Mathematical and Computational Applications, Vol. 10, No. 1, pp. 155-163, 2005.

EXAMINING OF TETRAGONAL SURFACE PATCHES WITH THEIR GEOMETRIC PROPERTIES AND RELATIONSHIPS

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Abstract- In this paper, the geometric properties and relationships of the rectangular are applied to a tetragonal surface patch. Then, O. Bonnet integral formula is generalized for the tetragonal surface patchs which are bounded with a curvatural polygon.

Key words- Surface Geometry, tetragonal surface patch, rectangular.

1. INTRODUCTION

This article presents the geometric properties and relationships of rectangulars in a plane to apply tetragonal patches on surface. Therefore, in the second section; using some formulas of the surface geometry, [3], [6], [8], and especially J. Liouville and O. Bonnet integral formulas, [1], [2], [4], [5], are reminded. In the third section, the method of examining of rectangulars in the plane, [7], is generalized to a tetragonal surface patch which is divided to *mxn* sub-patches. Then, using formulas that are given in the second section, O. Bonnet integral formula is generalized for the area of the tetragonal surface patch.

2. SURFACE GEOMETRY

Definition 2.1

A subset $M \subset \boldsymbol{R}^3$ is a regular surface if, for each $p \in M$, there exists a neighborhood V in $\boldsymbol{R}^3$ and a map $r : U \to V \cap M$ of an open set $U \subset \boldsymbol{R}^3$ onto $V \cap M \subset \boldsymbol{R}^3$ such that

1- r is differentiable. This means that if we write

$$r(u, v)=(x(u, v), y(u, v), z(u, v)), \qquad (u, v) \in U$$

the function $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivatives of all orders in U.

2- r is homeomorphism. Since r is continuous by condition 1, this means that r has an inverse $r^{-1}: V \cap M \to U$ which is continuous; that is, r^{-1} is the restriction of a continuous map $F : W \subset \boldsymbol{R}^3 \to \boldsymbol{R}^2$ defined on an open set W containing $V \cap M$.

3- (The regularity condition) For each $q \in U$, the differential $dr_q : \boldsymbol{R}^2 \to \boldsymbol{R}^3$ is one-to-one.

The mapping r is called a parameterization or a system of (local) coordinates in (a neighborhood of) p. The neighborhood $V \cap M$ of p in M is called a coordinate neighborhood.

To give condition 3 a more familiar form, let us compute the matrix of the linear map dr_q in the canonical bases $e_1 = (1,0)$, $e_2 = (0,1)$ of $\boldsymbol{R}^2$ with coordinates (u, v) and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ of $\boldsymbol{R}^3$, with coordinates (x, y, z).

Explicitly, for each point (u_0, v_0) in u the curve

$$u \to r\,(u, v_0)$$

is called the u-parameter curve, $v = v_0$, of x; and the curve

$$v \to r\,(u_0, v)$$

is the v-parameter curve, $u = u_0$.

By the definition of differential

$$dr_q(e_1) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \vec{r}_u\,, \qquad dr_q(e_2) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \vec{r}_v\,,$$

[3], [8].

Definition 2.2 (The First and Second Fundamental Form of the Surface)

Let M be a surface is given by $\vec{r} = \vec{r}\,(u, v)$, then u and v are called the curvilinear coordinates of this surface. The vector equation of the curve on surface and its differential are

$$\vec{r} = \vec{r}\,[u(t); \mathrm{v(t)}],$$
$$d\vec{r} = \vec{r}_u\,du + \vec{r}_v\,dv.$$

The first fundamental form, as ds is differential of its length, is

$$\mathrm{I} = ds^2 = d\vec{r}^{\,2} = \vec{r}_u^{\,2}\,du^2 + 2\vec{r}_u\,\vec{r}_v\,du\,dv + \vec{r}_v^{\,2}\,dv^2$$
$$= E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

The unit normal vector of the surface and its differential are

$$\vec{N} = \vec{N}\,(u,v),$$
$$d\vec{N} = \vec{N}_u\,du + \vec{N}_v\,dv,$$

then the second fundamental form is

$$\mathrm{II} = -\,d\vec{N} \,.\, d\vec{r} = -(\vec{N}_u\,du + \vec{N}_v\,dv)(\vec{r}_u\,du + \vec{r}_v\,dv),$$

[2], [3], [5], [6].

Definition 2.3

If tangent vector α at a point P of a (α) curve on a $M \subset E^3$ surface is parallel to one of the principal directions of M surface, then (α) is called a principal curve or curvature line.

When Frenet and Darboux frames on curvature lines are taken as $(\vec{t}, \vec{n}, \vec{b})$, $(\vec{t}, \vec{N}, \vec{B})$ respectively, we have

$$\frac{d\vec{N}}{ds} = -\vec{t}\,k_n\,,$$

$$\tau = -\frac{d\theta}{ds}\,,$$

where s is the arc length of the curve, θ is the angle between principal normal and surface normal, k_n is normal curvature and τ is the torsion, [3], [6].

Theorem 2.1

Let $v = const$ and $u = const$ parameter curves on a surface be curvature lines, as k_{n1}, k_{n2} are principal curvatures, we have (Olinde Rodrigues formulas)

$$\vec{N}_u = -k_{n1}\vec{r}_u\,,$$
$$\vec{N}_v = -k_{n2}\vec{r}_v\,,$$

[2],[3],[6].

Theorem 2.2

Let us assume that φ is the angle between the unit tangent vector $\vec{t}$ that passes through a point P of a (α) curve on a M surface and the first of $\vec{t}_1$ and $\vec{t}_2$ unit tangent vectors of the parameter curves at this point, we have

$$k_{n1}\cos\varphi = k_n\cos\varphi - \tau_g\sin\varphi\,,$$
$$k_{n2}\sin\varphi = k_n\sin\varphi + \tau_g\cos\varphi\,,$$

where k_{n1} and k_{n2} are principal curvatures and the normal curvatures corresponding to $\vec{t}$ is k_n and geodesic torsion is τ_g, [1].

Theorem 2.3

Let us assume that the trihedrons of $u = const$ and $v = const$ parameter curves, which are perpendicular to each other, passes through a point P on a surface M and any curve of the surface passes through P are $(\vec{t}_1,\vec{N},\vec{B}_1)$, $(\vec{t}_2,\vec{N},\vec{B}_2)$ and $(\vec{t},\vec{N},\vec{B})$ respectively. If φ is the angle between the unit vectors $\vec{t}$ and $\vec{t}_1$, then we have

$$\frac{d\vec{t}_1}{ds} = (k_{n1}\cos\varphi - \tau_{g2}\sin\varphi)\vec{N} + (k_{g1}\cos\varphi + k_{g2}\sin\varphi)\vec{t}_2\,,$$
$$\frac{d\vec{t}_2}{ds} = (k_{n2}\sin\varphi - \tau_{g1}\cos\varphi)\vec{N} - (k_{g1}\cos\varphi + k_{g2}\sin\varphi)\vec{t}_1\,,$$

where k_{g1}, k_{g2} and τ_{g1}, τ_{g2} are geodesic curvatures and torsions that belongs to parameter curves, respectively, [1].

Theorem 2.4 (Formula of J. Liouville)

Let $\vec{t}_1$ and $\vec{t}_2$ be the tangent directions of v = const, u = const parameter curves which are perpendicular to each other at a point P on the surface $\vec{r} = \vec{r}$ (u, v) and the geodesic curvature of these directions are k_{g1} and k_{g2}, respectively. Let $\vec{t}$ be a tangent direction of any curve through point P on the surface. Then, we get the relation of the geodesic curvature of $\vec{t}$ direction

$$k_g = \frac{d\varphi}{ds} + k_{g1}\cos\varphi + k_{g2}\sin\varphi\,,$$

where φ is the angle between $\vec{t}$ and $\vec{t}_1$, [3],[4].

Theorem 2.5 (O.Bonnet's Integral Formula)

Let A be a region with single dependence of M surface that is given by $\vec{r} = \vec{r}\,(u,\ v)$ occurs regular points. Let (α) be a curve, which is continuous and without multiple point, where is limit to this surface area, and k_g be geodesic curvature of the surface through this curve. Also, let K be Gauss curvature which is belong to points on this surface region and if the arc element of the curve is ds and the surface element is $d\sigma$ which is belong to region that occurs the given surface, then we have

$$\int_{(\alpha)} k_g ds = 2\pi - \iint_{(A)} K d\sigma \ , \qquad K = k_1 k_2 . \tag{1}$$

The integral is taken on positive direction on the curve (α), [2],[4].

3. THE TETRAGONAL SURFACE PATCH AND ITS EXAMINING

3.1 Definition of Tetragonal Surface Patches

Describe a tetragonal surface patch by giving the coordinate curve of one of its vertices and the lengths of its two principal curves. Consider only tetragonal surface patches whose sides are parallel to the coordinate axes.

Denote a tetragonal surface patch symbolically with the boldface capital letters ***TSP***. Again, indicate a specific tetragonal by adding an alphabetic or numeric subscript : ***TSP$_1$***, for example. Thus, to specify a tetragonal surface patch, we write

$$\boldsymbol{TSP} = \boldsymbol{TSP}(u_0, v_0, a_u, a_v)$$

where u_0 and v_0 are the coordinate curves of the lower left or minimum vertex and a_u and a_v are the length of the principal curves.

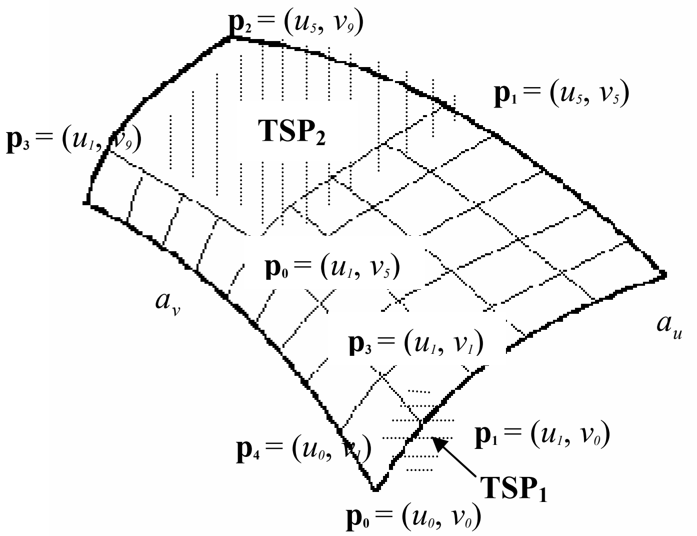

Figure 1 Tetragonal surface patch defined on the surface.

If the tetragonal surface patch divided to *mxn* sub-patches, then the length of the sides are *m* and *n*. So, to specify a tetragonal surface patch, we write

TSP = ***TSP***(u_i, v_j, a_u, a_v)

where u_i, v_j $(0 \le i \le m$ and $0 \le j \le n)$ are the coordinates of the lower left vertex and a_u, a_v $(0 \le a_u \le m\text{-}i$ and $0 \le a_v \le n\text{-}j)$ are the length of the sides.

The coordinates of its maximum vertex are simply (u_{i+a_u}, v_{j+a_v}). In counter-clockwise order from the minimum vertex, they are $\mathbf{p}_1 = (u_i, v_j)$, $\mathbf{p}_2 = (u_{i+a_u}, v_j)$, $\mathbf{p}_3 = (u_{i+a_u}, v_{j+a_v})$, $\mathbf{p}_4 = (u_i, v_{j+a_v})$.

Again use a lowercase boldface **p** to denote a point and a subscript to indicate a specific point, as in Figure 1. Note that $\mathbf{p}_{\mathbf{min}} = \mathbf{p}_0$ and $\mathbf{p}_{\mathbf{max}} = \mathbf{p}_2$.

3.2 Generalization of The O. Bonnet Integral Formula For Tetragonal Surface Patches

If (α) curve is a curvatural polygon which is formed of some regular arc pieces coming together as shown in Figure 2, the function φ at edge points of this curvatural polygon makes a stepping as much as the external angel at the edge point, for instance the angle φ is passing from $\overset{\frown}{p_4 p_1}$ arc to $\overset{\frown}{p_1 p_2}$ arc at the point p_1. So, if α_1 is the angle between the tangents of $\overset{\frown}{p_4 p_1}$ and $\overset{\frown}{p_1 p_2}$ arc, then its external angel is $\beta_1 = \pi - \alpha_1$. Hence, if (α) curve is composed of n curve parts, we have

$$\int_{(\alpha)} d\varphi = 2\pi - \sum_{i=1}^{n} \beta_i \ .$$

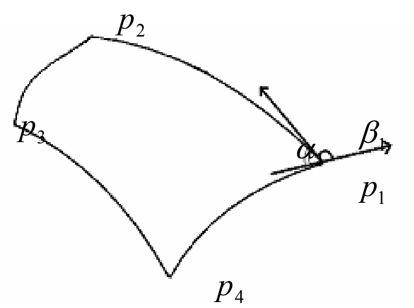

Figure 2

However, if $\beta_i = \pi - \alpha_i$, we get

$$\int_{(\alpha)} d\varphi = 2\pi - \sum_{i=1}^{n} (\pi - \alpha_i) = (2-n)\pi + \sum_{i=1}^{n} \alpha_i \ .$$

Then, the integral formula of O.Bonnet (9) is formed by

$$\int_{(\alpha)} k_g ds = (2-n)\pi + \sum_{i=1}^{n} \alpha_i - \iint_{(A)} K d\sigma \ , \qquad (2)$$

where n is the number of curvatural polygon's edges.

In E^3, we find

$$< r_{u1}, r_{v1} >= \sqrt{E_1} \cdot \sqrt{G_1} \cdot \cos\beta_1 = F_1 \quad \text{and} \quad \beta_1 = \arccos\frac{F_1}{\sqrt{E_1 G_1}},$$

for the edge point p_1 of the curvatural quadrangle $p_1 p_2 p_3 p_4$ which is surround the surface patch. If $\alpha_1 = \pi - \beta_1$, then we have

$$\alpha_1 = \pi - \arccos\frac{F_1}{\sqrt{E_1 G_1}}.$$

If similar operations made for the other edge points p_2, p_3 and p_4 of the surface patch, then we get

$$\alpha_i = \pi - \arccos\frac{F_i}{\sqrt{E_i G_i}}, \qquad i = 2,3,4.$$

All of them put in their places in equation (10), we find

$$\int_{(\alpha)} k_g ds = 2\pi - \sum_{i=1}^{4} \arccos\frac{F_i}{\sqrt{E_i G_i}} - \iint_{(A)} K d\sigma$$

which is the area of the tetragonal surface patch.

If we take a plane instead of a surface, then Gaussian curvature is

$$K = k_{n1} \cdot k_{n2} = 0$$

and the geodesic curvature of the plane curves is equal to common curvature. Thus, the relation of (9) is

$$\int_{(\alpha)} k_1 ds = 2\pi .$$

From the definition of curvature in the plane curves, if the angle element between the tangent at one point of the curve and the tangent at the point being infinitely close given point is $d\varphi$ and the arc element between these points is ds, then we have

$$k_1 = \frac{d\varphi}{ds} \qquad \text{and} \qquad \int k_1 ds = \int d\varphi = 2\pi .$$

Therefore, if we take the curvatural polygon in the plane which is defined by equation (10), the relation becomes

$$\int k_1 ds = (2-n)\pi + \sum_{i=1}^{n} \alpha_i .$$

If we take the polygon as a linear polygon, then $k_1 \to 0$ and we get

$$\sum \alpha_i = (n-2)\pi .$$

To find the length of the domain of the tetragonal surface patch, we firstly find

$$s_1 = \int_0^1 \|r_u\| du \quad \text{and} \quad s_2 = \int_0^1 \|r_u\| du$$

for the curves of $v = 0$ and $v = 1$, respectively. Similarly we find

$$s_3 = \int_0^1 \|r_v\| dv \quad \text{and} \quad s_4 = \int_0^1 \|r_v\| dv$$

for the curves of $u = 0$ and $u = 1$, respectively. Then, we find

$$\text{Perimeter} = \sum_{i=1}^{4} s_i \ .$$

3.3 The Relationships And Algorithms of The Tetragonal Surface Patch With a Point And The Other Tetragonal Surface Patches

The geometric center of a patch is at the point

$$p_{CG} = \left(u_{i+{a_u}/{2}}, v_{j+{a_v}/{2}} \right).$$

Many computer graphics problems require that you determine if a given point is inside, outside, or on the boundary of a closed shape. Given a point $\boldsymbol{p} = (u, v)$, test its coordinates against those of $\boldsymbol{p_{min}}$ and $\boldsymbol{p_{max}}$. Figure 3, if the point is inside the patch, then both of the following conditions must be satisfied :

$$u_{min} < u < u_{max} \quad \text{and} \quad v_{min} < v < v_{max}.$$

If the point is outside the patch then at least one of the following conditions is true :

$$u < u_{min} \quad \text{or} \quad u > u_{max}\ , \quad v < v_{min} \quad \text{or} \quad v > v_{max}.$$

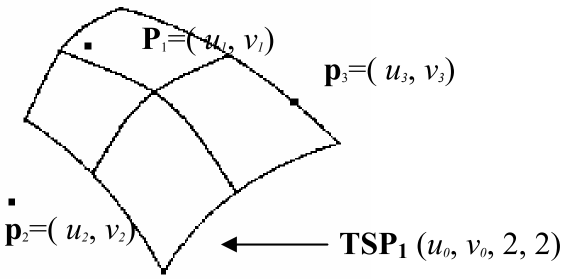

Figure 3 Point classification with respect to a tetragonal surface patch.

If the point is on the boundary of the patch, then

$$\left.\begin{array}{c} u = u_{\min} \\ or \\ u = u_{\max} \end{array}\right\} \quad \text{and} \quad v_{min} < v < v_{max}$$

or

$$\left.\begin{array}{c} v = v_{\min} \\ or \\ v = v_{\max} \end{array}\right\} \quad \text{and} \quad u_{min} < u < u_{max}\ .$$

Let $p_{min,1}$ $(u_{min,1}, v_{min,1})$ and $p_{max,1}$ $(u_{max,1}, v_{max,1})$ are the minimum and maximum points of the patch $\boldsymbol{TSP_1}$, respectively. Also, $a_{u,1}$ and $a_{v,1}$ are the length of the sides. Similarly, let $p_{min,2}$ $(u_{min,2}, v_{min,2})$ and $p_{max,2}$ $(u_{max,2}, v_{max,2})$ are the minimum and maximum points of the patch $\boldsymbol{TSP_2}$, respectively. Also, $a_{u,2}$ and $a_{v,2}$ are the length of the sides.

1. ***TSP₁*** is inside ***TSP₂*** if and only if

$$u_{\min,1} \geq u_{\min,2} \qquad \text{and} \qquad u_{\max,1} \leq u_{\max,2}$$
$$v_{\min,1} \geq v_{\min,2} \qquad\qquad\qquad v_{\max,1} \leq v_{\max,2}$$

Note that if the above equation is true, then there are also true that $a_{u,1} < a_{u,2}$ and $a_{v,1} < a_{v,2}$, see Figure 4(a).

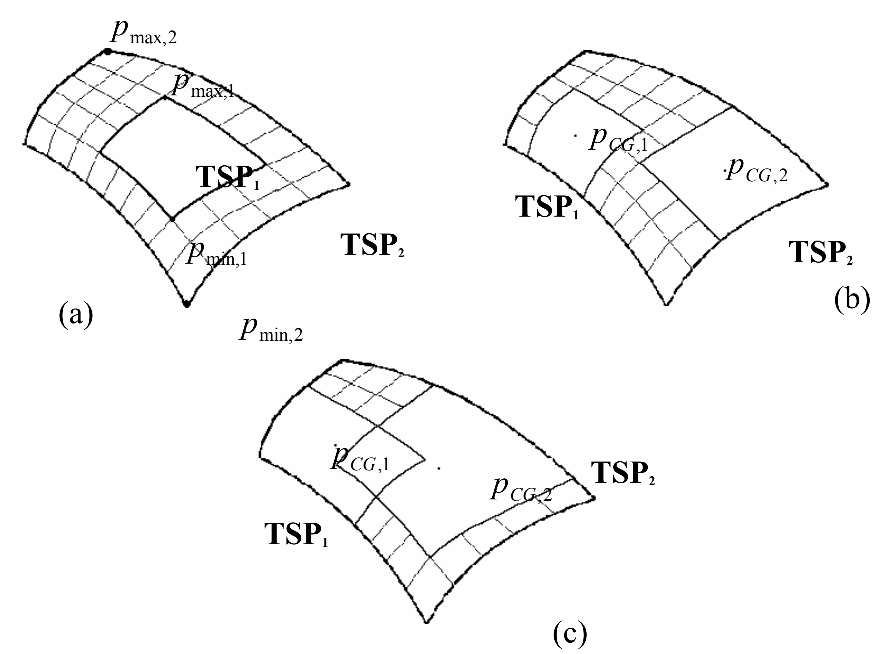

Figure 4 Relationships between two tetragonal surface patches.

2. ***TSP₁*** is outside ***TSP₂*** if and only if

$$\left|u_{CG,1} - u_{CG,2}\right| \geq \frac{a_{u,1} + a_{u,2}}{2}, \qquad \left|v_{CG,1} - v_{CG,2}\right| \geq \frac{a_{v,1} + a_{v,2}}{2},$$

where $(u_{CG,1}, v_{CG,1})$ and $(u_{CG,2}, v_{CG,2})$ are the geometric centers of the ***TSP₁*** and ***TSP₂***, respectively (Figure 4(b)).

3. In Figure 4(c), ***TSP₁*** is intersect ***TSP₂*** if and only if

$$\frac{\left|a_{u,1} - a_{u,2}\right|}{2} < \left|u_{CG,1} - u_{CG,2}\right| < \frac{a_{u,1} + a_{u,2}}{2}$$

and

$$\frac{\left|a_{v,1} - a_{v,2}\right|}{2} < \left|v_{CG,1} - v_{CG,2}\right| < \frac{a_{v,1} + a_{v,2}}{2},$$

4. Now consider the conditions describing two patches that share a boundary. How many ways can you imagine this condition occurring? Figure 5 shows four ways that two square patch can share a *v* = *const* boundary. Express these as

$$\left.\begin{array}{ll} v_{\max,1} = v_{\min,2} & or \\ v_{\max,1} = v_{\max,2} & or \\ v_{\min,1} = v_{\min,2} & or \\ v_{\min,1} = v_{\max,2} & \end{array}\right\} \quad \text{and} \quad \left|u_{CG,1} - u_{CG,2}\right| < \frac{a_{u,1} + a_{u,2}}{2} \; .$$

Figure 5 shows four ways that two patches can share an *u* = *const* boundary. Express these as

$$\left.\begin{array}{ll} u_{\min,1} = u_{\max,2} & or \\ u_{\min,1} = u_{\min,2} & or \\ u_{\max,1} = u_{\max,2} & or \\ u_{\max,1} = u_{\min,2} & \end{array}\right\} \quad \text{and} \quad \left|v_{CG,1} - v_{CG,2}\right| < \frac{a_{v,1} + a_{v,2}}{2} \; .$$

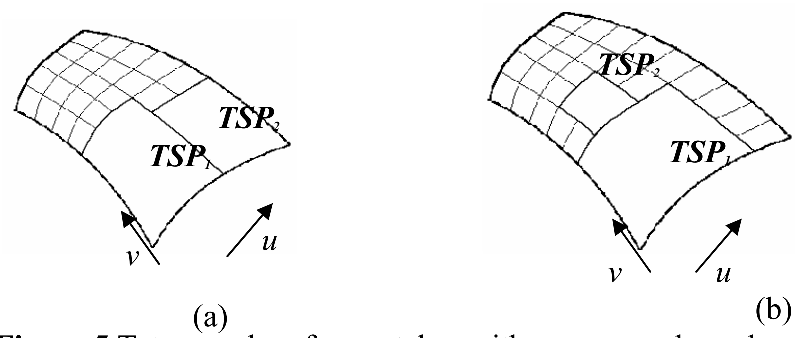

Figure 5 Tetragonal surface patches with a common boundary.

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