimate:

$$\|M_{\vec{G}}\|_{p, \text{rest}} = \max_{A \subset V} \frac{\|M_{\vec{G}}(\chi_A)\|_{\ell^p(V)}}{\|\chi_A\|_{\ell^p(V)}},$$

where $\vec{G} = (V, E)$. From the definition we see that

$$\|M_{\vec{G}}\|_{p, \text{rest}} \leq \|M_{\vec{G}}\|_p. \quad (28)$$

We have the following estimate for $\|M_{\vec{S}_{I,n}}\|_{p, \text{rest}}$.

Theorem 5. Let $0 < p < \infty$ and $n \geq 2$. Then

$$\|M_{\vec{S}_{I,n}}\|_{p, \text{rest}} = \begin{cases} 2^{1/p}, & \log_2(n-1) \leq p < \infty; \\ \left(1 + \frac{n-1}{2^p}\right)^{1/p}, & 0 < p < \log_2(n-1). \end{cases}$$

Proof. Without loss of generality, we may assume that $\vec{S}_{I,n} = (V, E)$, where $V = \{1, \dots, n\}$ and $E = \{2 \rightarrow 1, \dots, n \rightarrow 1\}$. For $A \subset V$ with $|A| = k < n$. Please note that $\|\chi_A\|_{\ell^p(V)} = k^{1/p}$.

We consider two cases:

Case 1: $1 \in A$.

We have

$$M_{\vec{S}_{I,n}}\chi_A(i) = \begin{cases} 1, & i \in A; \\ \frac{1}{2}, & i \notin A. \end{cases}$$

Therefore,

$$\|M_{\vec{S}_{I,n}}\chi_A\|_{\ell^p(V)} = \left(\sum_{j=1}^n (M_{\vec{S}_{I,n}}\chi_A(j))^p\right)^{1/p} = \left(k + \frac{n-k}{2^p}\right)^{1/p}.$$

It follows that

$$\frac{\|M_{\vec{S}_{I,n}}\chi_A\|_{\ell^p(V)}}{\|\chi_A\|_{\ell^p(V)}} = \left(1 + \frac{1}{2^p} \left(\frac{n}{k} - 1\right)\right)^{1/p}.$$

Case 2: $1 \notin A$.

We have

$$M_{\vec{S}_{I,n}}\chi_A(i) = \begin{cases} 1, & i \in A \cup \{1\}; \\ 0, & i \notin A, i \neq 1. \end{cases}$$

Therefore,

$$\|M_{\vec{S}_{I,n}}\chi_A\|_{\ell^p(V)} = \left(\sum_{j=1}^n (M_{\vec{S}_{I,n}}\chi_A(j))^p\right)^{1/p} = (k+1)^{1/p}.$$

It follows that

$$\frac{\|M_{\vec{S}_{I,n}}\chi_A\|_{\ell^p(V)}}{\|\chi_A\|_{\ell^p(V)}} = \left(1 + \frac{1}{k}\right)^{1/p}.$$

Hence, we have

$$\begin{aligned}\|M_{\overrightarrow{S_{I,n}}}\|_{p,\text{rest}} &= \max \left\{ \left(1 + \frac{n-1}{2^p}\right)^{1/p}, 2^{1/p} \right\} \\ &= \begin{cases} 2^{1/p}, & \log_2(n-1) \leq p < \infty; \\ \left(1 + \frac{n-1}{2^p}\right)^{1/p}, & 0 < p < \log_2(n-1). \end{cases}\end{aligned}$$

This yields the conclusion of Theorem 5. $\square$

Remark 3. Let $\overrightarrow{S_{I,n}} = (V, E)$ with $V = \{1, \dots, n\}$ and $E = \{2 \rightarrow 1, \dots, n \rightarrow 1\}$ and $A \subset V$. Then

(i) When $\log_2(n-1) \leq p < \infty$, then $\|M_{\overrightarrow{S_{I,n}}}\chi_A\|_{\ell^p(V)} = 2^{1/p}\|\chi_A\|_{\ell^p(V)}$ if and only if $A = \{i\}$ for some $i \in \{2, \dots, n\}$.

(ii) When $0 < p < \log_2(n-1)$, then $\|M_{\overrightarrow{S_{I,n}}}\chi_A\|_{\ell^p(V)} = (1 + \frac{n-1}{2^p})^{1/p}\|\chi_A\|_{\ell^p(V)}$ if and only if $A = \{1\}$.

Theorem 6. Let $1 < p \leq \infty$. Then

$$\|M_{\overrightarrow{S_{O,n}}}\|_{p,\text{rest}} = \begin{cases} \left(1 + \frac{(n-1)^{p-1}}{n^p}\right)^{1/p}, & 1 < p < \infty; \\ \left(1 + \frac{1}{n^p}\right)^{1/p}, & 0 < p \leq 1. \end{cases}$$

Proof. Without loss of generality, we may assume that $\overrightarrow{S_{O,n}} = (V, E)$, where $V = \{1, \dots, n\}$ and $E = \{1 \rightarrow 2, \dots, 1 \rightarrow n\}$. For $A \subset V$ with $|A| = k < n$. It is clear that $\|\chi_A\|_{\ell^p(V)} = k^{1/p}$. We consider two cases:

Case 1: $1 \in A$.

We have

$$M_{\overrightarrow{S_{O,n}}}\chi_A(i) = \begin{cases} 1, & i \in A; \\ 0, & i \notin A. \end{cases}$$

Therefore,

$$\|M_{\overrightarrow{S_{O,n}}}\chi_A\|_{\ell^p(V)} = \left(\sum_{j=1}^n (M_{\overrightarrow{S_{O,n}}}\chi_A(j))^p\right)^{1/p} = k^{1/p}.$$

It follows that

$$\frac{\|M_{\overrightarrow{S_{O,n}}}\chi_A\|_{\ell^p(V)}}{\|\chi_A\|_{\ell^p(V)}} = 1.$$

Case 2: $1 \notin A$.

We have

$$M_{\overrightarrow{S_{O,n}}}\chi_A(i) = \begin{cases} 1, & i \in A; \\ \frac{k}{n}, & i \notin A, i = 1; \\ 0, & i \notin A, i \neq 1. \end{cases}$$

Therefore,

$$\|M_{\overrightarrow{S_{O,n}}}\chi_A\|_{\ell^p(V)} = \left(\sum_{j=1}^n (M_{\overrightarrow{S_{O,n}}}\chi_A(j))^p\right)^{1/p} = \left(k + \frac{k^p}{n^p}\right)^{1/p}.$$

It follows that

$$\frac{\|M_{\overrightarrow{S_{O,n}}}\chi_A\|_{\ell^p(V)}}{\|\chi_A\|_{\ell^p(V)}} = \left(1 + \frac{k^{p-1}}{n^p}\right)^{1/p}.$$

Hence, we have

$$\|M_{\vec{S}_{O,n}}\|_{p,\text{rest}} = \max_{1 \leq k \leq n-1} \left(1 + \frac{k^{p-1}}{n^p}\right)^{1/p} = \begin{cases} \left(1 + \frac{(n-1)^{p-1}}{n^p}\right)^{1/p}, & 1 < p < \infty; \\ \left(1 + \frac{1}{n^p}\right)^{1/p}, & 0 < p \leq 1. \end{cases}$$

This proves Theorem 6. $\square$

Remark 4. Let $\vec{S}_{O,n} = (V, E)$ with $V = \{1, \dots, n\}$ and $E = \{1 \rightarrow 2, \dots, 1 \rightarrow n\}$ and $A \subset V$. Then

- (i) When $1 < p < \infty$, then $\|M_{\vec{S}_{O,n}}\chi_A\|_{\ell^p(V)} = \left(1 + \frac{(n-1)^{p-1}}{n^p}\right)^{1/p} \|\chi_A\|_{\ell^p(V)}$ if and only if $A = \{2, \dots, n\}$.
- (ii) When $0 < p \leq 1$, then $\|M_{\vec{S}_{O,n}}\chi_A\|_{\ell^p(V)} = \left(1 + \frac{1}{n^p}\right)^{1/p} \|\chi_A\|_{\ell^p(V)}$ if and only if $A = \{i\}$ for some $i \in \{2, \dots, n\}$.

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