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# Exact Solutions of Non-Linear Evolution Models in Physics and Biosciences Using the Hyperbolic Tangent Method

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**Abstract:** There has been considerable interest in seeking exact solutions of non-linear evolution equations that describe important physical and biological processes. Nonetheless, it is a difficult undertaking to determine closed form solutions of mathematical models that describe natural phenomena. This is because of their high non-linearity and the huge number of parameters of which they consist. In this article we determine, using the hyperbolic tangent (tanh) method, travelling wave solutions to non-linear evolution models of interest in biology and physics. These solutions have recognizable properties expected of other solutions and thus can be used to deduce properties of the general solutions.

**Keywords:** hyperbolic tangent method; non-linear evolution models; travelling wave solutions

## 1. Introduction

The study of non-linear partial differential equations is an important area of research in applied mathematics, theoretical physics and engineering. There has been considerable interest in seeking exact solutions of non-linear evolution equations that describe important physical and dynamical processes. In the recent past, many powerful methods such as variational iteration method [1], homotopy analysis method [2], homology pertubation techniques [3], modified tanh-coth method [4], the Jacobi elliptic function method [5] and integral transform operators [6] have been used to obtain exact travelling wave solutions of non-linear problems.

In some conservative systems, solutions are found by direct integration, suitable transformation and other techniques. The original partial differential equation could also be solved with direct methods such as Hirota's bilinear technique [7], truncated painleve expansion [8] and direct algebraic method [9]. More precisely, there is no single method that can be used to handle all types of non-linear problems. A powerful and effective technique called the hyperbolic tangent (tanh) method helps in finding exact solutions of non-linear differential equations which allows all solitary and shock wave solutions to be obtained [10]. Moreover, the main advantage of this method is that it helps to find exact solutions of higher non-linear evolution equations which are of fundamental importance. Further, this technique is straightforward and only minimal algebra is required.

Most non-linear partial differential equations require powerful methods to solve for an explicit solution. Quite recently, travelling wave solutions of complex non-linear wave equations were found with the aid of the tanh method. The main idea of this method is to express the solution of the non-linear differential equation as a polynomial. It is based on the homogeneous balance principle [11].

The tanh method is based on priori assumption that travelling waves can be expressed in terms of hyperbolic functions [12]. This method was proposed for obtaining travelling wave solutions of non-linear waves that are essentially of a localized nature. It was later adopted to determine exact travelling wave solutions of generalised Hirota Satsuma coupled Korteweg-de Vries (KdV) system, the doubled Sine-Gordon equation and Schrondinger equation.

The tanh method was first presented by Malfliet [13]. Much work has been focused on the various extensions and applications of the method in Khater etal. [14]. Huibin and Kelin [12] introduced a power series in tanh as a possible solution and substituted this expansion directly into a higher-order KdV equation. Wei [15] used a discrete singular convolution algorithm for the integration of the sine-Gordon equation. Fan and Hon [16] introduced an extended tanh method where the solution is based on a series expansion of the Riccati equation. The double sine-Gordon equation, (2 + 1) dimensional sine-Gordon equation, and the coupled Schrodinger-KdV equation were handled by using the generalized tanh method in [16].

It is a difficult undertaking to determine analytical solutions of several mathematical models that describe natural phenomena. The aim of this study, therefore, is to determine analytic solutions to some useful mathematical models in biosciences and physics using the tanh method and to demonstrate how powerful, yet easy, the method is.

## 2. Preliminaries

In this section, we introduce the theory behind travelling waves and give a schematic outline of the hyperbolic tangent method.

### *2.1. Travelling Wave Solutions*

A travelling wave is one that advances in a particular direction, with an additional property of retaining a fixed shape. Travelling waves are associated with having a constant velocity throughout their propagation. Such waves are observed in many areas of science, like in combustion which may occur as a result of chemical reaction [17]. Examples of travelling waves in nature include the impulses found in the fibre nerves [18], the laws of conservation connecting to problems in fluid dynamics [19] and structures present in solid mechanics are also typically modelled as standing waves [20].

The phenomena of waves is also observed in many natural reaction, convection and diffusion processes. Studying this occurrence is a motivation to know the reason why travelling waves are essential in mathematical analysis. To this effect, analysis of travelling waves provides a means of finding closed form solutions of the equation. Moreover, travelling wave solutions are easier to analyse with recognizable properties expected of other solutions and thus can be used as a tool in comparison principles and to determine the properties of general solutions.

### *2.2. The Hyperbolic Tangent Method (Tanh Method)*

We now present a schematic outline to determine the main features of the technique and demonstrate the method using some examples. The reader may as well refer to [14,21–23] for the method description. The following are the steps taken in using the tanh method:

1. Suppose one needs to determine solitary wave solutions to a non-linear partial differential equation of the form

$$U_t = F(U, U_{xx}, U_{xxx}, \cdots) \text{ or } U_{tt} = F(U, U_x U_{xx}, U_{xxx}, \cdots). \tag{1}$$

The solution to Equation (1) is proposed to be a polynomial

$$F(Y) = \sum_{n=0}^{N} a_n Y^n. \tag{2}$$

A travelling wave solution requires the coordinates: $z = k(x - ct)$ and $u(x,t) = U(z)$, where $U(z)$ represents the, localized, wave solution which travels with a velocity $c$ and wave number $k$.

Without loss of generality, we define $k > 0$. Consequently, the PDEs are transformed into ODEs. That is, for example the first and second order, partial derivatives with respect to time and space become

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -kc\frac{dU}{dz}, \quad \frac{\partial u^2}{\partial t^2} = k^2c^2\frac{d^2U}{dz^2}, \\ \frac{\partial u}{\partial x} &= k\frac{dU}{dz}, \quad \frac{\partial^2 u}{\partial x^2} = k^2k^2\frac{d^2U}{dz^2}. \end{aligned}\right\} \tag{3}$$

2. The central step is the introduction of $Y = \tanh z$ as a new independent variable and the corresponding derivatives are then changed as follows:

$$Y' = \operatorname{sech}^2 z = 1 - \tanh^2 z = 1 - Y^2 \tag{4}$$

$$\frac{dF(Y)}{dz} = (1 - \tanh^2 z)\frac{dF(Y)}{dY} = (1 - Y^2)\frac{dF(Y)}{dY}. \tag{5}$$

$$\frac{d^2F(Y)}{dz^2} = (1 - Y^2)\frac{d}{dY}\left((1 - Y^2)\frac{dF(Y)}{dY}\right). \tag{6}$$

$$\frac{d^3F(Y)}{dz^3} = (1 - Y^2)\frac{d}{dY}\left[(1 - Y^2)\frac{d}{dY}\left((1 - Y^2)\frac{dF(Y)}{dY}\right)\right]. \tag{7}$$

Next, the degree of the polynomial (2) is determined by equating every two possible highest exponents in the equation to get a linear system for $N$ and then that system is solved rejecting any solution $N$ which is not a positive integer.

3. Determining the degree of the polynomial and coefficients $a_n, n = 0, 1, 2, \cdots N$. Solving the non-linear system is the most involving step. So the following assumptions are made:

   (i) All parameters in the problem are considered strictly positive. If some parameters are zero, we must calculate $N$ again because the polynomial might have changed.
   (ii) The coefficient of the highest power of $Y$ term must be non-zero
   (iii) The wave number $k$ is assumed to be positive.

4. Substitute the solutions for the coefficients and parameters into the original equation.

For clarity, let us look at some examples:

1. Korteweg-de Vries (KdV) equation

The KdV equation is one of the most famous non-linear partial differential equations. It was derived in fluid mechanics to describe shallow water waves in a rectangular channel. The equation is of the form:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + b\frac{\partial^3 U}{\partial x^3} = 0. \tag{8}$$

By setting $z = k(x - ct)$, the partial derivatives of Equation (8) are transformed into

$$\left.\begin{aligned} \frac{\partial U}{\partial t} &= -kc\frac{dU}{dz}, \ \frac{\partial U}{\partial x} = k\frac{dU}{dz} \\ \frac{\partial^2 U}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{k\partial U}{\partial z}\right) = k^2\frac{d^2U}{dz^2}, \frac{\partial^3 U}{\partial x^3} = k^3\frac{d^3U}{dz^3} \end{aligned}\right\} \tag{9}$$

Substituting each term in Equation (9) into Equation (8) one obtains

$$-kc\frac{dU}{dz} + kU\frac{dU}{dz} + bk^3\frac{d^3U}{dz^3} = 0. \tag{10}$$

Next, we introduce $Y = \tanh z$ and as previously derived, one obtains

$$
\begin{aligned}
&-kc(1-Y^2)\frac{dF(Y)}{dY}+kF(Y)(1-Y^2)\frac{dF(Y)}{dY}\\
&+bk^3(1-Y^2)\frac{d}{dY}\left[(1-Y^2)\frac{d}{dY}\left((1-Y^2)\frac{dF(Y)}{dY}\right)\right]=0
\end{aligned}
\tag{11}
$$

where $U = F(Y) = \sum_{n=0}^{N} a_n Y^n$. The next step is to substitute the derivative of the polynomial $F(Y)$ into Equation (11) and find the highest power of $Y$ in each term.

$$
\begin{aligned}
F(Y) &= \sum_{n=0}^{N} a_n Y^n\\
\frac{dF(Y)}{dY} &= \sum_{n=0}^{N} n a_n Y^{n-1}.
\end{aligned}
$$

For the first term in Equation (11), we have

$$
\begin{aligned}
-kc(1-Y^2)\frac{dF(Y)}{dY} &= -kc(1-Y^2)\left(\sum n a_n Y^{n-1}\right)\\
&= -kc\sum n a_n Y^{n-1} + kc\sum n a_n Y^{n+1}.
\end{aligned}
$$

The highest power of $Y$ in the first term is $Y^{N+1}$. For the second term in Equation (11), we have

$$
\begin{aligned}
kF(Y)(1-Y^2)\frac{dF(Y)}{dY} &= k\left(\sum a_n Y^n\right)(1-Y^2)\left(\sum a_n Y^{n-1}\right)\\
&= k\left(\sum\sum n a_n^2 Y^{2n-1} - \sum\sum n a_n^2 Y^{2n+1}\right).
\end{aligned}
$$

The highest power of $Y$ in the second term is $Y^{2N+1}$. For the third term in (11), we start from the inner, right, derivative to the outer, left, derivative such that

$$
\begin{aligned}
(1-Y^2)\frac{dF(Y)}{dY} &= (1-Y^2)\left(\sum n a_n Y^{n-1}\right).\\
\frac{d}{dY}\left((1-Y^2)\frac{dF(Y)}{dY}\right) &= \left[\sum n(n-1)a_n Y^{n-1} - \sum n(n+1)a_n Y^n\right].
\end{aligned}
$$

$$
\begin{aligned}
&(1-Y^2)\frac{d}{dY}\left((1-Y^2)\frac{dF(Y)}{dY}\right)\\
&=\sum n(n-1)a_n Y^{n-2} - \sum n(n-1)a_n Y^n - \sum n(n+1)a_n Y^n + \sum n(n+1)a_n Y^{n+2}.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dY}\left[(1-Y^2)\frac{d}{dY}\left((1-Y^2)\frac{dF(Y)}{dY}\right)\right]\\
&=\sum n(n-2)(n-1)a_n Y^{n-3} - \sum n(n)(n-1)a_n Y^{n-1} - \sum n(n)(n+1)a_n Y^{n-1}\\
&+\sum n(n+2)(n+1)a_n Y^{n+1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(1-Y^2)\frac{d}{dY}\left[(1-Y^2)\frac{d}{dY}\left((1-Y^2)\frac{dF(Y)}{dY}\right)\right]\\
&=(1-Y^2)[\sum n(n-2)(n-1)a_n Y^{n-3} - \sum n(n)(n-1)a_n Y^{n-1} - \sum n(n)(n+1)a_n Y^{n-1}\\
&+\sum n(n+2)(n+1)a_n Y^{n+1}]\\
&=\sum n(n-2)(n-1)a_n Y^{n-3} - \sum n(n-2)(n-1)a_n Y^{n-1} - \sum n(n)(n-1)a_n Y^{n-1}\\
&+\sum n(n)(n-1)a_n Y^{n+1} - \sum n(n)(n+1)a_n Y^{n-1} + \sum n(n)(n+1)a_n Y^{n+2}\\
&+\sum n(n+2)(n+1)a_n Y^{n+1} - \sum n(n+2)(n+1)a_n Y^{n+3}.
\end{aligned}
$$

We can notice that the highest power of $Y$ in the third term is $Y^{N+3}$. Taking the highest possible exponent of $Y$, we have $Y^{2N+1} = Y^{N+3}$ implying that $2N + 1 = N + 3$. Thus $N = 2$. The solution, $F(Y) = \sum_{n=0}^{2} a_n Y^n$, is therefore of the form $F(Y) = a_0 + a_1 Y + a_2 Y^2$. Substituting this solution into Equation (11), one obtains $F(Y) = a_0 + a_2 Y^2$ where $a_0 = 8bk^2$ and $a_2 = -12bk^2$. The values of $k$ and $c$ are arbitrary. We can convince ourselves that

$$\begin{aligned} F(Y) &= 8bk^2 - 12bk^2 Y^2 = 4bk^2(2 - 3\tanh z), \\ U(x,t) &= 4bk^2\left(2 - 3\tanh^2\left[k(x - ct)\right]\right). \end{aligned}$$

2. Burgers equation

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} - a\frac{\partial^2 U}{\partial x^2} = 0 \tag{12}$$

The Burgers equation is another famous non-linear PDE used in modelling various phenomena in applied mathematics. The positive parameter $a$ is a diffusion constant which represents the dissipative effect of $U$. In order to determine closed form solutions of Burgers equation, the same schematic outline in the previous example is followed. Firstly, Equation (12) is transformed into

$$-kc\frac{dU}{dz} + kU\frac{dU}{dz} - ak^2\frac{d^2U}{dz^2} = 0. \tag{13}$$

Introducing $Y = \tanh z$ and $U := F(Y) = \sum_{n=0}^{N} a_n Y^N$ yields

$$\begin{aligned} -kc(1 - Y^2)\frac{dF(Y)}{dY} + kF(Y)(1 - Y^2)\frac{dF(Y)}{dY} \\ -ak^2(1 - Y^2)\frac{d}{dY}\left[(1 - Y^2)\frac{d}{dY}\left((1 - Y^2)\frac{dF(Y)}{dY}\right)\right] = 0. \end{aligned} \tag{14}$$

After substituting the derivatives of the polynomial $F(Y)$ into (14) and balancing the highest powers of $Y$, we arrive at $Y^{2N+1}$ in the second term and third term $Y^{N+2}$. Hence $2N + 1 = N + 2$ and therefore in this example $N = 1$. Hence the polynomial solution takes the form

$$F(Y) = a_0 + a_1 Y. \tag{15}$$

Substituting Equation (15) into (14), the solution obtained is $F(Y) = a_0 - 2akY$ where $a_0 = r_{15}$, $a_1 = r_{16}$, $k = -\frac{r_{16}}{2a}$. The constants $r_{15}$ and $r_{16}$ remain arbitrary and the values are greater than zero. Lastly, we need to find the value of $a_0$ and require that the solution vanishes for $z \longrightarrow \infty$ as $Y \longrightarrow 1$. So we let $F(Y) = 0$ such that $a_0 = 2ak$ since $Y \longrightarrow 1$. Thus

$$\begin{aligned} F(Y) &= 2ak - 2akY = 2ak(1 - Y) \\ U(x,t) &= 2ak\left[1 - \tanh\ k(x - ct)\right]. \end{aligned}$$

This is a well known shock wave solution for burgers equation [22].

In the next section, we further demonstrate how to use the tanh method in determining solutions to even complex evolution mathematical models, thus highlighting how useful and powerful the method is.

## 3. Travelling Wave Solutions to Selected Models

In this section we determine closed form solutions to selected, mathematical models in the literature using the tanh method. We as well, for some of the models, prove the existence of travelling wave solutions. We consider the following models:

1. FitzHungh-Nagumo equation [24];
2. Korteweg-de Vries-burgers equation [12,23];
3. Melanoma model [25];
4. Microbial growth model [26];
5. Tumour-immune interaction model [27].

### *3.1. FitzHungh-Nagumo Equation*

The FitzHungh-Nagumo equation is a common approximation to describe nerve fibre propagation. The model is of the form

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + U(1-U)(a-U) = 0, \tag{16}$$

where $a$ is a real constant. If $a = -1$, one gets the real Newell-Whitehead equation describing the dynamical behaviour near the bifurcation point for the Rayleigh-Benard convection of binary fluid mixtures [9]. We will firstly transform Equation (16) into a system of order differential equations and prove the existence of travelling wave solutions by analysing the system's phase space.

**Lemma 1.** *By transforming the system* (16) *into a system of first order ODES and using phase space analysis, the minimum wave speed of invasion for the FitzHungh-Nagumo equation is:*

$$c_{min} = \frac{2\sqrt{a}}{k}. \tag{17}$$

**Proof.** Equation (16) is transformed to an ODE by letting $z = k(x - ct)$ thus obtaining

$$k^2 \frac{d^2 U}{dz^2} = -kc\frac{dU}{dz} + U(1-U)(a-U). \tag{18}$$

Equation (18) is then transformed into a system of autonomous ODEs by letting

$$\left.\begin{array}{ll} \frac{dU}{dz} & = V \quad \text{such that} \\ k^2 \frac{dV}{dz} & = -kcV + U(1-U)(a-U). \end{array}\right\} \tag{19}$$

The equilibrium points, $U, V$ of the system (19) are $(0,0)$, $(1,0)$ and $(a,0)$. The Jacobian matrix of the system (19) is

$$J(U,V) = \begin{pmatrix} 0 & 1 \\ 3U^2 - 2(1-a)U + a & -kc \end{pmatrix}. \tag{20}$$

The Jacobian matrices at the points $(0,0)$,$(1,0)$ and $(a,0)$ respectively are

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ a & -kc \end{pmatrix}. \tag{21}$$

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ 3a+1 & -kc \end{pmatrix}. \tag{22}$$

$$J(a,0) = \begin{pmatrix} 0 & 1 \\ 5a^2 - a & -kc \end{pmatrix}. \tag{23}$$

The eigenvalues at $J(0,0)$ are

$$\lambda_{(0,0)} = \frac{-kc \pm \sqrt{k^2c^2 - 4a}}{2}. \tag{24}$$

The eigenvalues at $J(1,0)$ are

$$\lambda_{(1,0)} = \frac{-kc \pm \sqrt{k^2c^2 + 12a + 4}}{2} \tag{25}$$

and those at $J(a,0)$ are

$$\lambda_{(a,0)} = \frac{-kc \pm \sqrt{k^2c^2 + 20a^2 - 4a}}{2}. \tag{26}$$

A travelling wave solution would be the heteroclinic connection joining the point $(0,0)$ to $(1,0)$ or $(a,0)$. It therefore means that the solution leaving the equilibrium point $(0,0)$ must not oscillate around it implying that the eigenvalues of $J(0,0)$ must not be complex. Imposing this condition implies that $k^2c^2 - 4a > 0$ which means that the minimum wave speed $c_{\min} = \frac{2\sqrt{a}}{k}$. This implies that the FitzHungh-Nagumo equation exhibits travelling wave solutions for $c \geq c_{\min}$. $\square$

We now proceed to determine the travelling wave solutions. Introducing $Y = \tanh z$ and $U := F(Y) = \sum_{n=0}^{N} a_n Y^N$, one obtains

$$\begin{aligned} -kc(1-Y^2)\frac{dF(Y)}{dY} - k^2(1-Y^2)\frac{d}{dY}\left[(1-Y^2)\frac{dF(Y)}{dY}\right] \\ + F(Y)(1-F(Y))(a-F(Y)) = 0. \end{aligned} \tag{27}$$

After substituting the derivatives of the polynomial $F(Y)$ into (27) and balancing the highest powers of $Y$, we arrive at $Y^{N+2}$ in the second term and the third term $Y^{3N}$ from (27), $3N = N+2$ and therefore in this example $N = 1$. The polynomial, thus, takes the form $F(Y) = a_0 + a_1 Y$ and the solution obtained is $F(Y) = \frac{1}{2} - \frac{1}{2}Y$ where $a_0 = \frac{1}{2}$, $a_1 = -\frac{1}{2}$, $k = \frac{1}{2\sqrt{2}}$, $c = \frac{2a-1}{\sqrt{2}}$.

The closed form solution of the Fitzhugh-Nagumo equation is thus given by:

$$U(x,t) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{\sqrt{2}} - \frac{2a-1}{\sqrt{2}}t\right)\right].$$

### 3.2. Korteweg-de Vries-Burgers Equation

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + b\frac{\partial^3 U}{\partial x^3} - a\frac{\partial^2 U}{\partial x^2} = 0 \tag{28}$$

This equation is familiar in fluid mechanics. It describes shallow water waves in an elastic tube with dissipation. To determine a closed form solution of this equation, like with all the other examples, we let $z = k(x - ct)$ such that (28) is transformed to

$$-kc\frac{dU}{dz} + Uk\frac{dU}{dz} + b\frac{d^3U}{dz^3} - a\frac{d^2U}{dz^2} = 0. \tag{29}$$

By following the same procedure as in Section 3.1 we obtain a limiting speed $c_{\min} = \frac{a^2}{4bc}$. Substituting $Y = \tanh z$ and $U := F(Y) = \sum_{n=0}^{N} a_n Y^N$ into Equation (29) yields

$$\begin{aligned} &-kc(1-Y^2)\frac{dF(Y)}{dY} + F(Y)k(1-Y)(1-Y^2)\frac{dF(Y)}{dY} \\ &+ bk^3(1-Y^2)\frac{d}{dY}\left[(1-Y^2)\frac{d}{dY}\left[(1-Y^2)\frac{dF(Y)}{dY}\right]\right] \\ &- a(1-Y^2)\frac{d}{dY}\left[(1-Y^2)\frac{dF(Y)}{dY}\right] = 0. \end{aligned} \tag{30}$$

Substituting the derivatives of the polynomial $F(Y)$ into (30) and balancing the highest powers of $Y$, we get $Y^{2N+1}$ in the second term and third term $Y^{N+3}$ such that $N = 2$. The solution is determined to be

$$F(Y) = 36k^2b(1-Y)\left(1+\frac{1}{3}Y\right) \tag{31}$$

where $a_0 = 36\,r_{27}^2 r_{28},\ a_1 = 12\,r_{27}^2 r_{28},\ k = r_{27},\ c = 24\,r_{27}^2 r_{28},\ a = 10\,r_{27}^3 r_{28},\ b = r_{28}$. The coefficients $r_{27}$ and $r_{28}$ remain arbitrary and the values are greater than zero. We require that the solution vanishes for $z \longrightarrow \infty$ as $Y \longrightarrow 1$ because $F(Y) = (1-Y)^m \sum_{n=0}^{N-m} a_n Y^N$ due to the boundary condition. We take the case $m = 1$ to get this solution but $N$ remains the way we had it earlier. Therefore

$$F(Y) = 36k^2b(1-Y)\left(1+\frac{1}{3}Y\right) \quad \text{with} \quad c = 24bk^2.$$

Thus

$$\begin{aligned} F(Y) &= 36k^2b\left[1-\tanh z\left(1+\frac{1}{3}\tanh z\right)\right] \\ U(x,t) &= 36k^2b\left[1-\tanh k(x-ct)\left(1+\frac{1}{3}\tanh k(x-ct)\right)\right]. \end{aligned}$$

This represents a particular combination of a solitary wave in the first term with Burgers shock wave in the second term.

### 3.3. Melanoma Model

We determine closed form solutions for a melanoma growth model originally presented in [25]. The model describes haptotatic cell invasion by a skin cancer type known as melanoma. The model, in dimensionless form, is derived as:

$$\frac{\partial E}{\partial t} = -E^2U \tag{32}$$

$$\frac{\partial U}{\partial t} = U(1-U) - \frac{\partial}{\partial x}\left(U\frac{\partial E}{\partial x}\right) \tag{33}$$

with boundary conditions

$$\lim_{x\to-\infty} E(t,x) = 0,\ \lim_{x\to\infty} E(t,x) = \tilde{E},\ \lim_{x\to-\infty} U(t,x) = 1,\ \lim_{x\to\infty} U(t,x) = 0$$

and $x \in \mathbb{R}, t \in \mathbb{R}^+$. Here $E(t,x)$ represents the extracellular matrix concentration and $U(t,x)$ is the invasive tumor population. It is important to note that we ignore the Protease density as it is assumed to be constant. By letting $z = k(x-ct)$, $Y = \tanh z$, $E(Y) = F_1(Y) = \sum_{n=0}^{N} a_n Y^N$ and $U(Y) = F_2(Y) = \sum_{n=0}^{N} b_n Y^N$, the model Equation (32) are transformed to

$$\begin{aligned} -kc(1-Y^2)\frac{dF_1(Y)}{dY} &= \frac{F_1^2(Y)F_2(Y)}{kc} \\ -kc(1-Y^2)\frac{dF_2(Y)}{dY} &= F_2(1-F_2(y)) - k^2c^2(1-Y^2)^2\frac{dF_1(Y)}{dY}\frac{dF_2(Y)}{dY} \\ &\quad - k^2F_2(Y)(1-Y^2)\left[\frac{d}{dY}\left[(1-Y^2)\frac{dF_1(Y)}{dY}\right]\right]. \end{aligned} \tag{34}$$

Following the procedure as in the previous examples, the travelling wave solution permitted by (32) are given by

$$U(t,x) = \frac{1}{2} - \frac{1}{2}\tanh\left[k(x - \frac{1}{2k}t)\right], \quad E(t,x) = 0$$

where $k$ is arbitrary.

*3.4. Microbial Growth Model*

$$\begin{aligned} \frac{\partial S}{\partial t} &= \rho\frac{\partial^2 S}{\partial x^2} - \alpha\frac{\partial S}{\partial x} - f(S)P \\ \frac{\partial P}{\partial t} &= d\frac{\partial^2 P}{\partial x^2} - \alpha\frac{\partial P}{\partial x} + [f(S) - K]P \end{aligned} \tag{35}$$

The model (35) with $\alpha = 0$ was introduced in [28] to study a population model with diffusion. This is a model for microbial growth in a flow reactor. This system with $\alpha = 0$ and $f(S) = S$, is a simple diffusive epidemic model in which $S$ and $P$ represents the densities of susceptible and infective population. For $\alpha = 0$ and $K = 0$, Equation (35) also serves as a model for single-stage reaction of first order combustion [29]. Quite recently, this model was derived in [26] to study a single population microbial growth for limiting nutrient in a flow reactor. We use the hyperbolic tangent method to determine closed form solutions to the model Equation (35).

We transform the PDEs in (35) to ODEs in Equation (36) by letting $z = k(x - ct)$.

$$\left.\begin{aligned} kc\tfrac{dS}{dz} + \rho k^2\tfrac{d^2S}{dz^2} - \alpha k\tfrac{dS}{dz} - SP &= 0 \\ kc\tfrac{dP}{dz} + dk^2\tfrac{d^2P}{dz^2} - \alpha k\tfrac{dP}{dz} + [S - K]P &= 0. \end{aligned}\right\} \tag{36}$$

By following the same procedure as in Section 3.1, one obtains a limiting speed $c_{\min}$ given by

$$c_{\min} = \frac{2\sqrt{Kdk}}{k^2} - \alpha k^2. \tag{37}$$

We thus deduce that travelling wave solutions to the model Equation (35) only exist for $c \geq c_{\min}$.

Next, we introduce $Y = \tanh z$ and Equation (36) is transformed into

$$\left.\begin{aligned} kc(1-Y^2)\tfrac{dF_1(Y)}{dY} + \rho k^2(1-Y^2)\tfrac{d^2F_1(Y^2)}{dY} - \alpha k(1-Y^2)\tfrac{dF_1(Y)}{dY} - F_1(Y)F_2(Y) &= 0 \\ kc(1-Y^2)\tfrac{dF_2(Y)}{dY} + dk^2(1-Y^2)\tfrac{d^2F_2(Y)}{dY^2} - \alpha k(1-Y^2)\tfrac{dF_2(Y)}{dY} - [F_1(Y) - K]F_2(Y) &= 0 \end{aligned}\right\} \tag{38}$$

where $F_1(Y) = \sum_{n=0}^{N} a_n Y^n$ and $F_2(Y) = \sum_{n=0}^{N} b_n Y^n$. By substituting the derivatives of the polynomials $F_1(Y)$ and $F_2(Y)$ in (38), finding the highest power of $Y$ in each term of the system and balancing the highest powers of $Y$. We arrive at $Y^{N+2}$ in the second term and the forth term $Y^{2N}$. Hence $N + 2 = 2N$ and therefore in this example $N = 2$.

One thus obtains the flowing solutions

$$S(x,t) = a_0 + a_1 \tan k(x - ct) + a_2 \tanh^2 k(x - ct) \text{ and } P(x,t) = b_0 + b_2 \tanh^2 k(x - ct)$$

where

$$a_0 = -\frac{K\left((3d-2)^2 - \rho^2\right)}{\rho^2},\ a_1 = \frac{3K(3d-2)}{\rho},\ a_2 = \frac{3K(3d-2)^2}{\rho^2}$$

$$b_0 = \frac{3K(3d-2)^2}{(d\rho^2)},\ b2 = -\frac{3K(3d-2)^2}{(d\rho^2)} \quad \text{with}$$

$$c = \sqrt{\left(\frac{\alpha^2 - 9d^2K}{18d^3}\right)} \quad \text{and} \quad k = \sqrt{\frac{K}{2d}}.$$

These solutions represent shock waves observed in reaction-diffusion equations. Next, we determine travelling wave solutions to an even more complicated model for tumour-immune growth.

### *3.5. Tumour-Immune Interaction Model*

Malinzi and Amima [27] analysed a moving boundary problem to investigate cancer dormancy. In one dimension, and with a constant radius, a simplified form of the model takes the form

$$\begin{aligned}
\frac{\partial X}{\partial t} &= \psi_x \frac{\partial^2 X}{\partial r^2} - \beta X \frac{\partial^2 U}{\partial r^2} - \beta \frac{\partial X}{\partial r}\frac{\partial U}{\partial r} + \alpha_1 X(1-\alpha_2 X) + \frac{fXY}{\eta + Y} - \phi_1 XY, \\
\frac{\partial Y}{\partial t} &= \psi_y \frac{\partial^2 Y}{\partial r^2} + \beta_1 Y(1-\beta_2 Y) - \phi_2 XY, \\
\frac{\partial U}{\partial t} &= \psi_u \frac{\partial^2 Y}{\partial r^2} + \phi_3 XY - d_1 U.
\end{aligned} \tag{39}$$

By letting $z = k(r - ct)$, the model Equation (39) is transformed to

$$\begin{aligned}
kc\frac{dX}{dz} + \psi_x k^2 \frac{d^2X}{dz^2} - \beta k^2\left(X\frac{d^2U}{dz^2} + \frac{dU}{dz}\frac{dX}{dz}\right) + \varphi_1 X(1-\varphi_2 X) + \frac{\delta XY}{\gamma + Y} - \nu_1 XY = 0, \\
kc\frac{dY}{dz} + \psi_y k^2 \frac{d^2Y}{dz^2} + \sigma_1 Y(1-\sigma_2 Y) - \nu_2 XY = 0, \\
kc\frac{dU}{dz} + \psi_u k^2 \frac{d^2U}{dz^2} + \nu_3 XY - \mu_1 U = 0
\end{aligned} \tag{40}$$

with boundary conditions:

$$\lim_{t \to -\infty}(X, Y, U) = X^0 \quad \text{and} \quad \lim_{t \to -\infty}(X, Y, U) = X^1$$

where $X^0 = (0, 0, 0)$ or $X^0 = (1/\varphi_2, 0, 0)$ and $X^1 := (X^*, Y^*, U^*)$ is the tumour endemic state.

**Lemma 2.** *By transforming the system (40) into a system of first order differential equations and using phase space analysis, the minimum wave speeds of invasion for the tumour and immune travelling wave fronts, respectively, are:*

$$c^y_{min} = \frac{2}{k^2}\sqrt{\sigma_1 \psi_y} \quad \textit{and} \quad c^x_{min} = 2\sqrt{\varphi_1 \psi_x}. \tag{41}$$

**Proof.** Equation (40) are transformed into first order differential equations by letting $y_1 = dX/dz$, $y_2 = dY/dz$, $y_3 = dU/dz$ to get

$$\begin{aligned}
\frac{dy_1}{dt} &= \frac{1}{k^2\psi_x}\left(\beta k^2 (X y_3 + y_1 y_3) - kcy_1 - \varphi_1 X(1-\varphi_2 X) - \frac{\delta XY}{\gamma + Y} + \nu_1 XY\right), \\
\frac{dX}{dt} &= x_1, \\
\frac{dy_2}{dt} &= \frac{1}{k^2\psi_y}\Big(-kcy_2 - \sigma_1 y(1-\sigma_2) + \nu_2 XY\Big), \\
\frac{dy}{dt} &= y_2, \\
\frac{dy_3}{dt} &= \frac{1}{k^2\psi_u}\Big(-kcy_3 - \nu_3 XY + \mu_1 U\Big), \\
\frac{du}{dz} &= y_3.
\end{aligned} \tag{42}$$

The travelling wave solutions are trajectories connecting equilibrium $X^0$ to $X^1$. Just as discussed before, the solution leaving $X^0$ is therefore not required to oscillate, that is, the eigenvalues of the Jacobian matrix of (42) evaluated at $X^0$ must not have complex roots.

$$J(X^0) = \begin{pmatrix} \frac{c}{(-\psi_x)k} & \frac{\psi_1}{(-\psi_x)k^2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{k^3 c}{\psi_y} & -\frac{k^2\sigma_1}{\psi_y} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{k^3 c}{\psi_u} & \frac{k^2\mu_1}{\psi_u} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{43}$$

The eigenvalues are:

$$\begin{gathered} -\frac{k^3 c + \sqrt{k^4 c^2 - 4\,\psi_y\sigma_1}k}{2\,\psi_y}, -\frac{k^3 c - \sqrt{k^4 c^2 - 4\,\psi_y\sigma_1}k}{2\,\psi_y}, \frac{c - \sqrt{-4\,\psi_1\psi_x + c^2}}{2\,2\psi_x}, \\ \frac{c + \sqrt{-4\,\psi_1\psi_x + c^2}}{2\,2\psi_x}, -\frac{k^3 c + \sqrt{k^4 c^2 + 4\,\mu_1\psi_u}k}{2\,\psi_u}, -\frac{k^3 c - \sqrt{k^4 c^2 + 4\,\mu_1\psi_u}k}{2\,\psi_u}. \end{gathered} \tag{44}$$

The conditions

$$\sqrt{k^4 c^2 - 4\,\psi_y\sigma_1}k \quad \text{and} \quad \sqrt{-4\,\psi_1\psi_x + c^2} \tag{45}$$

must determine the minimum wave speeds since all the other eigenvalues are real except for those with these terms. That is $k^4c^2 - 4\,\psi_y\sigma_1 \geq 0$ and $-4\,\psi_1\psi_x + c^2 \geq 0$ which gives

$$c^y_{\min} = \frac{2}{k^2}\sqrt{\sigma_1\psi_y} \quad \text{and} \quad c^x_{\min} = 2\sqrt{\varphi_1\psi_x}.$$

□

To determine the closed form solutions, we introduce the transformation $W = \tanh(z)$ and let the solutions take the following forms

$$\begin{aligned} X(W) = F_1(W) &= (1-W)^{p_1}(1+W)^{q_1}\sum_{i=0}^{N-p_1-q_1} a_i W^n, \\ Y(W) = F_2(W) &= (1-W)^{p_2}(1+W)^{q_2}\sum_{i=0}^{N-p_2-q_2} b_i W^n, \\ U(W) = F_3(W) &= (1-W)^{p_3}(1+W)^{q_3}\sum_{i=0}^{N-p_3-q_3} c_i W^n, \end{aligned} \tag{46}$$

where $p_1(\neq 0) + q_1(\neq 0) = 2, \cdots, N$, $p_2(\neq 0) + q_2(\neq 0) = 2, \cdots, N$, $p_3(\neq 0) + q_3(\neq 0) = 2, \cdots, N$.

Equation (40) become

$$
\begin{aligned}
kc(1-W^2)\frac{dF_1}{dW} + \psi_x k^2(1-W^2)\frac{d}{dW}\left((1-W^2)\frac{dF_1}{dW}\right)& \\
-\beta k^2 F_1(W)(1-W^2)\frac{d}{dW}\left((1-W^2)\frac{dF_3}{dW}\right) - \beta(k^2c^2(1-W^2)^2\frac{dF_1}{dW}\frac{dF_3}{dW})& \\
+\varphi_1 F_1(W)(1-\varphi_2 F_1(W)) + \frac{\delta F_1(W)F_2(W)}{\gamma + F_2(W)} - \nu_1 F_1(W)F_2(W) &= 0, \\
kc(1-W^2)\frac{dF_2}{dW} + \psi_y k^2(1-W^2)\frac{d}{dW}\left((1-W^2)\frac{dF_2}{dW}\right)& \\
+\sigma_1 F_2(W)(1-\sigma_2 F_2(W)) - \nu_2 F_1(W)F_2(W) &= 0, \\
kc(1-W^2)\frac{dF_3}{dW} + \psi_u k^2(1-W^2)\frac{d}{dW}\left((1-W^2)\frac{dF_3}{dW}\right) + F_1(w)F_2(W) - \mu_1 F_3(W) &= 0.
\end{aligned}
\tag{47}
$$

The substitution of $F_i(W)$ in Equation (47) yields $p_i = q_i = 1, i = 1,2$, that is,

$$F_1(W) = a_0(1-W)^2, \quad F_2(W) = b_0(1-W)^2, \quad F_3(W) = c_0(1-W)^2. \tag{48}$$

The following results are obtained after substituting (48) into (47):
Either

$$a_0 = \frac{1}{4\varphi_2}, \quad b_0 = 0, \quad c = \frac{1}{2\sqrt{6}}\sqrt{\frac{\varphi_1}{\psi_x}}, \quad c = \frac{5}{\sqrt{6}}\sqrt{\varphi_1\psi_x}, \quad c_0 = 0 \tag{49}$$

or

$$a_0 = 0, \quad b_0 = \frac{1}{4\sigma_2}, \quad k = \frac{1}{2\sqrt{6}}\sqrt{\frac{\sigma_1}{\psi_y}}, \quad c = \frac{5}{\sqrt{6}}\sqrt{\sigma_1\psi_y}, \quad c_0 = 0. \tag{50}$$

The result in (49) would imply immune cells attacking tumour cells with a wave front

$$X(t,r) = \frac{1}{4\varphi_2}\Big(1 - \tanh\left(k(r-ct)\right)\Big)^2, \text{ where } k = \frac{1}{2\sqrt{6}}\sqrt{\frac{\varphi_1}{\psi_x}} \text{ and } c = \frac{5}{\sqrt{6}}\sqrt{\varphi_1\psi_x}$$

and that in (50) would imply that tumour cells attack immune cells with a wave front

$$Y(t,r) = \frac{1}{4\sigma_2}\Big(1 - \tanh\left(k(r-ct)\right)\Big)^2, \text{ where } k = \frac{1}{2\sqrt{6}}\sqrt{\frac{\sigma_1}{\psi_y}} \text{ and } c = \frac{5}{\sqrt{6}}\sqrt{\sigma_1\psi_y}.$$

Both the travelling wave solutions in Equations (49) and (50) and the minimum wave speeds in Equation (41) are characterized by the cell carrying capacities, diffusion constants. Malinzi and Amima [27] note that these results imply that the cell invasion dynamics are mainly driven by their motion and growth rates. Thus, treatment options should strive to improve immune recognition of tumour cells, reduce the intrinsic growth rate of tumour cells and increase that for immune cells.

## 4. Conclusions

Most non-linear partial differential equations require powerful tools to determine exact solutions. Recently, travelling wave solutions of complex non-linear equations have been determined using the tanh method [14,21–23,30]. We have demonstrated how effective the tanh method is in determining closed form solutions to even highly non-linear evolution models; that is, melanoma model [25], microbial growth model [26] and a tumour-immune interaction model [27]. By elaborating all the steps involved, we further demonstrated how straightforward and concise the method is in comparison to other existing techniques. A possible extension of this work is to compare the solutions generated here using the tanh method with those generated by other methods in the literature.

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