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Total Controllability of the Second Order Semi-Linear Differential Equation with Infinite Delay and Non-Instantaneous Impulses

Dimplekumar N. Chalishajar [1,*] and Avadhesh Kumar [2]

[1] Department of Applied Mathematics, Virginia Military Institute, Lexington, VA 24450, USA
[2] School of Basic Sciences, Indian Institute of Technology Mandi, Kamand 175005, India; soni.iitkgp@gmail.com
* Correspondence: chalishajardn@vmi.edu or dipu17370@gmail.com; Tel.: +1-540-464-7493

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Abstract: In this manuscript, a stronger concept of exact controllability called *Total Controllability* has been introduced. Sufficient conditions have been established for the total controllability of the proposed problem. The proposed control problem is a second-order semi-linear differential equation with infinite delay and non-instantaneous impulses. The tools for study include the strongly continuous cosine family and Sadovskii's fixed point theorem. The cosine family and the nonlinear function associated with the system are assumed to be non-compact. In addition, the total controllability of an integrodifferential problem has been investigated. Finally, an example is provided to illustrate the analytical findings.

Keywords: infinite delay; non-instantaneous impulses; Sadovskii's fixed point theorem; total controllability

MSC: 34K30; 34K45; 34G20; 47D09

1. Introduction

A number of evolving processes such as shocks, harvesting, and natural disasters are generally subjected to abrupt changes. These phenomena involve perturbations with negligible duration in comparison with the duration of an entire evolution. Sometimes, abrupt changes may stay for finite time intervals. Such impulses are known as non-instantaneous impulses. The study of non-instantaneous impulsive differential equations have significant applications in different areas, for example, in hemodynamical equilibrium and the theory of rocket combustion. A familiar application of non-instantaneous impulse is the introduction of insulin into the bloodstream. It produces an abrupt change in the bloodstream. The consequent absorption is a gradual process that remains active over a finite time span. Recently, existence, the uniqueness of solutions and controllability of impulsive problems with non-instantaneous impulses have been investigated by Muslim et al. [1,2].

The research on non-instantaneous impulses is of current interest to many authors. Several new research papers have recently been published on non-instantaneous impulsive systems [3–6]. Hernández et al. [7] studied mild and classical solutions for the impulsive differential equation with non-instantaneous impulses, which is of the form:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), \ t \in (s_i, t_{i+1}], \ i = 0, 1, ..., m, \\ x(t) = g_i(t, x(t)), \ t \in (t_i, s_i], \ i = 1, 2, ..., m, \\ x(0) = x_0 \in X. \end{cases} \tag{1}$$

In [8], Wang and Fečkan gave a remark on the impulsive condition of the system (1):

$$x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, ..., m, \tag{2}$$

where $g_i \in C([t_i, s_i] \times X, X)$ with positive constants L_{g_i}, $i = 1, ..., m$ such that

$$\|g_i(t, x_1) - g_i(t, x_2)\| \le L_{g_i}\|x_1 - x_2\|, \quad \forall\, x_1, x_2 \in X.$$

It follows from Theorem 2.1 in [7] that $\max\{L_{g_i} : i = 1, ..., m\} < 1$ is a necessary condition. Then, Banach's fixed point theorem gives a unique $z_i \in C([t_i, s_i], X)$ so that $z = g_i(t, z)$ if and only if $z = z_i(t)$. Thus, (2) is equivalent to

$$x(t) = z_i(t), \quad t \in (t_i, s_i], \quad i = 1, 2, ..., m, \tag{3}$$

which does not depend on the state $x(\cdot)$. Thus, it is necessary to modify (2) and consider the condition

$$x(t) = g_i(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, 2, ..., m. \tag{4}$$

Hence, $x(t_i^+) = g_i(t_i, x(t_i^-))$, $i = 1, 2, ..., m$. The symbols $x(t_i^+) := \lim_{\epsilon \to 0^+} x(t_i + \epsilon)$ and $x(t_i^-) := \lim_{\epsilon \to 0^-} x(t_i + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_i$ respectively. Motivated by the above remark, Wang and Fečkan [8] have shown existence, uniqueness and stability of solutions of such general class of impulsive differential equations.

Delay differential equations or functional differential equations have been frequently used in the modeling of scientific phenomena for many years. Often, it has been assumed that the delays are either fixed constants or distributed delays. In recent years, many papers have been published on time delay of both kinds, finite as well as infinite [9–13] .

Controllability is a mathematical problem, in which one can drive the given state of some dynamical systems by means of a control parameter presented in the equation. Many authors dealt with controllability problems that can be found in many recently published papers [14–17]. In [9], Arthi and Balachandran have established controllability results of second order instantaneous impulsive systems with infinite delay. Selvia and Arjunanb [13] have studied controllability for the impulsive differential systems with instantaneous impulses and finite delay. Up to now, the abundance of controllability results have only been available for the instantaneous impulsive systems. Recently, Wang et al. [18] discussed controllability of fractional non-instantaneous impulsive differential inclusions. However, Wang et al. [18] achieved exact controllability by only applying control in the last subinterval of time. However, in this manuscript, the control is applied for each subinterval of time, due to which the concept of total controllability arises. Moreover, none of the research papers have so far discussed the controllability of the non-instantaneous impulsive differential equation with infinite delay. Therefore, this manuscript is devoted to the study of total controllability for the following second-order semi-linear differential equation with infinite delay and non-instantaneous impulses in a Banach space X:

$$\begin{cases} x''(t) = Ax(t) + Bu(t) + f(t, x_t),\ t \in (s_i, t_{i+1}],\ i = 0, 1, ..., m, \\ x(t) = J_i^1(t, x(t_i^-)),\ t \in (t_i, s_i],\ i = 1, 2, ..., m, \\ x'(t) = J_i^2(t, x(t_i^-)),\ t \in (t_i, s_i],\ i = 1, 2, ..., m, \\ x(\theta) = \phi(\theta) \in \mathcal{B}_h,\ \ \theta \in (-\infty, 0],\ \ x'(0) = \xi \in X, \end{cases} \tag{5}$$

where $x(t)$ is a state function with time interval $0 = s_0 = t_0 < t_1 < s_1 < t_2, ..., t_m < s_m < t_{m+1} = T < \infty$. The control function $u(\cdot) \in L^2(J_1 := \bigcup_{i=0}^{m}[s_i, t_{i+1}], U)$, where U is a Banach space. Let B be a bounded linear operator from U to X. Let $x_t : (-\infty, 0] \to X;\ x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space $\mathcal{B}_h$. Consider the state function $x \in C((t_i, t_{i+1}], X), i = 0, 1, ..., m$ and there exist $x(t_i^-)$ and $x(t_i^+)$, $i = 1, 2, ..., m$ with $x(t_i^-) = x(t_i)$. The functions $J_i^1(t, x(t_i^-))$ and $J_i^2(t, x(t_i^-))$ represent non-instantaneous impulses during the intervals $(t_i, s_i]$ $, i = 1, 2, ..., m.$ The operator A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $\mathcal{C}(t)$ on X. J_i^1, J_i^2 and f are suitable functions. These functions will be explained briefly in the subsequent sections.

In many problems, such as the transverse motion of an extensible beam and the vibration of hinged bars, we deal with several partial differential equations. These partial differential equations can be formulated as the second-order abstract differential equations in the infinite dimensional spaces. The theory of strongly continuous cosine family is used to study the second-order abstract differential equations. Many authors utilize this strongly continuous cosine family tool to investigate the existence and uniqueness of the solutions and various types of controllability for second-order nonlinear in abstract spaces [19–21].

The manuscript proceeds as follows. In Sections 1 and 2, the introduction, notations, and results are given which will be required for the later sections. In Sections 3 and 4, total controllability of the problem (5) and integrodifferential problem are investigated respectively. In Section 5, an example is given to show the application of the obtained abstract results.

2. Preliminaries and Assumptions

In this section, some useful properties of the theory of cosine family and definitions are briefly reviewed.

Definition 1. *A one parameter family $\mathcal{C}(t)$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family, if and only if*

(i) $\mathcal{C}(s+t)+\mathcal{C}(s-t) = 2\mathcal{C}(s)\mathcal{C}(t)$ *for all* $s,t \in \mathbb{R}$,
(ii) $\mathcal{C}(0) = I$,
(iii) $\mathcal{C}(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed point* $x \in X$.

$\mathcal{S}(t)$ is the sine function associated with the strongly continuous cosine family $\mathcal{C}(t)$, which is defined by

$$\mathcal{S}(t)x = \int_0^t \mathcal{C}(s)x\,ds, \quad x \in X, \quad t \in \mathbb{R}.$$

$D(A)$ be the domain of the operator A, which is defined by

$$D(A) = \{x \in X : \mathcal{C}(t)x \text{ is twice continuously differentiable in t}\}.$$

$D(A)$ is the Banach space endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ for all $x \in D(A)$. Let set E be defined as

$$E = \{x \in X : \mathcal{C}(t)x \text{ is once continuously differentiable in t}\},$$

which is a Banach space endowed with norm $\|x\|_E = \|x\| + \sup_{0\leq t\leq 1} \|A\mathcal{S}(t)x\|$ for all $x \in E$.

With the help of $\mathcal{C}(t)$ and $\mathcal{S}(t)$, an operator valued function is defined as

$$\bar{h}(t) = \begin{bmatrix} \mathcal{C}(t) & \mathcal{S}(t) \\ A\mathcal{S}(t) & \mathcal{C}(t) \end{bmatrix}.$$

Operator valued function $\bar{h}(t)$ is a strongly continuous group of bounded linear operators in the space $E \times X$ generated by the operator

$$\bar{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

defined in $D(A) \times E$. It follows that $A\mathcal{S}(t) : E \to X$ is a bounded linear operator and that $A\mathcal{S}(t)x \to 0$ as $t \to 0$, for each $x \in E$. If $x : [0,\infty) \to X$ is locally integrable function then

$$y(t) = \int_0^t \mathcal{S}(t-s)x(s)ds$$

defines an E valued continuous function, which is a consequence of the fact that

$$\int_0^t \bar{h}(t-s)\begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t \mathcal{S}(t-s)x(s)ds \\ \int_0^t \mathcal{C}(t-s)x(s)ds \end{bmatrix}.$$

This defines an $(E \times X)$ valued continuous function. For more details on cosine family theory, see [22–24].

We define the abstract phase space $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \to (0, \infty)$ is a continuous function with $l = \int_{-\infty}^0 h(s)ds < +\infty$.

Let $\mathcal{B}_h$ be defined as

$$\mathcal{B}_h := \Big\{\phi : (-\infty, 0] \to X \text{ such that, for any } r > 0, \phi(\theta) \text{ is bounded and measurable function on } [-r, 0] \text{ and } \int_{-\infty}^0 h(s) \sup_{s \le \theta \le 0} |\phi(\theta)| ds < +\infty\Big\}.$$

The phase space $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \le \theta \le 0} |\phi(\theta)| ds, \quad \forall\, \phi \in \mathcal{B}_h.$$

Hence, we can prove that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Lemma 1 ([12]). *Suppose $x \in \mathcal{B}_h$; then, for each $t \in J$, $x_t \in \mathcal{B}_h$. Moreover,*

$$l|x(t)| \le \|x_t\|_{\mathcal{B}_h} \le l \sup_{s \in [0,t]} (|x(s)| + \|x_0\|_{\mathcal{B}_h}),$$

where $l = \int_{-\infty}^0 h(s)ds < +\infty$.

Let $PC([0, T], X)$ be the space of piecewise continuous functions. $PC([0, T], X) = \{x : [0, T] \to X : x \in C((t_k, t_{k+1}], X), k = 0, 1, ..., m$ and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1, 2, ..., m\}$. It can be easily proved that $PC([0, T], X)$ for all $t \in [0, T]$, is a Banach space endowed with the supremum norm.

In the following definition, the concept of the mild solution for the problem (5) is introduced.

Definition 2. *A function $x : (-\infty, T] \to X$ is called a mild solution of the impulsive problem (5), if it satisfies the following relations:*
$x(\theta) = \phi(\theta) \in \mathcal{B}_h$, $\theta \in (-\infty, 0]$, $x'(0) = \xi$, and $x \in PC([0, T], X)$,
the non-instantaneous impulse conditions
$x(t) = J_i^1(t, x(t_i^-))$, $\quad t \in (t_i, s_i]$, $\quad i = 1, 2, ..., m$,
$x'(t) = J_i^2(t, x(t_i^-))$, $\quad t \in (t_i, s_i]$, $\quad i = 1, 2, ..., m$
and $x(t)$ is the solution of the following integral equations

$$\begin{aligned} x(t) &= \mathcal{C}(t)\phi(0) + \mathcal{S}(t)\xi + \int_0^t \mathcal{S}(t-s)[Bu(s) + f(s, x(s+\theta))]ds, \\ &\quad \forall\, t \in [0, t_1],\ \theta \in (-\infty, 0], \\ x(t) &= \mathcal{C}(t-s_i)J_i^1(s_i, x(t_i^-)) + \mathcal{S}(t-s_i)J_i^2(s_i, x(t_i^-)) + \int_{s_i}^t \mathcal{S}(t-s)[Bu(s) \\ &+ f(s, x(s+\theta))]ds,\ \forall\, t \in [s_i, t_{i+1}],\ i = 1, 2, ..., m,\ \theta \in (-\infty, 0]. \end{aligned} \tag{6}$$

Definition 3. *System (5) is said to be exactly controllable on $[0,T]$, if for the initial state $x(0) = \phi(0) \in \mathcal{B}_h$ and arbitrary final state $fs(t_{m+1}) \in X$, there exists a control $u \in L^2(J_1, U)$ such that the mild solution (6) satisfies $x(t_{m+1}) = fs(t_{m+1})$.*

Definition 4. ***(Total Controllability)*** *System (5) is said to be totally controllable on $[0,T]$, if for the initial state $x(0) = \phi(0) \in \mathcal{B}_h$ and arbitrary final state $fs(t_{i+1}) \in X$ of each sub-interval $[s_i, t_{i+1}]$, there exists a control $u \in L^2(J_1, U)$ such that the mild solution (6) satisfies $x(t_{i+1}) = fs(t_{i+1})$ where $i = 0, 1, ..., m$.*

Lemma 2. *(Sadovskii's fixed point theorem) [25]. Suppose that M is a nonempty, closed, bounded and convex subset of a Banach space X and $\Gamma : M \subseteq X \to X$ is a condensing operator. Then, the operator Γ has a fixed point in M.*

In order to prove the total controllability for problem (5), the following assumptions are taken:

- **(A1)** A is the infinitesimal generator of a strongly continuous cosine family $\mathcal{C}(t)$. In addition, there exists a constant N_1 such that $\|\mathcal{C}(t)\| \leq N_1$, for all $t \in J$.
- **(A2)** $f : J_1 \times \mathcal{B}_h \to X$, $J_1 = \bigcup_{i=0}^{m} [s_i, t_{i+1}]$ is a continuous function and there exist positive constant K_1 such that
$$\|f(t, x_1) - f(t, x_2)\| \leq K_1 \|x_1 - x_2\|_{\mathcal{B}_h}$$
for every $x_1, x_2 \in \mathcal{B}_h$, $t \in J_1$. In addition, there exists a positive constant N such that $N = \max_{t \in J_1} \|f(t, 0)\|$.
- **(A3)** B is continuous operator from U to X and the linear operator $W_{s_i}^{t_{i+1}} : L^2(J_1, U) \to X$ is defined by $W_{s_i}^{t_{i+1}} u = \int_{s_i}^{t_{i+1}} \mathcal{S}(t_{i+1} - s) Bu(s) ds$, $i = 0, 1, 2, ..., m$. It has a bounded invertible operator $(W_{s_i}^{t_{i+1}})^{-1}$ which takes values in $L^2(J_1, U)/Ker W_{s_i}^{t_{i+1}}$ and there exist positive constants K_2^i such that $\|(W_{s_i}^{t_{i+1}})^{-1}\| \leq K_2^i$.
- **(A4)** There exist positive constants $C_{J_i^1}$ and $C_{J_i^2}$, $i = 1, 2, ..., m$ such that $C_{J_i^1} = \max_{t \in I_i} \|J_i^1(t, 0)\|$ and $C_{J_i^2} = \max_{t \in I_i} \|J_i^2(t, 0)\|$, where $I_i := [t_i, s_i]$.
- **(A5)** $J_i^k \in C(I_i \times X, X)$ and there are positive constants $L_{J_i^k}$, $i = 1, 2, ..., m$, $k = 1, 2$, such that $\|J_i^k(t, x_1) - J_i^k(t, x_2)\| \leq L_{J_i^k} \|x_1 - x_2\|$, $\forall\, t \in I_i$ and $x_1, x_2 \in X$.

3. Total Controllability

In this section, total controllability of the system (5) is investigated.

Let us define

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{C}(t_{i+1} - s_i)(J_i^1(s_i, x(t_i^-))) + \mathcal{S}(t_{i+1} - s_i)(J_i^2(s_i, x(t_i^-))), & t \in J_1, \end{cases}$$

where $J_0^1(t, \cdot) = \phi(0)$, $J_0^2(t, \cdot) = \xi$. Let $x(t) = z(t) + \tilde{\phi}(t)$, $t \in (-\infty, 0] \cup J_1$. It is clear that x satisfies (6), if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_{s_i}^{t} S(t - s)[Bu(s) + f(s, z_s + \tilde{\phi}_s)] ds, \ \forall\, t \in J_1.$$

For each positive number δ, we define a ball

$$\mathcal{W}_\delta = \{z : (-\infty, T] \to X : z|_J \in PC(J, X) : \|z\| \leq \delta\},$$

then, for each δ, it is obviously a bounded, closed, and convex set in $PC(J, X)$. For the sake of convenience, we denote $c_1 = \sup_{t \in J} \|\tilde{\phi}_t\|_{\mathcal{B}_h}$

Lemma 3. *If all the assumptions (A1)–(A4) are satisfied, then the control function for the problem (5) has an estimate* $\|u(t)\| \leq Q_i$, $\forall\, t \in [s_i, t_{i+1}]$, $\quad i = 0, 1, 2, ..., m$, *where*

$$\begin{aligned} Q_i &= K_2^i \Big[\|fs(t_{i+1})\| + N_1[L_{J_i^1}(\delta + c_1) + C_{J_i^1}] + N_1 T[L_{J_i^2}(\delta + c_1) + C_{J_i^2}] \\ &+ N_1 T^2 \Big[K_1(l\delta + c_1) + N \Big] \Big]. \end{aligned}$$

Proof. The feedback control function for $t \in [s_i, t_{i+1}]$, $i = 0, 1, 2, ..., m$, is defined as follows:

$$\begin{aligned} u(t) = (W_{s_i}^{t_{i+1}})^{-1} \Big[fs(t_{i+1}) - \mathcal{C}(t_{i+1} - s_i)(J_i^1(s_i, x(t_i^-))) - \mathcal{S}(t_{i+1} - s_i)(J_i^2(s_i, x(t_i^-))) \\ - \int_{s_i}^{t_{i+1}} \mathcal{S}(t_{i+1} - s) f(s, x_s) ds \Big](t), \end{aligned} \tag{7}$$

where $fs(t_{i+1})$ is arbitrary final state of each sub-interval $[s_i, t_{i+1}]$, $i = 0, 1, ..., m$. By mild solution (6), the final state at $t = t_{i+1}$ is

$$\begin{aligned} x(t_{i+1}) &= \mathcal{C}(t_{i+1} - s_i)(J_i^1(s_i, x(t_i^-))) + \mathcal{S}(t_{i+1} - s_i)(J_i^2(s_i, x(t_i^-))) \\ &+ \int_{s_i}^{t_{i+1}} \mathcal{S}(t_{i+1} - s) f(s, x_s)) ds + \int_{s_i}^{t_{i+1}} \mathcal{S}(t_{i+1} - s) Bu(s) ds \\ &= \mathcal{C}(t_{i+1} - s_i)(J_i^1(s_i, x(t_i^-))) + \mathcal{S}(t_{i+1} - s_i)(J_i^2(s_i, x(t_i^-))) \\ &+ \int_{s_i}^{t_{i+1}} \mathcal{S}(t_{i+1} - s) f(s, x_s) ds + W_{s_i}^{t_{i+1}} u \\ &= \mathcal{C}(t_{i+1} - s_i)(J_i^1(s_i, x(t_i^-))) + \mathcal{S}(t_{i+1} - s_i)(J_i^2(s_i, x(t_i^-))) \\ &+ \int_{s_i}^{t_{i+1}} S(t_{i+1} - s) f(s, x_s) ds + W_{s_i}^{t_{i+1}} (W_{s_i}^{t_{i+1}})^{-1} \Big[fs(t_{i+1}) \\ &- \mathcal{C}(t_{i+1} - s_i)(J_i^1(s_i, x(t_i^-))) - \mathcal{S}(t_{i+1} - s_i)(J_i^2(s_i, x(t_i^-))) \\ &- \int_{s_i}^{t_{i+1}} \mathcal{S}(t_{i+1} - s) f(s, x_s) ds \Big] \\ &= fs(t_{i+1}). \end{aligned}$$

Hence, control function (7) is suitable for the problem (5), for every $t \in [s_i, t_{i+1}]$ and $i = 0, 1, 2, ..., m$. The estimate of control function $u(t)$ is given by

$$\begin{aligned} \|u(t)\| &\leq K_2^i \Big[\|fs(t_{i+1})\| + N_1 \|J_i^1(s_i, z(t_i^-) + \tilde{\phi}(t_i^-))\| + N_1 T \|J_i^2(s_i, z(t_i^-) + \tilde{\phi}(t_i^-))\| \\ &+ N_1 T \int_{s_i}^{t_{i+1}} \|f(s, z_s + \tilde{\phi}_s)\| ds \Big] \\ &\leq K_2^i \Big[\|fs(t_{i+1})\| + N_1[L_{J_i^1}(\delta + c_1) + C_{J_i^1}] + N_1 T[L_{J_i^2}(\delta + c_1) + C_{J_i^2}] \\ &+ N_1 T^2 \Big[K_1(l\delta + c_1) + N \Big] \Big]. \end{aligned}$$

Hence, the required estimate for control (7) is obtained. □

Theorem 1. *If all the assumptions (A1)–(A5) are satisfied, then the second order control problem (5) is totally controllable on* $[0, T]$ *provided that*

$$L_F = \max_{1\le i\le m}\left\{N_1T^2K_1l,\ L_{J_i^1}[N_1L_{J_i^1}+N_1TL_{J_i^2}+N_1T^2K_1l]\right\}<1. \tag{8}$$

Proof. Let $\tilde{K}=\|B\|$ and a map $\mathcal{F}: PC(J,X)\to PC(J,X)$ be defined as

$$\begin{aligned}
(\mathcal{F}z)(t) &= J_i^1\left(t,\int_{s_{i-1}}^{t_i}\mathcal{S}(t_i-s)[Bu(s)+f(s,z_s+\tilde{\phi}_s)]ds\right),\ \forall\ t\in(t_i,s_i]\ i=1,2,...,m;\\
(\mathcal{F}z)(t) &= \int_0^t \mathcal{S}(t-s)[Bu(s)+f(s,z_s+\tilde{\phi}_s)]ds,\ \forall\ t\in[0,t_1];\\
(\mathcal{F}z)(t) &= \int_{s_i}^t \mathcal{S}(t-s)[Bu(s)+f(s,z_s+\tilde{\phi}_s)]ds,\ \forall\ t\in(s_i,t_{i+1}]\ i=1,2,...,m.
\end{aligned}$$

In order to apply Sadovskii's fixed point theorem, first we need to show that there exists a positive number δ such that $\mathcal{F}(\mathcal{W}_\delta)\subseteq\mathcal{W}_\delta$. If it is not true, then for each positive number δ, there exists a function $z^\delta(\cdot)\in\mathcal{W}_\delta$, but $\mathcal{F}(z^\delta)\notin\mathcal{W}_\delta$, that is, $\|(\mathcal{F}z^\delta)(t)\|>\delta$ for some $t\in J$. For $t\in(s_i,t_{i+1}]$, $i=1,2,...,m$, we have

$$\begin{aligned}
\delta<\|(\mathcal{F}z^\delta)(t)\| &\le \int_{s_i}^t\|\mathcal{S}(t-s)Bu(s)\|ds+\int_{s_i}^t\|\mathcal{S}(t-s)f(s,z_s^\delta+\tilde{\phi}_s)\|ds\\
&+ N_1T^2\tilde{K}Q_i+N_1T^2[K_1(l\delta+c_1)+N].
\end{aligned}$$

Dividing both sides by δ and taking limit as $\delta\to\infty$, we get

$$\begin{aligned}
1 &< N_1L_{J_i^1}+N_1TL_{J_i^2}+N_1T^2\tilde{K}[N_1L_{J_i^1}+N_1TL_{J_i^2}+N_1T^2K_1l]\\
&+ N_1T^2K_1l\\
&= (1+N_1T^2\tilde{K})[N_1L_{J_i^1}+N_1TL_{J_i^2}+N_1T^2K_1l].
\end{aligned}$$

For $t\in[0,t_1]$, we have

$$\begin{aligned}
1<\|(\mathcal{F}z^\delta)(t)\| &\le \|\mathcal{C}(t)\phi(0)\|+\|\mathcal{S}(t)\xi\|+\int_0^t\|\mathcal{S}(t-s)Bu(s)\|ds\\
&+ \int_0^t\|\mathcal{S}(t-s)f(s,z_s^\delta+\tilde{\phi}_s)\|ds\\
&\le N_1\|\phi(0)\|+N_1T\|\xi\|+N_1T^2\tilde{K}Q_0+N_1T^2[K_1(l\delta+c_1)+N]
\end{aligned}$$

Dividing both sides by δ and taking limit as $\delta\to\infty$, we get

$$\begin{aligned}
1 &< N_1T^2\tilde{K}N_1T^2K_1l+N_1T^2K_1l\\
&= (1+N_1T^2\tilde{K})N_1T^2K_1l.
\end{aligned}$$

Similarly, for $t\in(t_i,s_i]$, $i=1,2,...,m$, we have the following estimate

$$1<L_{J_i^1}(1+N_1T^2\tilde{K})[N_1L_{J_i^1}+N_1TL_{J_i^2}+N_1T^2K_1l].$$

After summarizing the above three inequalities, we get the following inequality

$$1<\beta,$$

where

$$\beta = \min_{1\le i\le m}\left\{N_1T^2K_1l,\ L_{J_i^1}[N_1L_{J_i^1}+N_1TL_{J_i^2}+N_1T^2K_1l]\right\}.$$

This contradicts condition (8). Hence, for some positive number δ, $\mathcal{F}(\mathcal{W}_\delta) \subseteq \mathcal{W}_\delta$. Next, we show that $\mathcal{F}$ is the contraction operator. For any $z, y \in \mathcal{W}_\delta,\ t \in (s_i, t_{i+1}],\ i = 1, 2, ..., m$,

$$\begin{aligned}
\|(\mathcal{F}z)(t) - (\mathcal{F}y)(t)\| \quad &\leq \quad \tilde{K}K_2^i N_1 T^2 \Big\{ N_1 L_{J_i^1} \|z(t_i^-) - y(t_i^-)\| + N_1 T L_{J_i^2} \|z(t_i^-) - y(t_i^-)\| \\
&+ \quad K_1 \int_{s_i}^{T} \|\mathcal{S}(T-\eta)\| \|z_\eta - y_\eta\| d\eta \Big\} + K_1 \int_{s_i}^{t} \|\mathcal{S}(t-s)\| \|z_s - y_s\| ds \\
&\leq \quad \tilde{K}K_2^i N_1 T^2 \Big\{ (N_1 L_{J_i^1} + N_1 T L_{J_i^2}) \|z(t_i^-) - y(t_i^-)\| \\
&+ \quad K_1 N_1 T^2 l \|z - y\| \Big\} + K_1 N_1 T^2 l \|z - y\|..
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(\mathcal{F}z) - (\mathcal{F}y)\| \quad &\leq \quad \tilde{K}K_2^i N_1 T^2 \Big\{ (N_1 L_{J_i^1} + N_1 T L_{J_i^2}) \|z - y\| \\
&+ \quad K_1 N_1 T^2 l \|z - y\| \Big\} + K_1 N_1 T^2 l \|z - y\| \\
&\leq \quad (1 + \tilde{K}K_2^i N_1 T^2) \Big[(N_1 L_{J_i^1} + N_1 T L_{J_i^2}) \\
&+ \quad K_1 N_1 T^2 l \Big] \|z - y\|.
\end{aligned}$$

For $t \in [0, t_1]$,

$$\begin{aligned}
\|(\mathcal{F}z)(t) - (\mathcal{F}y)(t)\| \quad &\leq \quad \tilde{K}K_2^0 N_1 T^2 K_1 \int_0^T \|\mathcal{S}(T-\eta)\| \|z_\eta - y_\eta\| d\eta \\
&+ \quad K_1 \int_0^t \|\mathcal{S}(t-s)\| \|z_s - y_s\| ds \\
&\leq \quad \tilde{K}K_2^0 N_1 T^2 K_1 N_1 T^2 l \|z - y\| \\
&+ \quad K_1 N_1 T^2 l \|z - y\|.
\end{aligned}$$

Therefore,

$$\|(\mathcal{F}z) - (\mathcal{F}y)\| \quad \leq \quad (1 + \tilde{K}K_2^0 N_1 T^2) K_1 N_1 T^2 l \|z - y\|.$$

Similarly, for $t \in (t_i, s_i], i = 1, 2, ..., m$,

$$\begin{aligned}
&\|(\mathcal{F}z)(t) - (\mathcal{F}y)(t)\| \\
&\quad \leq L_{J_i^1} \tilde{K}K_2^i N_1 T^2 \Big\{ N_1 L_{J_i^1} \|z(t_{i-1}^-) - y(t_{i-1}^-)\| + N_1 T L_{J_i^2} \|z(t_{i-1}^-) - y(t_{i-1}^-)\| \\
&\qquad + K_1 \int_{s_{i-1}}^{T} \|\mathcal{S}(T-\eta)\| \|z_\eta - y_\eta\| d\eta \Big\} + K_1 \int_{s_{i-1}}^{t_i} \|\mathcal{S}(t_i - s)\| \|z_s - y_s\| ds \\
&\leq L_{J_i^1} \tilde{K}K_2^i N_1 T^2 \Big\{ (N_1 L_{J_i^1} + N_1 T L_{J_i^2}) \|z(t_{i-1}^-) - y(t_{i-1}^-)\| \\
&\qquad + K_1 N_1 T^2 l \|z - y\| \Big\} + K_1 N_1 T^2 l \|z - y\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|(\mathcal{F}z)-(\mathcal{F}y)\| &\leq L_{J_i^1}\tilde{K}K_2^i N_1 T^2\Big\{(N_1 L_{J_i^1}+N_1 T L_{J_i^2})\|z-y\| \\ &+ K_1 N_1 T^2 l\|z-y\|\Big\} + K_1 N_1 T^2 l\|z-y\| \\ &\leq L_{J_i^1}(1+\tilde{K}K_2^i N_1 T^2)\Big[(N_1 L_{J_i^1}+N_1 T L_{J_i^2}) \\ &+ K_1 N_1 T^2 l\Big]\|z-y\|.\end{aligned}$$

After summarizing the above three inequalities, we get the following inequality

$$\|(\mathcal{F}z)-(\mathcal{F}y)\| \leq L_F\|z-y\|.$$

From condition (8), we conclude that $\mathcal{F}$ is a strict contraction mapping on $\mathcal{W}_\delta$. Therefore, $\mathcal{F}$ is a condensing map on $\mathcal{W}_\delta$. Hence, the application of Sadovskii's fixed point theorem results in a mild solution z. Defining $x(t)=z(t)+\tilde{\phi}(t)$, $t\in(-\infty,T]$, we obtain that $z(\cdot)$ is a mild solution of the problem (5). Thus, system (5) is totally controllable. □

4. Integrodifferential Equation

In this section, a control system represented by an integrodifferential equation in the Banach space X is considered as follows:

$$\begin{cases} x''(t)=Ax(t)+Bu(t)+f(t,x_t)+\int_0^t \kappa(t-s)g(s,x_s)ds,\ t\in\bigcup_{i=0}^m(s_i,t_{i+1}], \\ x(t)=J_i^1(t,x(t_i^-)),\quad t\in\bigcup_{i=1}^m(t_i,s_i], \\ x'(t)=J_i^2(t,x(t_i^-)),\quad t\in\bigcup_{i=1}^m(t_i,s_i], \\ x(\theta)=\phi(\theta)\in\mathcal{B}_h,\ \ \theta\in(-\infty,0],\ \ x'(0)=\xi\in X, \end{cases} \tag{9}$$

where $x(t)$ is a state function with time interval $0=s_0=t_0<t_1<s_1<t_2,...,t_m<s_m<t_{m+1}=T$. The control function $u(\cdot)\in L^2(J_1,U)$, where U is a Banach space. Let A be the infinitesimal generator of a strongly continuous cosine family of linear operators $\mathcal{C}(t)$ on X. Let $B:U\to X$ be a bounded linear operator.

In order to prove the controllability of the integrodifferential equation (9), the following conditions are required:

(A6) The real-valued function κ is piecewise continuous on $[0,T]$ and there exists a positive constant Ω such that $\Omega=\int_0^T|\kappa(s)|ds$.

(A7) $g:J_1\times\mathcal{B}_h\to X$, is a continuous function and there exist positive constant K_2 such that

$$\|g(t,x_1)-g(t,x_2)\| \leq K_2\|x_1-x_2\|_{\mathcal{B}_h}$$

for every $x_1,x_2\in\mathcal{B}_h$, $t\in J_1$. In addition, there exists a positive constant $\tilde{N}$ such that $\tilde{N}=\max_{t\in J_1}\|g(t,0)\|$.

Definition 5. *A function $x:(-\infty,T]\to X$ is called a mild solution of the impulsive problem (9), if it satisfies the following relations:*
$x(\theta)=\phi(\theta)\in\mathcal{B}_h$, $\theta\in(-\infty,0]$, $x'(0)=\xi$ and $x\in PC([0,T],X)$,
the non-instantaneous impulse conditions
$x(t)=J_i^1(t,x(t_i^-))$, $t\in(t_i,s_i]$, $i=1,2,...,m$,

$x'(t) = J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, 2, ..., m,$
and $x(t)$ is the solution of the following integral equations

$$
\begin{aligned}
x(t) &= \mathcal{C}(t)\phi(0) + \mathcal{S}(t)\xi + \int_0^t \mathcal{S}(t-s)\Big[Bu(s) + f(s, x(s+\theta)) \\
&+ \int_0^s \kappa(s-\eta)g(\eta, x(\eta+\theta))d\eta\Big]ds, \ \forall\ t \in [0, t_1],\ \theta \in (-\infty, 0], \\
x(t) &= \mathcal{C}(t-s_i)(J_i^1(s_i, x(t_i^-))) + \mathcal{S}(t-s_i)(J_i^2(s_i, x(t_i^-))) \\
&+ \int_{s_i}^t \mathcal{S}(t-s)\Big[Bu(s) + f(s, x(s+\theta)) \\
&+ \int_0^s \kappa(s-\eta)g(\eta, x(\eta+\theta))d\eta\Big]ds, \ \forall\ t \in [s_i, t_{i+1}], \quad i = 1, 2, ..., m,\ \theta \in (-\infty, 0].
\end{aligned} \tag{10}
$$

Theorem 2. *If all the assumptions (A1)–(A7) are satisfied, then the second order integrodifferential control problem (9) is totally controllable on $[0, T]$ provided that*

$$\tilde{L}_F = \max_{1\le i\le m}\Big\{N_1T^2l(K_1 + \Omega K_2),\, L_{J_i^1}\Big[N_1L_{J_i^1} + N_1TL_{J_i^2} + N_1T^2l(K_1 + \Omega K_2)\Big]\Big\} < 1.$$

Proof. The feedback control function for $t \in [s_i, t_{i+1}],\ i = 0, 1, 2, ..., m,$ is defined as

$$
\begin{aligned}
u(t) &= (W_{s_i}^{t_{i+1}})^{-1}\Big[fs(t_{i+1}) - \mathcal{C}(t_{i+1}-s_i)(J_i^1(s_i, x(t_i^-))) - \mathcal{S}(t_{i+1}-s_i)(J_i^2(s_i, x(t_i^-))) \\
&- \int_{s_i}^{t_{i+1}} \mathcal{S}(t_{i+1}-s)\Big\{f(s, x_s) + \int_0^s \kappa(s-\eta)g(\eta, x_\eta)d\eta\Big\}ds\Big](t).
\end{aligned}
$$

The estimate of control function $u(t)$ is obtained as

$$\|u(t)\| \le \tilde{Q}_i,$$

where

$$
\begin{aligned}
\tilde{Q}_i &= K_2^i\Big[\|fs(t_{i+1})\| + N_1[L_{J_i^1}(\tilde{\delta} + c_1) + C_{J_i^1}] + N_1T[L_{J_i^2}(\tilde{\delta} + c_1) + C_{J_i^2}] \\
&+ N_1T^2\Big[K_1(l\tilde{\delta} + c_1) + N + \Omega[K_2(l\tilde{\delta} + c_1) + \tilde{N}]\Big]\Big].
\end{aligned}
$$

Let $x(t) = \tilde{z}(t) + \tilde{\phi}(t)$ where $t \in (-\infty, 0] \cup J_1$. It is clear that x satisfies (10), if and only if $\tilde{z}$ satisfies $\tilde{z}_0 = 0$ and

$$
\begin{aligned}
\tilde{z}(t) &= \int_{s_i}^t \mathcal{S}(t-s)\Big[Bu(s) + f(s, \tilde{z}_s + \tilde{\phi}_s) \\
&+ \int_0^s \kappa(s-\eta)g(\eta, \tilde{z}_\eta + \tilde{\phi}_\eta)d\eta\Big]ds, \ \forall\ t \in J_1.
\end{aligned}
$$

For each positive number $\tilde{\delta}$, we define a ball

$$\tilde{\mathcal{W}}\delta = \{\tilde{z} : (-\infty, T] \to X : \tilde{z}|_J \in PC(J, X) : \|\tilde{z}\| \ \le \tilde{\delta}\}.$$

A map $\tilde{\mathcal{F}} : PC(J, X) \to PC(J, X)$ is defined as

$$\begin{aligned}
(\tilde{\mathcal{F}}\tilde{z})(t) &= \int_{s_{i-1}}^{t_i} \mathcal{S}(t_i - s)\Big[Bu(s) + f(s, \tilde{z}_s + \tilde{\phi}_s) \\
&+ \int_0^s \kappa(s-\eta)g(\eta, \tilde{z}_\eta + \tilde{\phi}_\eta)d\eta\Big]ds\Big), \ \forall\, t \in (t_i, s_i] \ \ i = 1, 2, ..., m; \\
(\tilde{\mathcal{F}}\tilde{z})(t) &= \int_0^t \mathcal{S}(t - s)\Big[Bu(s) + f(s, \tilde{z}_s + \tilde{\phi}_s) \\
&+ \int_0^s \kappa(s-\eta)g(\eta, \tilde{z}_\eta + \tilde{\phi}_\eta)d\eta\Big]ds, \ \forall\, t \in [0, t_1]; \\
(\tilde{\mathcal{F}}\tilde{z})(t) &= \int_{s_i}^t \mathcal{S}(t - s)\Big[Bu(s) + f(s, \tilde{z}_s + \tilde{\phi}_s) \\
&+ \int_0^s \kappa(s-\eta)g(\eta, \tilde{z}_\eta + \tilde{\phi}_\eta)d\eta\Big]ds, \ \forall\, t \in (s_i, t_{i+1}] \ \ i = 1, 2, ..., m.
\end{aligned}$$

The following inequality can be easily obtained

$$\|(\tilde{\mathcal{F}}\tilde{z}) - (\tilde{\mathcal{F}}\tilde{y})\| \le \tilde{L}_F\|\tilde{z} - \tilde{y}\|.$$

Hence, Theorem 2 is the consequence of Theorem 1. □

5. Application

Example : A forced string equation in the space $X = \mathbb{R}$ is considered as follows:

$$\begin{cases}
x''(t) + a_1 x(t) + \frac{x_t}{(16+t)(1+x_t)} = g(t), \quad t \in J_1 = (0, \frac{\pi}{4}] \cup (\frac{\pi}{2}, \pi], \\
x(t) = \frac{1}{10}\sin(t)x(\frac{\pi}{4}^-), \quad t \in J_2 = (\frac{\pi}{4}, \frac{\pi}{2}], \\
x'(t) = \frac{1}{10}\cos(t)x(\frac{\pi}{4}^-), \quad t \in J_2 = (\frac{\pi}{4}, \frac{\pi}{2}], \\
x(c_1) = \phi(c_1), \ \ c_1 \in (-\infty, 0], \ \ x'(0) = 1,
\end{cases} \tag{11}$$

where $a_1 \in (0, \infty)$, $\phi : (-\infty, 0] \to \mathbb{R}$, $x_t = x(t + c_1)$ and $g \in C(J_1, \mathbb{R})$. Let operator A be defined as follows:

$$Ax = -a_1 x \quad \text{with} \quad D(A) = \mathbb{R}.$$

Here, clearly the value $-a_1$ behaves like an infinitesimal generator of a strongly continuous cosine family $\mathcal{C}(t) = \cos\sqrt{a_1}t$. The associated sine family is given by $\mathcal{S}(t) = \frac{1}{\sqrt{a_1}}\sin\sqrt{a_1}t$. Let x_t represent delay in the state of the differential Equation (11). Non-instantaneous impulses $\frac{1}{10}\sin(t)x(\frac{\pi}{4}^-)$ and $\frac{1}{10}\cos(t)x(\frac{\pi}{4}^-)$ are created when the bob on the string is pushed extreme in the interval $(\frac{\pi}{4}, \frac{\pi}{2}]$.

Comparing with(5), we have

$$f(t, x_t) = \frac{x_t}{(16+t)(1+x_t)}, \ J_1^1(t, x(t_1^-)) = \frac{1}{10}\sin(t)x(\frac{\pi}{4}^-)$$

and $J_1^2(t, x(t_1^-)) = \frac{1}{10}\cos(t)x(\frac{\pi}{4}^-)$. Furthermore, we have $B = I$ (identity operator) and control function $u(t) = g(t)$. Let the final state on sub-intervals $[0, \pi/4]$ and $[\pi/2, \pi]$ are $fs(\pi/4)$ and $fs(\pi)$, respectively. The control function is defined by

$$u(t) = \begin{cases} u_1(t), & t \in [0, \pi/4], \\ u_2(t), & t \in [\pi/2, \pi], \end{cases}$$

where

$$\begin{aligned} u_1(t) &= (W_0^{\pi/4})^{-1}\Big[fs(\pi/4) - \cos\sqrt{a_1}(\pi/4) - \frac{1}{\sqrt{a_1}}\sin\sqrt{a_1}(\pi/4) \\ &- \frac{1}{\sqrt{a_1}}\int_0^{\pi/4}\sin\sqrt{a_1}(\pi/4 - s)f(s,(s+c_1))ds\Big](t), \\ u_2(t) &= (W_{\pi/2}^{\pi})^{-1}\Big[fs(\pi) - \frac{1}{10}\sin(\pi/2)x(\frac{\pi}{4}^-)\cos\sqrt{a_1}(\pi - \pi/2) \\ &- \frac{1}{\sqrt{a_1}}\frac{1}{10}\cos(\pi/2)x(\frac{\pi}{4}^-)\sin\sqrt{a_1}(\pi - \pi/2) \\ &- \frac{1}{\sqrt{a_1}}\int_{\pi/2}^{\pi}\sin\sqrt{a_1}(\pi - s)f(s,(s+c_1))ds\Big](t), \end{aligned}$$

and

$$W_0^{\pi/4}u_1 = \frac{1}{\sqrt{a_1}}\int_0^{\pi/4}\sin\sqrt{a_1}(\pi/4 - s)Bu_1(s)ds, \quad W_{\pi/2}^{\pi}u_2 = \frac{1}{\sqrt{a_1}}\int_{\pi/2}^{\pi}\sin\sqrt{a_1}(\pi - s)Bu_2(s)ds.$$

In addition, we get $|f(t,\psi_1) - f(t,\psi_2)| \le \frac{1}{16}|\psi_1 -, \psi_2|$, $|J_1^1(t,x_1) - J_1^1(t,x_2)| \le \frac{1}{10}|x_1 - x_2|$ and $|J_1^2(t,x_1) - J_1^2(t,x_2)| \le \frac{1}{10}|x_1 - x_2|$; therefore, (A2) and (A5) are satisfied with $K_1 = \frac{1}{16}$, $L_{J_1^1} = \frac{1}{10}$ and $L_{J_1^2} = \frac{1}{10}$. Moreover, $|\cos\sqrt{a_1}t| \le 1$, therefore (A1) is satisfied with $N_1 = 1$. By Lemma 1, we calculate $l = \int_\infty^0 h(s)ds$, where $h(t) = e^t$ be a continuous function that results in $l = 1$. Furthermore, we have

$$\begin{aligned} L_F &= \max\Big\{N_1T^2K_1l,\ L_{J_1^1}[N_1L_{J_1^1} + N_1TL_{J_1^2} + N_1T^2K_1l]\Big\} \\ &= \max\Big\{1.\pi^2.\frac{1}{16}.1,\ \frac{1}{10}\Big[1.\frac{1}{10} + 1.\pi.\frac{1}{10} + 1.\pi^2.\frac{1}{16}.1\Big]\Big\} \\ &< 1. \end{aligned}$$

Hence, the condition in Theorem 1 is satisfied. Therefore, Theorem 1 can be applied to problem (11).

6. Conclusions

In this manuscript, total controllability of second order semi-linear systems with infinite delay and non-instantaneous impulses is investigated. The proposed problem is simply concerned with time delay, which is unbounded. However, in the real-world problems, there are more complicated situations in which the delays depend on the unknown functions. Modeled equations of these problems are frequently called equations with state-dependent delay. Hence, the above results can be extended to study the controllability of various types of semi-linear dynamical systems with state-dependent delay.

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